A SIMPLE TUTORIAL ON DISCONTINUOUS GALERKIN METHODS

Jennifer Proft
CERMICS, ENPC

Outline

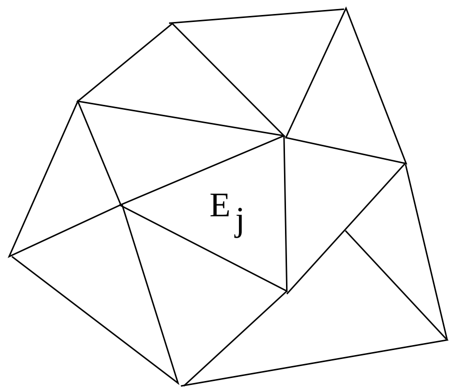
- Definitions
- A simple example
- Issues
- Historical development

  elliptic equations

  hyperbolic equations
- Discontinuous vs. continuous Galerkin
- Further discussion: convergence results, a posteriori results, current research

- Discontinuous Galerkin methods are those finite element methods which utilize element-wise discontinuous basis functions in the choice of approximation spaces.
- Let $\mathcal{P}^k$ define the space of polynomials of degree k on element E_j. A typical finite element subspace (broken space) is defined as

$$\mathcal{D}_k(\mathcal{E}_h) \equiv \{v : v|_{E_j} \in \mathcal{P}^k(E_j), \forall j = 1, .., N_h\}$$

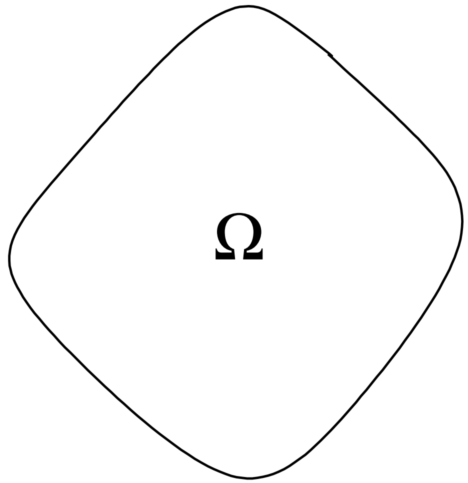


- Define the "broken" norm for positive integer m

$$|||\phi|||_{H^m} \equiv \left(\sum_{j=1}^{N_h} \|\phi\|^2_{H^m(E_j)} \right)^{1/2}$$

Consider the model problem on bounded domain

$$\begin{aligned} -\triangle u &= f && \text{in } \Omega \\ u &= u_D && \text{on } \partial\Omega_D \\ \nabla u \cdot n &= g \cdot n && \text{on } \partial\Omega_N \end{aligned}$$



We assume that $f \in L^2(\Omega)$ and $u_D \in H^{3/2}(\partial\Omega_D)$ and $g \in H^{1/2}(\partial\Omega_N)$. Then, there exists a unique solution $u \in H^1(\Omega)$.

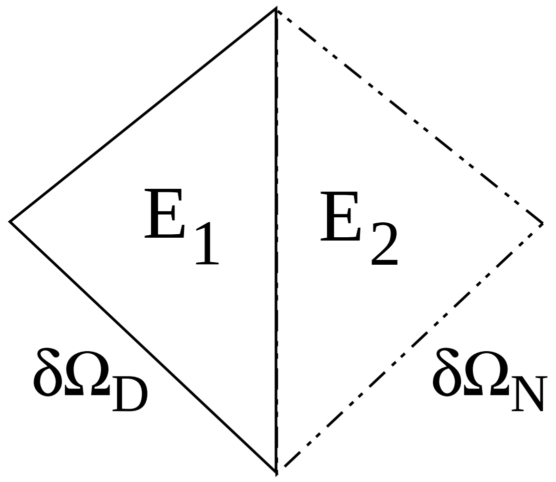
Weak variational formuations use integration by parts (Green's equation)

$$-\int_R \triangle u = \int_R \nabla u \cdot \nabla v - \int_{\partial R} \nabla u \cdot n\, v$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \, \nabla v = \int_\Omega f \, v + \int_{\partial\Omega_N} g \cdot n \, v$$



Discrete weak formulation

Find $u_h \in W_h \subset H_0^1(\Omega)$ such that for all $v \in W_h$

$$\int_\Omega \nabla u_h \, \nabla v = \int_\Omega f \, v + \int_{\partial\Omega_N} g \cdot n \, v$$

$$W_h(\Omega) \subset H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$$

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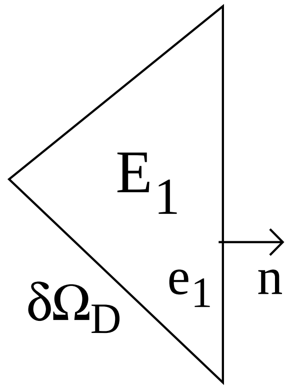
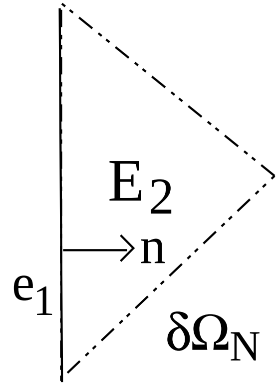
Weak formulation

Find $u \in H^1(E_1) \cup H^1(E_2)$ such that for all $v \in H^1(E_1) \cup H^1(E_2)$

$$\int_{E_1} \nabla u \, \nabla v - \int_{e_1} \nabla u \cdot n \, v - \int_{\partial\Omega_D} \nabla u \cdot n \, v = \int_{E_1} f \, v$$



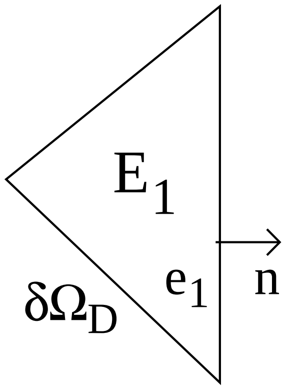
$$\int_{E_2} \nabla u \, \nabla v - \int_{e_1} \nabla u \cdot (-n) \, v = \int_{E_2} f \, v + \int_{\partial\Omega_N} g \cdot n \, v$$



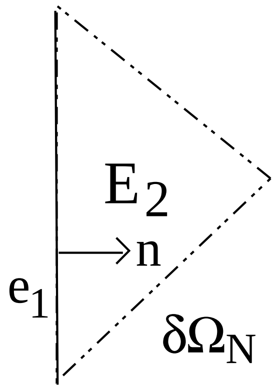
Discrete weak formulation

Find $u_h \in \mathcal{D}(\mathcal{E}_h)$ such that for all $v \in \mathcal{D}(\mathcal{E}_h)$

$$\int_{E_1} \nabla u_h \nabla v - \int_{e_1} \nabla u_h^- \cdot n\, v^- - \int_{\partial\Omega_D} \nabla u_h \cdot n\, v = \int_{E_1} f\, v$$



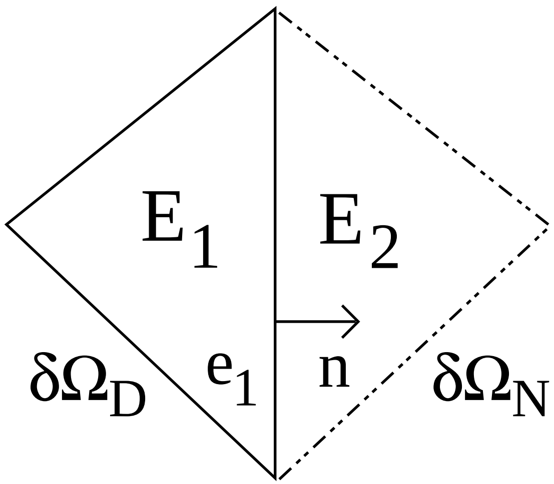
$$\int_{E_2} \nabla u_h \nabla v + \int_{e_1} \nabla u_h^+ \cdot n\, v^+ = \int_{E_2} f\, v + \int_{\partial\Omega_N} g \cdot n\, v$$



$$\mathcal{D}_k(\mathcal{E}_h) \equiv \{v : v|_{E_j} \in \mathcal{P}^k(E_j), \forall j = 1, .., N_h\}$$

Discrete weak formulation

Find $u_h \in \mathcal{D}(\mathcal{E}_h)$ such that for all $v \in \mathcal{D}(\mathcal{E}_h)$



$$\sum_{j=1}^{2} \int_{E_j} \nabla u_h \, \nabla v - \int_{e_1} [\nabla u_h \cdot n \, v]$$

$$- \int_{\partial\Omega_D} \nabla u_h^- \cdot n \, v^- = \sum_{j=1}^{2} \int_{E_j} f \, v + \int_{\partial\Omega_N} g \cdot n \, v$$

where the 'jump' is defined as $[w] = w^- - w^+$ for traces

$$v^-(x) = \lim_{\binom{s\to 0-}{k\to\infty}} \gamma_o v_k(x+sn_e), \qquad v^+(x) = \lim_{\binom{s\to 0+}{k\to\infty}} \gamma_o v_k(x+sn_e),$$

Some Issues

- How to incorporate dirichlet boundary conditions?
- What to do with jump on boundary term (i.e. choose fluxes)?
- Are these methods consistent? Stable? Convergent?
- How efficient are they to implement?
- How accurate are their solutions?
- How do these methods compare with other methods?
- Is there a unified framework for all DG methods?

Elliptic Equations

- First introduced by Nitsche in 1971, application to second-order elliptic equation
- Also known as (interior) penalty methods, attributed to Wheeler in 1978
- Vigorous development in past 5 years

Hyperbolic Equations

- First introduced by Reed and Hill in 1973, application to neutron transport problem
- First a priori analysis was published by LeSaint and Raviart in 1974
- Vigorous development in past 15 years (including Jaffré)

J. Nitsche, "Über ein Variationsprinzip zur Lösun von Dirichlet Problemen bei Verwendung von Teilöaumen, die keinen Randbedingungen unterworfen sind", *Abh. Math. Sem. Univ. Hamburg*, 36:9-15, 1971

I. Babuška, "The finite element method with interior penalty", *Math. Comp.*,27:221-228, 1973

W.Reed and T.Hill, "Triangular mesh methods for the neutron transport equation", *Los Alamos Scientific Laboratory*, LA-UR-73-479, Los Alamos, NM, 1973

P. LeSaint and P. Raviart, "On a finite element method for solving the neutron transport equations", in *Mathematical Aspects of Finite Elements in Partial Differential Equations*, Academic Press, 1974

DG on elliptic: Nitsche's method

Consider

$$\begin{aligned} -\triangle u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \qquad \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \nabla u \cdot n\, v = \int_\Omega f\, v$$

Since u vanishes on the boundary,

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \nabla u \cdot n\, v - \int_{\partial\Omega} \nabla v \cdot n\, u - \int_{\partial\Omega} \eta u\, v = \int_\Omega f\, v$$

Find $u_h \in W_h \subset H^1(\Omega)$ such that for all $v \in W_h$

$$\int_\Omega \nabla u_h \cdot \nabla v - \int_{\partial\Omega} \nabla u_h \cdot n\, v - \int_{\partial\Omega} \nabla v \cdot n\, u_h - \int_{\partial\Omega} \eta u_h\, v = \int_\Omega f\, v$$

consistency term

symmetry term

boundary penalty term

Mary Fanett Wheeler
Graduated 1971 Rice University (Houston, TX)
Professor at University of Texas at Austin
Likes scottie dogs and George W. Bush

Idea attributed to private communication by J. Douglas and T. Dupont.
Summarized and analyzed by D. Arnold in 1979 thesis.

- Suppose approximation space is discontinuous, then

$$\sum_{j=1}^{M} \int_{E_j} \nabla u \cdot \nabla v - \sum_{i=1}^{m} \int_{e_i} [\nabla u \cdot n_i \, v] - \int_{\partial\Omega} \nabla u \cdot n \, v = \sum_{j=1}^{M} \int_{E_j} f \, v$$

M. Wheeler, "An elliptic collocation-finite element method with interior penalties", *SIAM J. Numer. Anal*, 15:152-161, 1978

D. Arnold, "An interior penalty finite element method with discontinuous elements", *SIAM JNA*, 19(4):742-760, 1982

J. Douglas and T. Dupont, "Interior penalty procedures for elliptic and parabolic Galerkin methods", *Lecture Notes in Physics*, 58, Springer-Verlag, Berlin, 1976.

- Trick: Define average $w = \frac{1}{2}(w^- + w^+)$ and recall jump $[w] = w^- = w^+$ then

$$[a\,b] = [a]\{b\} + \{a\}[b]$$

- Apply to formulation

$$\sum_{j=1}^{M} \int_{E_j} \nabla u \cdot \nabla v - \sum_{i=1}^{m} \int_{e_i} [\nabla u \cdot n_i]\,\{v\} - \sum_{i=1}^{m} \int_{e_i} \{\nabla u \cdot n_i\}\,[v]$$

$$- \int_{\partial\Omega} \nabla u \cdot n\, v = \sum_{j=1}^{M} \int_{E_j} f\, v$$

- However, since $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$,

$$\begin{aligned} -\sum_{i=1}^{m} \int_{e_i} [\nabla u \cdot n_i]\,\{v\} &= 0, \\ -\sum_{i=1}^{m} \int_{e_i} \{\nabla v \cdot n_i\}\,[u] &= 0 \end{aligned}$$

Symmetric Interior Penalty DG

- Idea: add *interior penalty* term of the form

$$\sum_{i=1}^{m} \int_{e_i} \frac{\sigma}{|e_i|} [u]\,[v] = 0$$

$$\boxed{\begin{aligned}\sum_{j=1}^{M} \int_{E_j} \nabla u \cdot \nabla v - \sum_{i=1}^{m} \int_{e_i} \{\nabla u \cdot n_i\}\,[v] - \sum_{i=1}^{m} \int_{e_i} \{\nabla v \cdot n_i\}\,[u] \\ + \sum_{i=1}^{m} \int_{e_i} \frac{\sigma}{|e_i|} [u]\,[v] - \int_{\partial\Omega} \nabla u \cdot n\, v = \sum_{j=1}^{M} \int_{E_j} f\, v\end{aligned}}$$

Discrete Symmetric Interior Penalty DG

- Can weakly incorporate dirichlet boundary data $u = g$ on $\partial\Omega$

$$-\int_{\partial\Omega} (\nabla v \cdot n)\, u_h = -\int_{\partial\Omega} (\nabla v \cdot n)\, g$$

Find $u_h \in \mathcal{D}(\mathcal{E}_h)$ such that for all $v \in \mathcal{D}(\mathcal{E}_h)$

$$\boxed{\begin{aligned}\sum_{j=1}^{M} \int_{E_j} \nabla u_h \cdot \nabla v - \sum_{i=1}^{m} \int_{e_i} \{\nabla u_h \cdot n_i\}\, [v] - \sum_{i=1}^{m} \int_{e_i} \{\nabla v \cdot n_i\}\, [u_h] \\ + \sum_{i=1}^{m} \int_{e_i} \frac{\sigma}{|e_i|} [u_h]\, [v] - \int_{\partial\Omega} (\nabla u_h \cdot n)\, v - \int_{\partial\Omega} (\nabla v \cdot n)\, u_h \\ = \sum_{j=1}^{M} \int_{E_j} f\, v - \int_{\partial\Omega} \nabla v \cdot n\, g\end{aligned}}$$

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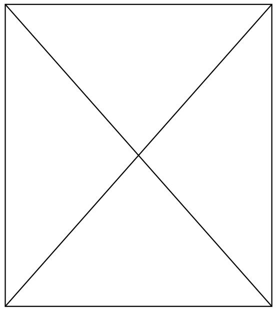
Non-Symmetric DG Method

J. Tinsley Oden
Graduated from Oklahoma State University in 1962
Director of TICAM at University of Texas at Austin
Likes hunting big game and has a ranch in Texas

Ivo Babuška
Graduated from Technical University, Prague in 1951
Professor at University of Texas at Austin
Has 4 PhD degrees and an arrest warrant.
Likes studing the history of mathematicians.



Carlos Baumann
Graduated from University of Texas in 1997
Researcher at Computational Mechanics, Inc in Austin
Oden's student at TICAM

Non-Symmetric Discontinuous Galerkin Method

Find $u_h \in \mathcal{D}(\mathcal{E}_h)$ such that for all $v \in \mathcal{D}(\mathcal{E}_h)$

$$\sum_{j=1}^{M} \int_{E_j} \nabla u_h \cdot \nabla v - \sum_{i=1}^{m} \int_{e_i} \{\nabla u_h \cdot n_i\} [v] + \sum_{i=1}^{m} \int_{e_i} \{\nabla v \cdot n_i\} [u_h]$$
$$- \int_{\partial\Omega} (\nabla u_h \cdot n) \, v + \int_{\partial\Omega} (\nabla v \cdot n) \, u_h$$
$$= \sum_{j=1}^{M} \int_{E_j} f \, v + \int_{\partial\Omega} \nabla v \cdot n \, g$$

- Form becomes positive-definite → inf-sup condition holds → unique solution and conditions of stability
- Consistent, locally mass-conservative formulation

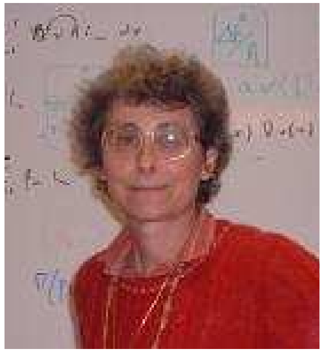
C. Baumann, "An h-p adaptive discontinuous finite element method for computational fluid dynamics", *PhD thesis*, The University of Texas at Austin, 1997.

J.T. Oden, I. Babuška, and C. Baumann, "A discontinuous hp finite element method for diffusion problems", *Journ. Comput. Phys.*, 146:491-519, 1998.

Non-Symmetric Interior Penalty DG Method

Béatrice Rivière
Graduated 2000 from University of Texas at Austin
Assistant Professor at University of Pittsburg
From Reunion Island, likes to eat Dodo au Vin

Mary Wheeler
(see page 12)



Vivette Girault
Professor at Paris VI
Visited University of Texas at Austin
Likes math a lot!

Non-Symmetric Interior Penalty DG Method

Find $u_h \in \mathcal{D}(\mathcal{E}_h)$ such that for all $v \in \mathcal{D}(\mathcal{E}_h)$

$$\sum_{j=1}^{M} \int_{E_j} \nabla u_h \cdot \nabla v - \sum_{i=1}^{m} \int_{e_i} \{\nabla u_h \cdot n_i\} [v] + \sum_{i=1}^{m} \int_{e_i} \{\nabla v \cdot n_i\} [u_h]$$
$$- \int_{\partial\Omega} (\nabla u_h \cdot n)\, v + \int_{\partial\Omega} (\nabla v \cdot n)\, u_h$$
$$+ \sum_{i=1}^{m} \int_{e_i} \frac{\sigma}{|e_i|} [u_h][v] + \int_{\partial\Omega} \frac{\sigma}{|e_i|} u_h\, v$$
$$= \sum_{j=1}^{M} \int_{E_j} f\, v + \int_{\partial\Omega} \frac{\sigma}{|e_i|} g\, v + \int_{\partial\Omega} \nabla v \cdot n\, g$$

- No longer locally mass-conservative, proved convergence rates.

B. Rivière, "Discontinuous Galerkin methods for solving the miscible displacement problem in porous media", *PhD Thesis*, The University of Texas at Austin, 2002.

B. Rivière, M. Wheeler, and V. Girault, "Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems, Part I", *Comput. Geo*, 8:337-360, 1999.

B. Rivière, M. Wheeler, and V. Girault, "A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems", *SIAM JNA*, 39(3):902-931, 2001.

DG on hyperbolic: Reed and Hill

- Consider the (linear) neutron transport equation for real number η and constant vector a
$$\eta u + \nabla \cdot (a\, u) = f \qquad \text{in } \Omega$$
Multiply by test function, integrate by parts over element E
$$\int_E \eta u\, v - \int_E u\, a \cdot \nabla v - \int_{\partial E} a \cdot n u^{\uparrow} = \int_E f v \qquad \text{in } \Omega$$
where upwind value $u^{\uparrow} = lim_{s \downarrow 0}(x - sa)$ is the value of u upstream in the characteristic direction a.
- LeSaint and Raviart immediately publish convergence results (1974), prove (suboptimal) h^k convergence in L^2 for general triangulations.
- Further results, extensions and numerics are published.

B. Cockburn, E. Karniadakis, C.W. Shu, "The Development of discontinuous Galerkin methods", In *Discontinuous Galerkin Methods*, Lecture Notes in Computational Science and Engineering, Springer, Berlin, 2000

DG on nonlinear hyperbolic systems

$$\int_E u_t\, v - \sum_{i=1}^{d} \int_E f_i(u) \cdot \nabla v + \sum_{i=1}^{d} \int_{\partial E} f_i(u) \cdot n\, v = 0$$

- *Time discretization*
  Problem: nonlinearities in flux function on boundary prevents element-by-element computation using characteristics

  implicit time discretization very inefficient

  space-time element construction

- *Analysis*
  Problem: nonlinear analysis extremely difficult

  only 3 results exist

Jerome Jaffré
Graduated from Paris VI in 1974
Researcher at INRIA Rocquencourt
Likes geosciences a lot!

J. Jaffré, C. Johnson, A. Szeppessy, "Convergence of the discontinuous Galerkin finite element method for hyperbolic conservation laws", *Math. Models and Meth. in Appl. Sci.*, 5:367-386, 1995

Development of LDG method

- *Cockburn & Shu*: RKDG methods
  - discretize in space via DG, polynomials of degree k
  - discretize in time via explicit TVD Runge-Kutta $(k+1)$
  - employ a general slope limiter to enforce maximum principle
- *Bassi & Rebay*: extension to Navier-Stokes
  - convection-diffusion type equations
  - rewrite as first order system
- *Cockburn & Shu*: LDG methods
  - extension to convection-diffusion systems
  - local solution (paralellizable)

F. Bassi and S. Rebay, "A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations", *J. Comput. Phys.*, 131:267-279, 1997

B. Cockburn and C.W. Shu, "The local discontinuous Galerkin method for time-dependent convection-diffusion systems", *SIAM JNA*, 35(6):2440-2463, 1998

LDG method on transport

Clint Dawson
Graduated 1988 from Rice University
Professor at University of Texas at Austin
Grew up on a farm in rural Texas

- *Cockburn & Dawson*
  - extension to multi-dimension tranport equation
  - diffusion tensor depends on (x,t) and allows for non-invertible diffusion tensors

$$\begin{aligned} \frac{\partial c}{\partial t} + \nabla \cdot (\boldsymbol{u} c) - \nabla \cdot (D \nabla c) &= 0 \qquad (\boldsymbol{x}, t) \in \Omega,\ t > 0, \\ (\boldsymbol{u} c - D \nabla c) \cdot \boldsymbol{n} &= (\boldsymbol{u} \cdot \boldsymbol{n}) \hat{g} \qquad \text{on } \partial\Omega_i \\ (-D \nabla c) \cdot \boldsymbol{n} &= 0 \qquad \text{on } \partial\Omega_o \end{aligned}$$

B. Cockburn and C. Dawson, "Some extensions of the local discontinuous Galerkin method for convection-diffusion equations in multidimensions",

LDG method on convection-diffusion

$$\begin{aligned}\frac{\partial c}{\partial t} + \nabla \cdot (\boldsymbol{u}c - D\nabla c) &= 0 \qquad (\boldsymbol{x}, t) \in \Omega,\ t > 0,\\ (\boldsymbol{u}c - D\nabla c)\cdot \boldsymbol{n} &= (\boldsymbol{u}\cdot\boldsymbol{n})\hat{g} \quad \text{on } \partial\Omega_i\\ (-D\nabla c)\cdot\boldsymbol{n} &= 0 \qquad\qquad \text{on } \partial\Omega_o\end{aligned}$$

Rewrite as first order system (let D=constant)

$$\begin{aligned}\frac{\partial c}{\partial t} + \nabla \cdot (\boldsymbol{u}c - \nabla \cdot (D\nabla c)) &= 0\\ \mathbf{z} &= -D\nabla c\\ (\boldsymbol{u}c - z)\cdot\boldsymbol{n} &= (\boldsymbol{u}\cdot\boldsymbol{n})\hat{g} \quad \text{on } \partial\Omega_i\\ \mathbf{z}\cdot\boldsymbol{n} &= 0 \qquad\qquad \text{on } \partial\Omega_o\end{aligned}$$

Apply DG to each equation separately

$$\int_E \frac{\partial c}{\partial t} v - \int_E (\boldsymbol{u}c + \mathbf{z}) \cdot \nabla v + \int_{\partial E} (\boldsymbol{u}c + \mathbf{z}) \cdot n\, v = \int_E f\, v$$

$$\int_E \mathbf{z}\, \boldsymbol{w} - \int_E Dc\, \nabla \cdot \boldsymbol{w} + \int_{\partial E} Dc\, \boldsymbol{w} \cdot \boldsymbol{n} = 0$$

$$\int_{\partial E \cap \partial \Omega_i} (\boldsymbol{u}c - z) \cdot \boldsymbol{n}\, v \;=\; \int_{\partial E \cap \partial \Omega_i} (\boldsymbol{u} \cdot \boldsymbol{n}) \hat{g}\, v$$

$$\int_{\partial E \cap \partial \Omega_o} \boldsymbol{u}c \cdot \boldsymbol{n}\, v \;=\; 0$$

$$-\sum_{j=1}^{M}\int_{E_j}\frac{\partial c_h}{\partial t}\,v-\sum_{j=1}^{M}\int_{E_j}(\boldsymbol{u}c_h+\mathbf{z}_h)\cdot\nabla v+\sum_{i=1}^{m}\int_{e_i}(\boldsymbol{u}c_h^{\uparrow}+\{\mathbf{z}_h\})\cdot\boldsymbol{n}\,[v]$$
$$+\int_{\partial\Omega_{\mathrm{o}}\cap\partial\Omega}(\boldsymbol{u}c_h\cdot\boldsymbol{n})\,v+\int_{\partial\Omega_{\mathrm{i}}\cap\partial\Omega}\hat{g}\boldsymbol{u}\cdot\boldsymbol{n}\,v=\sum_{j=1}^{M}\int_{E_j}f\,v$$

$$\sum_{j=1}^{M}\int_{E_j}\mathbf{z}_h\cdot\boldsymbol{w}_h+\sum_{j=1}^{M}\int_{E_j}Dc_h\,\nabla\cdot\boldsymbol{w}_h-\sum_{i=1}^{m}\int_{e_i}\{Dc_h\}\,[\boldsymbol{w}_h]\cdot\boldsymbol{n}_i=0$$

Features

DISCONTINUOUS GALERKIN FEM

- easily incorporates h- and p-adaptivity
- locally conservative and accurate
- easily handles complex geometry and nonmatching grids
- suppresses non-physical oscillations
- handles advection dominated flow regimes efficiently
- degrees of freedom associated with elements

CONTINUOUS GALERKIN FEM

- handles diffusion dominated flow regimes efficiently
- degrees of freedom usually associated with nodes