Parallel generation of ℓ–sequences

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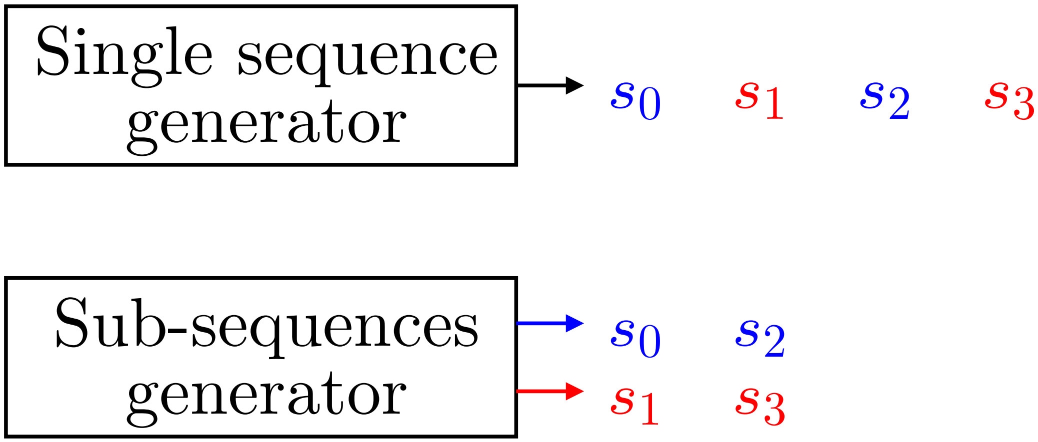
Outline

- **Introduction**
- **Parallel generation of m-sequences (LFSRs)**
  - Synthesis of sub-sequences
  - Multiple steps LFSR
- **Parallel generation of ℓ-sequences (FCSRs)**
  - Synthesis of sub-sequences
  - Multiple steps FCSR
- **Conclusion**

Part 1

Introduction

Sub-sequences generator



- Goal: **parallelism**
  - better throughput
  - reduced power consumption

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- $next^d(x_j)$: Cell connected to the output of x_j.

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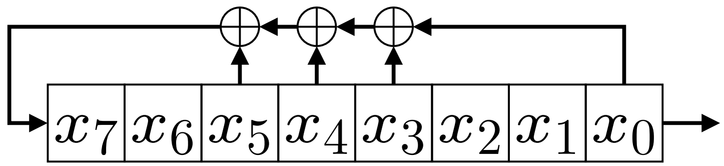
- Linear complexity: Size of smallest LFSR which generates S.

Fibonacci/Galois LFSRs

Fibonacci setup.



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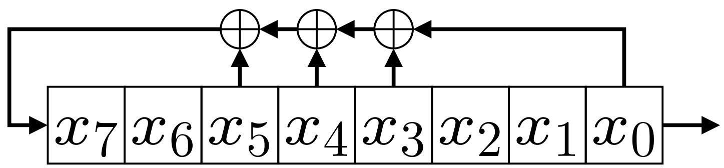

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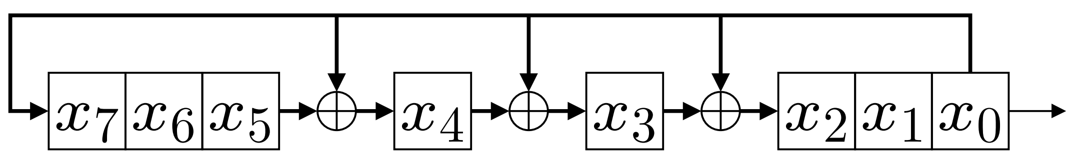

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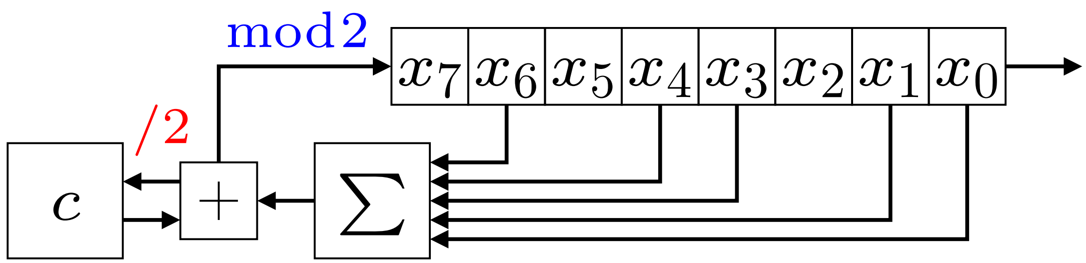

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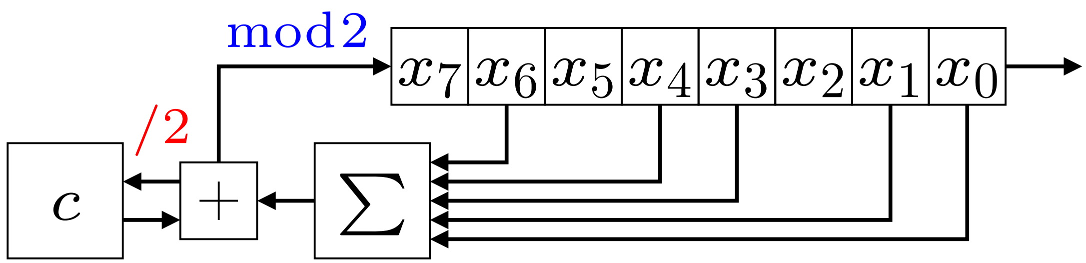

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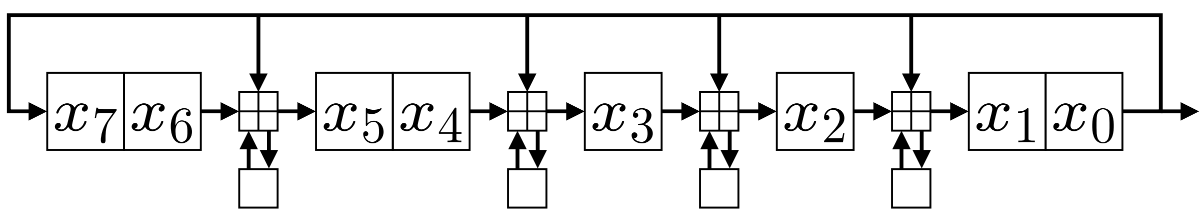
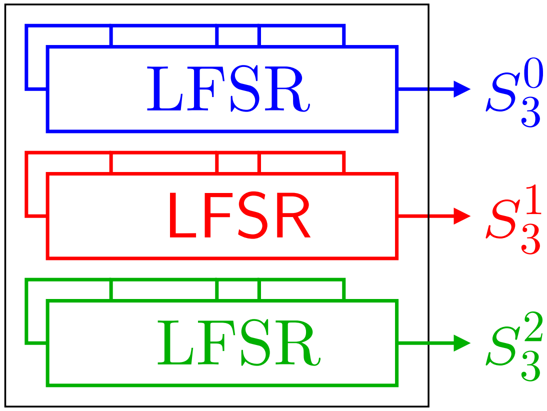


Part 2

Parallel generation of m-sequences (LFSRs)

Synthesis of Sub-sequences (1)



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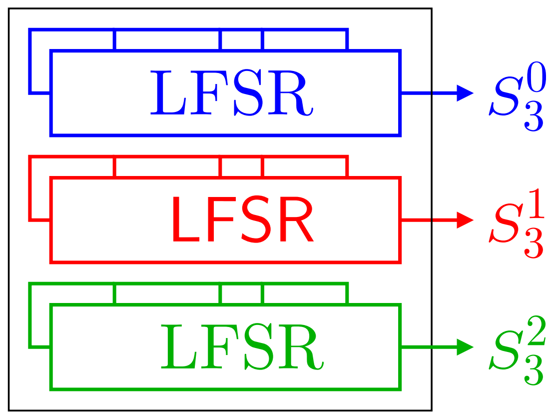

- Use Berlekamp-Massey algorithm to find the smallest LFSR for each sub-sequence.
- All sub-sequences are generated using d LFSRs defined by $Q^\star(x)$ but initialized with different values.

Synthesis of Sub-sequences (2)

Theorem [Zierler 59]: Let S be produced by an LFSR whose characteristic polynomial $Q(x)$ is irreducible in $\mathbf{F}_2$ of degree m. Let α be a root of $Q(x)$ and let T be the period of S. For $0 \leq i < d$, S_d^i can be generated by an LFSR with the following properties:

- The minimum polynomial of α^d in $\mathbf{F}_{2^m}$ is the characteristic polynomial $Q^\star(x)$ of the new LFSR with:
- Period $T^\star = \frac{T}{gcd(d,T)}$.
- Degree $m^\star$ is the multiplicative order of 2 in $\mathbf{Z}_{T^\star}$.

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[Lempel Eastman 71]

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- Duplication of the feedback:

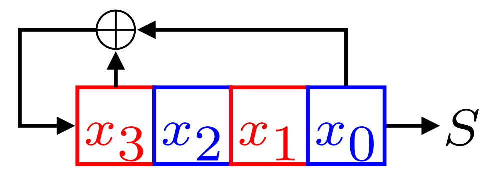
The sub-registers are linearly interconnected.

Fibonacci LFSR

$$next^1(x_0) = x_3$$
$$next^1(x_i) = x_{i-1} \text{ if } i \neq 0$$

$$(x_3)_{t+1} = (x_3)_t \oplus (x_0)_t$$
$$(x_i)_{t+1} = (x_{i-1})_t \text{ if } i \neq 3$$

1-decimation



$$\Downarrow$$

$$next^2(x_0) = x_2$$
$$next^2(x_1) = x_3$$
$$next^2(x_i) = x_{i-2} \text{ if } i > 1$$

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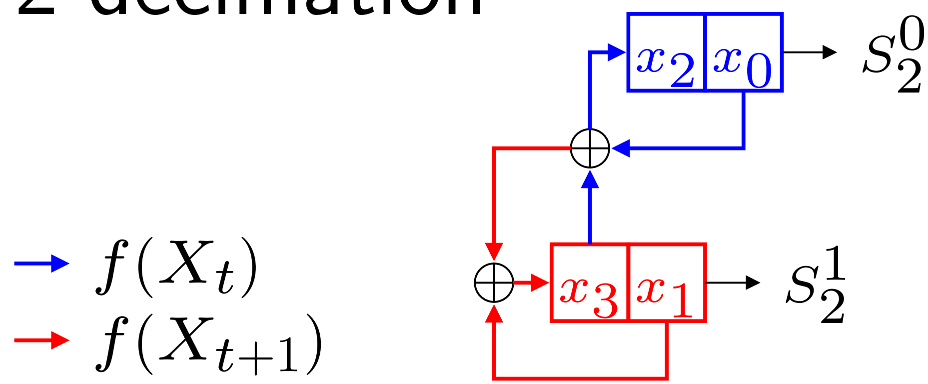

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  - Larger memory size: $d \times m^\star$
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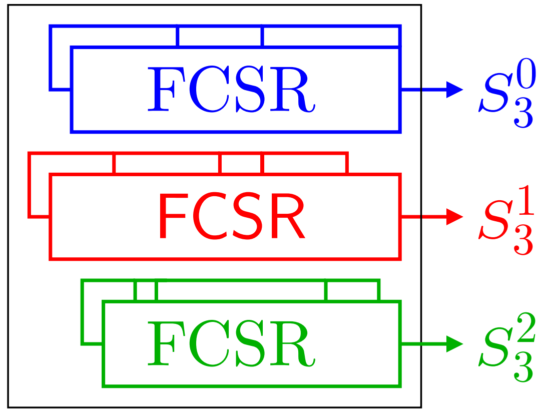
- **Multiple steps LFSR:**
  - Same memory size: m
  - More logic gates: $d \times wt(Q)$

Part 3

Parallel generation of ℓ-sequences (FCSRs)

Synthesis of Sub-sequences (1)



- We use an algorithm based on Euclid's algorithm **[Arnault Berger Necer 04]** or on lattice approximation **[Klapper Goresky 97]** to find the smallest FCSR for each sub-sequence.

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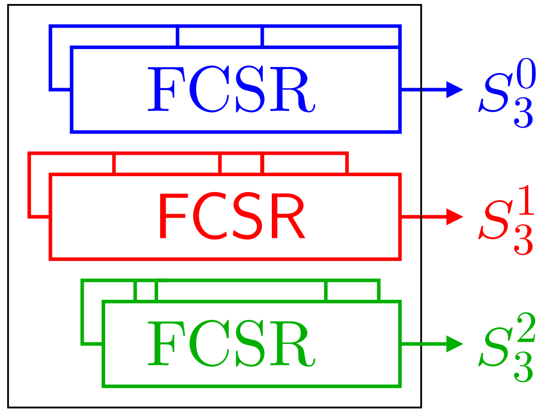

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  - If $\gcd(T,d) = 1$ then $T^\star = T$.
- If $\gcd(T,d) > 1$: $T^\star$ might depend on i!
  E.g. for $S = -1/19$ and $d = 3$: $T/gcd(T,d) = 6$.
  - S_3^0: The period $T^\star = 2$.
  - S_3^1: The period $T^\star = 6$.

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- **Conjecture [Goresky Klapper 97]:** Let S be an ℓ–sequence with connection integer $q = p^e$ and period T. Suppose p is prime and $q \notin \{5, 9, 11, 13\}$. For any d_1, d_2 relatively prime to T and incongruent modulo T and any i, j:

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- **Based on Conjecture:**
  - If q is prime and $gcd(T, d) = 1$ then $q^\star > q$.
  - Let q, p be prime and $T = q - 1 = 2p$:

    $1 \leq d < T$, and $d \neq p$ then $q^\star > q$.

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- We can use the following equations:

$$\begin{aligned}
(x_i)_{t+d} &= \begin{cases} g(X_{t+d-m+i}, c_{t+d-m+i}) \bmod 2 & \text{if } m-d < i < m \\ (x_{i+d})_t & \text{if } i \leq m-d \end{cases} \\
c_{t+d} &= g(X_{t+d-1}, c_{t+d-1})/2
\end{aligned}$$

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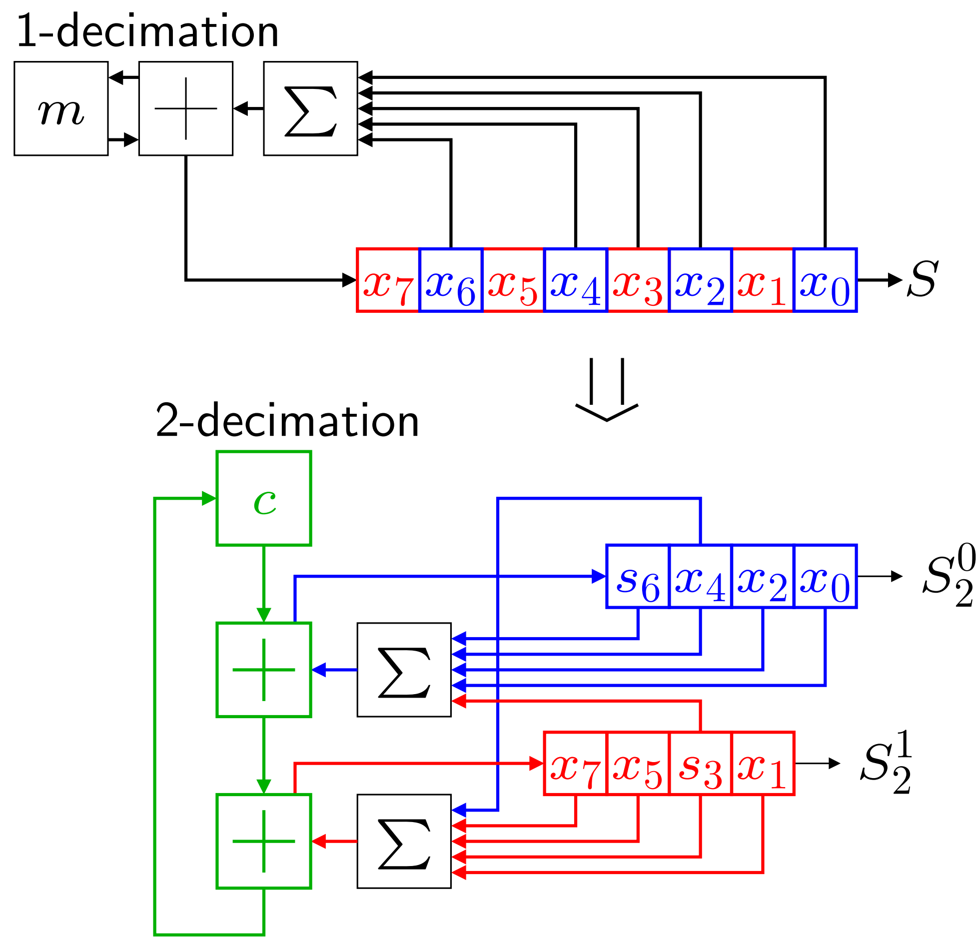

Galois FCSR

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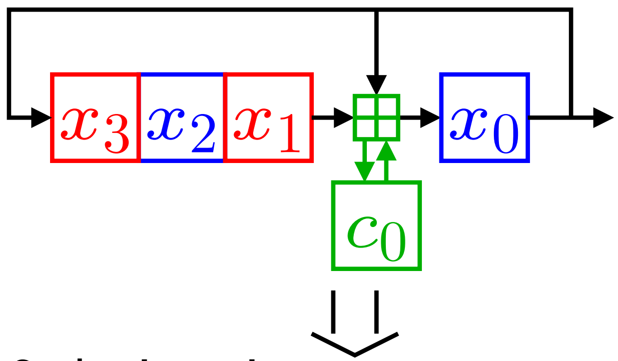

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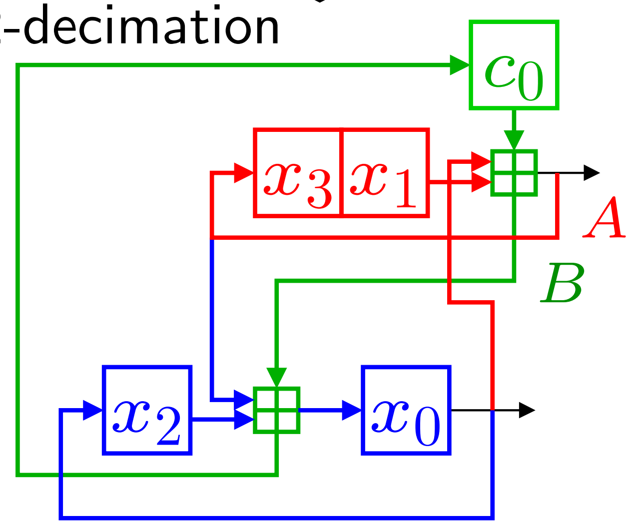

$$A = \boxplus\left[(x_0)_t, (x_1)_t, (c_0)_t\right] \bmod 2$$
$$B = \boxplus\left[(x_0)_t, (x_1)_t, (c_0)_t\right]_{\div 2}$$

$$(x_0)_{t+2} = \boxplus\left[A, B, (x_2)_t\right] \bmod 2$$
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$$(x_1)_{t+2} = (x_3)_t$$
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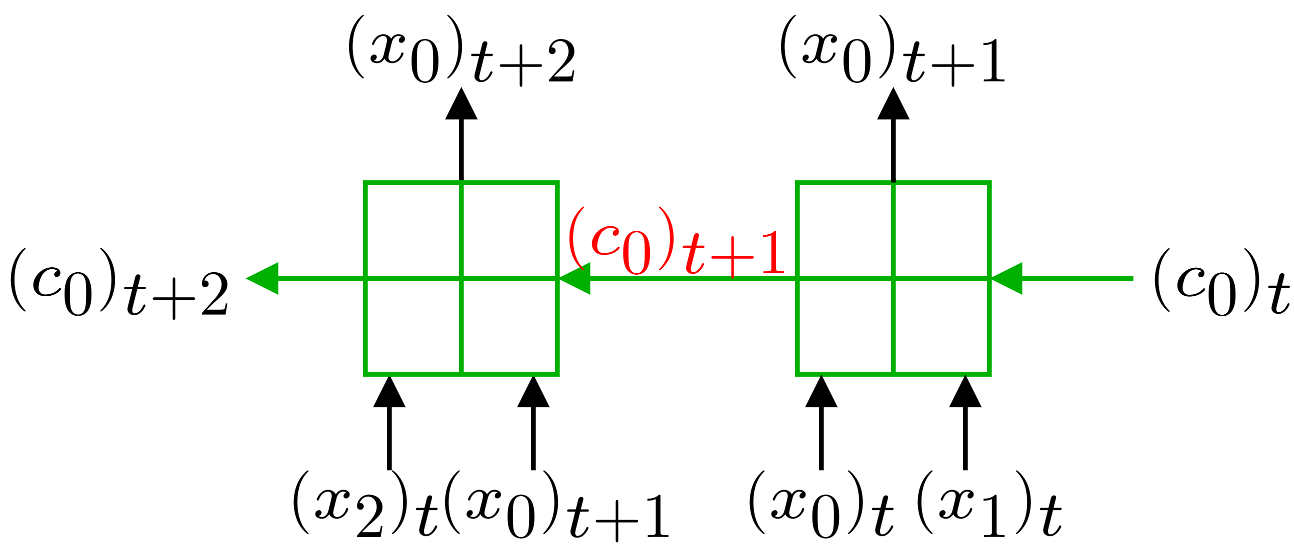

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- **Multiple steps FCSR:**
  - Same memory size.
  - Propagation of carry by well-known arithmetic circuits.

Part 4

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