Parallel generation of ℓ–sequences

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Kryptowochenende 2008

Princeton University INRIA

Outline

- Introduction

- LFSRs (Previous Results)
  - Synthesis of sub-sequences
  - Multiple steps LFSR

- FCSRs (Our Contribution)
  - Synthesis of sub-sequences
  - Multiple steps FCSR

- Conclusion

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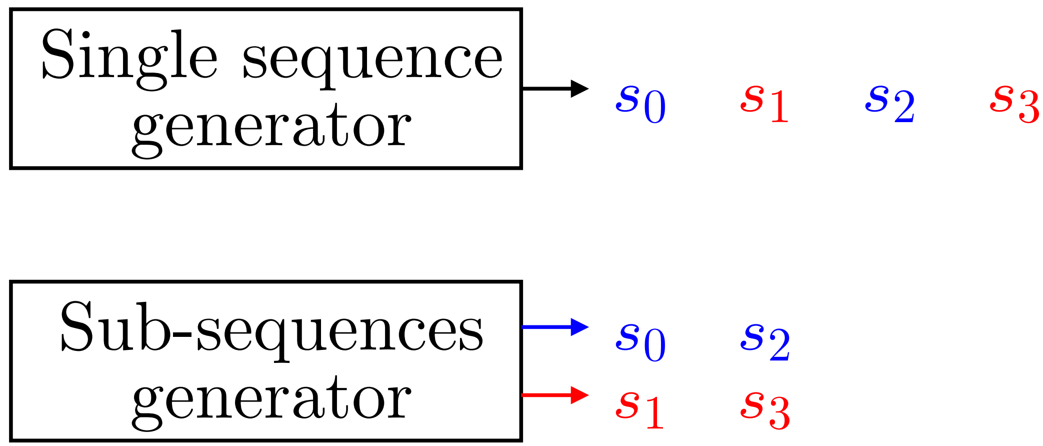
Part 1

Introduction

Sub-sequences generator



- Goal: **parallelism**
  - better throughput
  - reduced power consumption

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Notations

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- $next^d(x_j)$: Cell connected to the output of x_j.

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- Let $s(x) = \sum_{i=0}^{\infty} s_i x^i$ be the power series of $S = (s_0, s_1, s_2, \ldots)$. There exists two polynomials $p(x), q(x)$:

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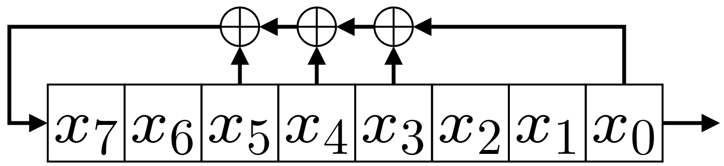
- m–sequence: S has maximal period of $2^m - 1$.
  (*iff* $q(x)$ is a primitive polynomial)
- Linear complexity: Size of smallest LFSR which generates S.

Fibonacci/Galois LFSRs

Fibonacci setup.



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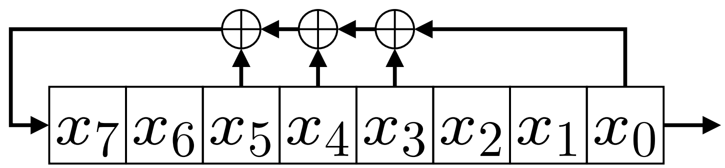

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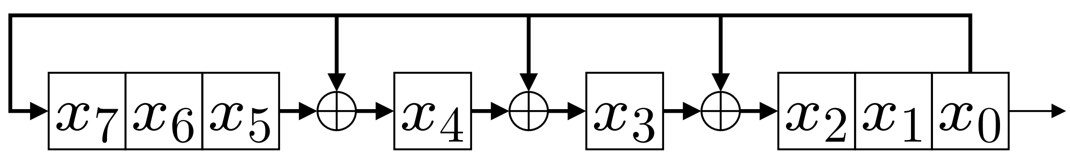

FCSRs
[Klapper Goresky 93]

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  (*iff* q is a prime power and $ord_q(2) = \varphi(q)$.)

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- 2–adic complexity: size of the smallest FCSR which produces S.

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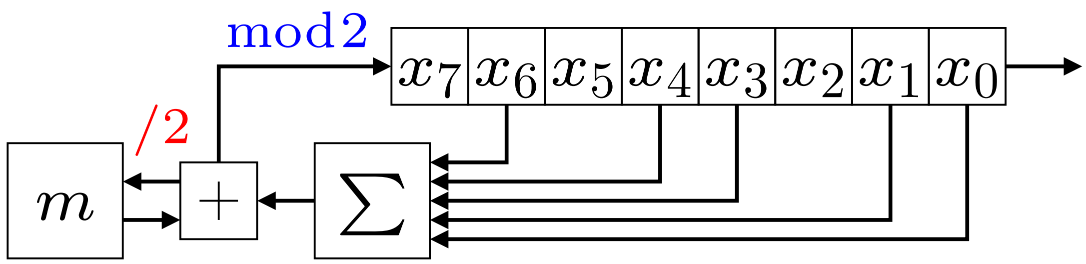

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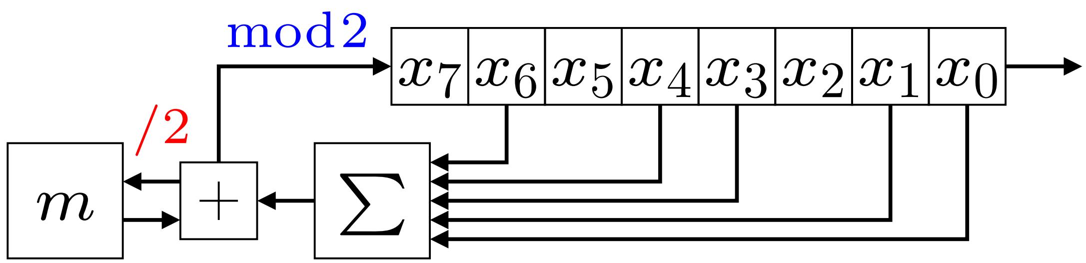

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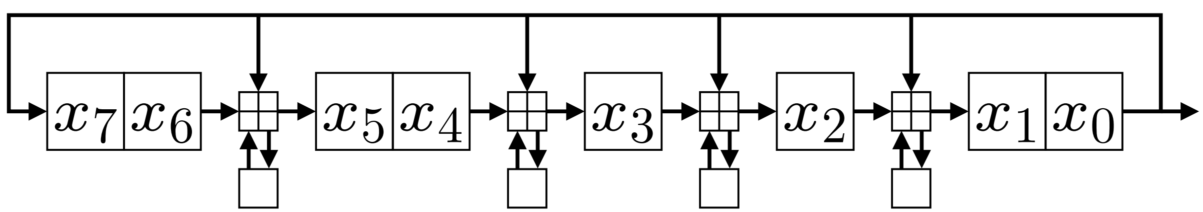

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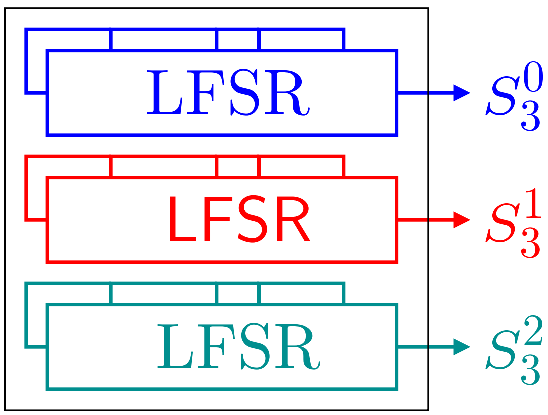
Part 2

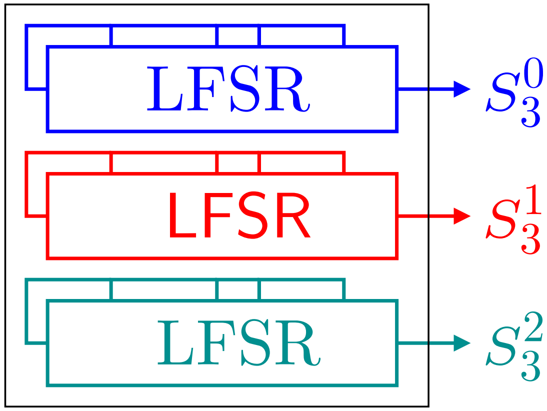
LFSRs (Previous Results)

Synthesis of Sub-sequences (1)



- Use Berlekamp-Massey algorithm to find the smallest LFSR for each subsequence.

Synthesis of Sub-sequences (1)



- Use Berlekamp-Massey algorithm to find the smallest LFSR for each subsequence.
- All sub-sequences are generated using d LFSRs defined by $Q^{\star}(x)$ but initialized with different values.

Synthesis of Sub-sequences (2)

Theorem [Zierler 59]: Let S be produced by an LFSR whose characteristic polynomial $Q(x)$ is irreducible in $\mathbf{F}_2$ of degree m. Let α be a root of $Q(x)$ and let T be the period of S. For $0 \le i < d$, S_d^i can be generated by an LFSR with the following properties:

- The minimum polynomial of α^d in $\mathbf{F}_{2^m}$ is the characteristic polynomial $Q^\star(x)$ of the new LFSR with:
- Period $T^\star = \frac{T}{gcd(d,T)}$.
- Degree $m^\star$ is the multiplicative order of 2 in $\mathbf{Z}_{T^\star}$.

Multiple steps LFSR
[Lempel Eastman 71]

- Clock d times the register in one cycle.

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$$x_i x_{i+d} \cdots x_{i+kd}$$

such that $0 \leq i < d$ and $i + kd < m$.

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such that $0 \leq i < d$ and $i + kd < m$.

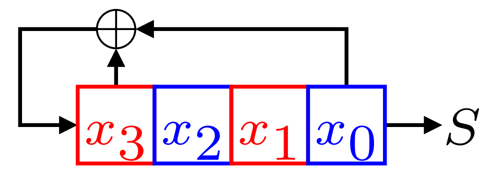
- Duplication of the feedback:

The sub-registers are linearly interconnected.

Fibonacci LFSR

$$next^1(x_0) = x_3$$
$$next^1(x_i) = x_{i-1} \text{ if } i \neq 0$$
$$(x_3)_{t+1} = (x_3)_t \oplus (x_0)_t$$
$$(x_i)_{t+1} = (x_{i-1})_t \text{ if } i \neq 3$$

1-decimation



$$\Downarrow$$

$$next^2(x_0) = x_2$$
$$next^2(x_1) = x_3$$
$$next^2(x_i) = x_{i-2} \text{ if } i > 1$$

$$(x_i)_{t+2} = (x_{i-2})_t \text{ if } i < 2$$
$$(x_2)_{t+2} = (x_3)_t \oplus (x_0)_t$$
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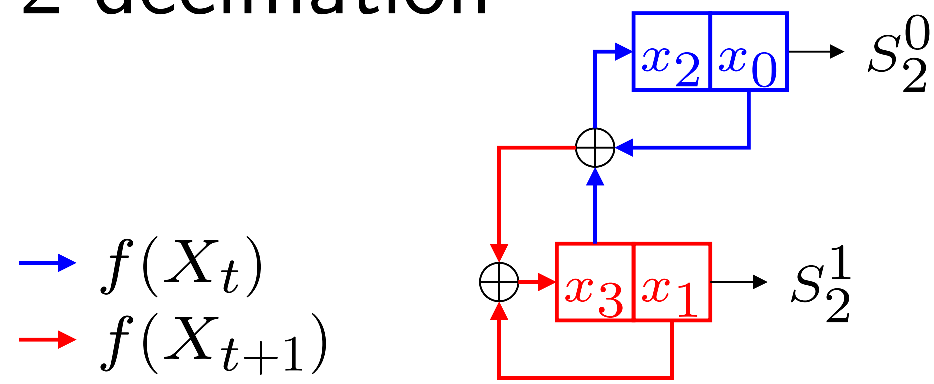
2-decimation



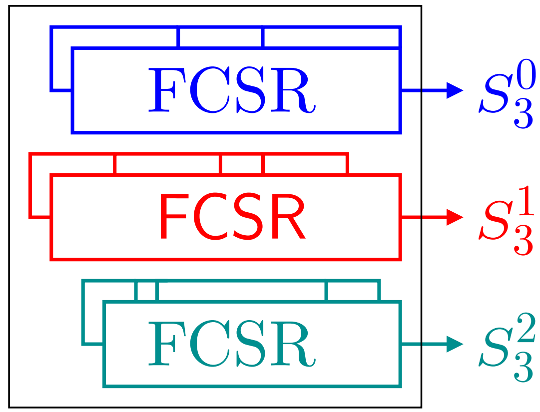
Comparison

Method	Memory cells	Logic Gates
LFSR synthesis	$d \times m^\star$	$d \times wt(Q^\star)$
Multiple steps LFSR	m	$d \times wt(Q)$

Part 3

FCSRs (Our Contribution)

Synthesis of Sub-sequences (1)



- We use an algorithm based on Euclid's algorithm **[Arnault Berger Necer 04]** to find the smallest FCSR for each sub-sequence.

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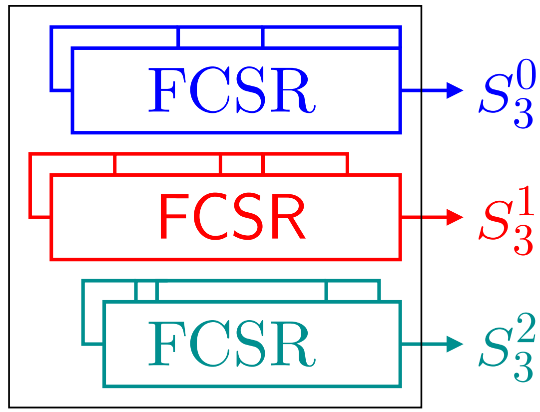

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- The sub-sequences do **not** have the same q.

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- A given S_d^i has period $T^\star$ and minimal connection integer $q^\star$.

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- **Period:** (True for all periodic sequences)
  - $T^\star \left| \frac{T}{\gcd(T,d)} \right.$,
  - If $\gcd(T,d) = 1$ then $T^\star = T$.
- If $\gcd(T,d) > 1$: $T^\star$ might depend on i!
  E.g. for $S = -1/19$ and $d = 3$: $T/gcd(T,d) = 6$.
  - S_3^0: The period $T^\star = 2$.
  - S_3^1: The period $T^\star = 6$.

Synthesis of Sub-sequences (3)

- **2-adic complexity [Goresky Klapper 97]:**
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- **Conjecture [Goresky Klapper 97]:** Let S be an ℓ–sequence with connection integer $q = p^e$ and period T. Suppose p is prime and $q \notin \{5, 9, 11, 13\}$. For any d_1, d_2 relatively prime to T and incongruent modulo T and any i, j:

  $S^i_{d_1}$ and $S^j_{d_2}$ are cyclically distinct.

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- **Based on Conjecture:** Let q and p be prime and $T = q - 1 = 2p$:

  If $1 \leq d < T$ and $d \neq p$ then $q^\star > q$.

Multiple steps FCSR

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- Propagation of the carry computation.

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- Let the feedback function be defined by

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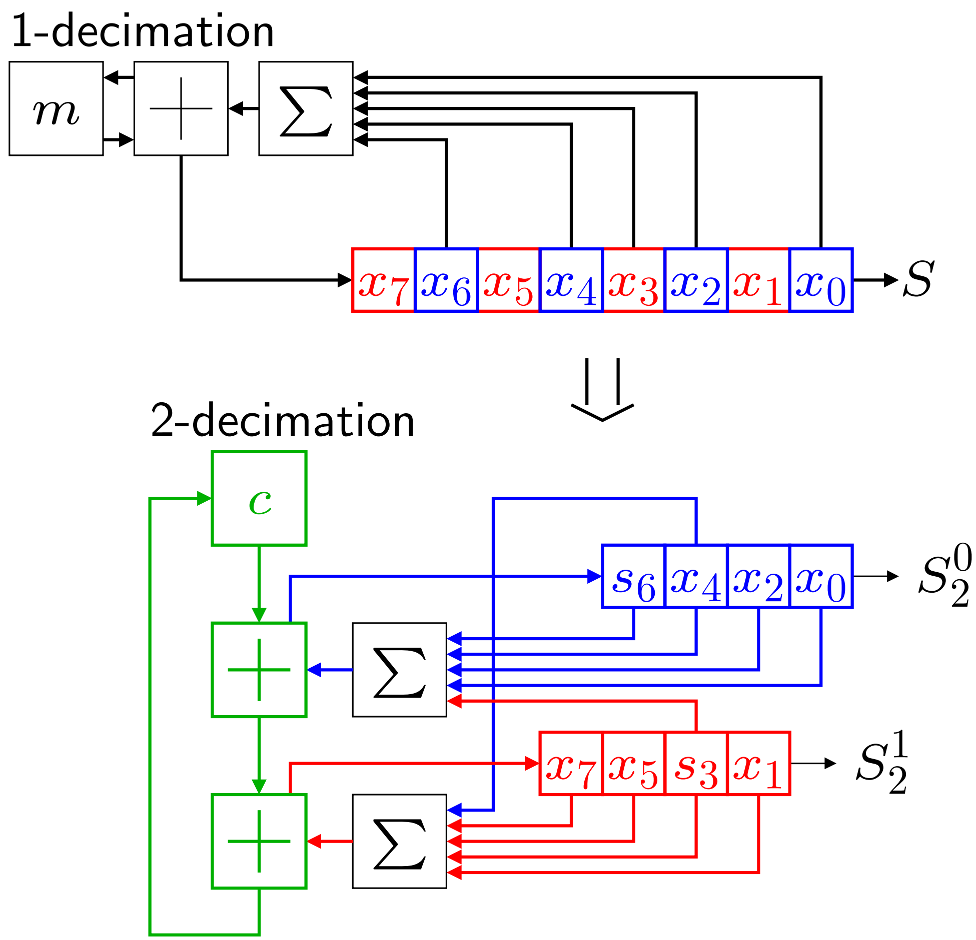
- We can use the following equations:

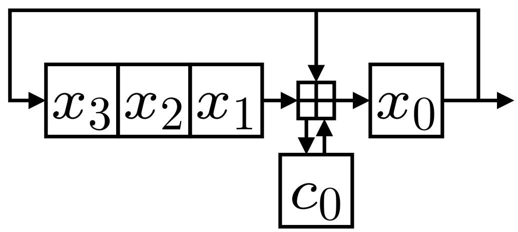
$$\begin{aligned}
(x_i)_{t+d} &= \begin{cases} g(X_{t+d-m+i}, c_{t+d-m+i}) \bmod 2 & \text{if } m-d \le i < m \\ (x_{i+d})_t & \text{if } i < m-d \end{cases} \\
c_{t+d} &= g(X_{t+d-1}, c_{t+d-1})/2
\end{aligned}$$

Fibonacci FCSR (2)



Galois FCSR (1)

- Example $q = -19$:



- Description at the bit-level:

$$\begin{cases} (x_0)_{t+1} &= (x_0)_t \oplus (x_1)_t \oplus (c_0)_t \\ (c_0)_{t+1} &= [(x_0)_t \oplus (x_1)_t]\,[(x_0)_t \oplus (c_0)_t] \oplus (x_0)_t \end{cases}$$

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- $d = 2$, description for the automaton at $t+1$ and $t+2$

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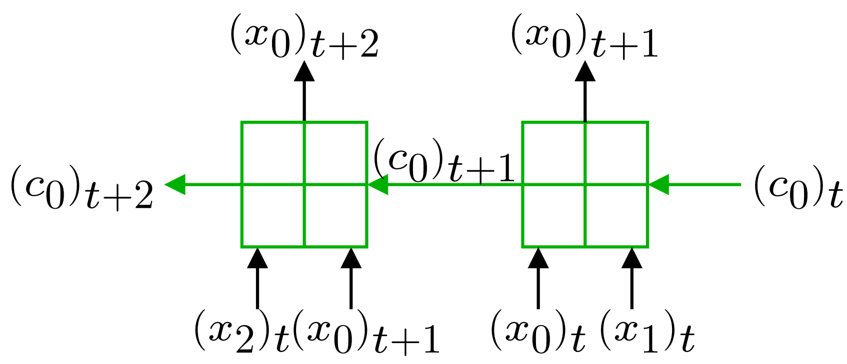
2-bit ripple carry adder



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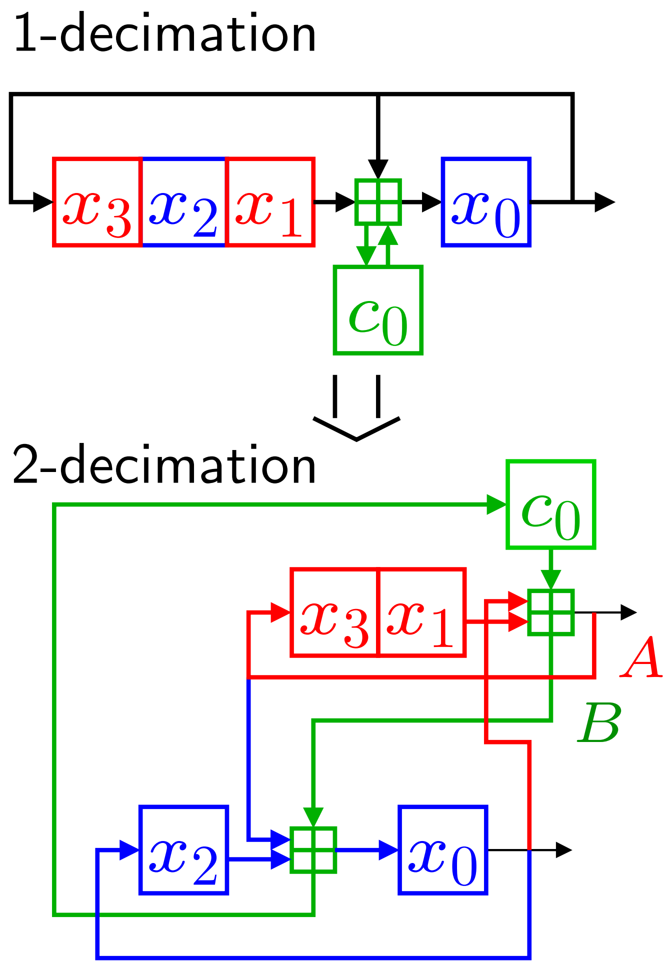

$$A = \boxplus\left[(x_0)_t, (x_1)_t, (c_0)_t\right] \bmod 2$$

$$B = \boxplus\left[(x_0)_t, (x_1)_t, (c_0)_t\right]_{\div 2}$$

$$(x_0)_{t+2} = \boxplus\left[A, B, (x_2)_t\right] \bmod 2$$

$$(c_0)_{t+2} = \boxplus\left[A, B, (x_2)_t\right]_{\div 2}$$

$$(x_1)_{t+2} = (x_3)_t$$

$$(x_2)_{t+2} = (x_0)_t$$

$$(x_3)_{t+2} = A$$

Comparison

- **Synthesis of Sub-sequences:**
  - Period: If $gcd(T, d) > 1$ it might depend on i.
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- **Multiple steps FCSR:**
  - Same memory size.
  - Propagation of carry by well-known arithmetic circuits.

Part 4

Conclusion

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