# A tutorial introduction to space-time coding: mathematical models, information theoretical aspects, and coding for MIMO channels

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**Lecture at the Conference on Coding and Cryptography**
**Carcans, France**

March 2008

# *Where can I download these slides?*

Slides on the web page

**http://www.josephboutros.org/C2**

Questions can be sent by email to

**boutros@tamu.edu**

# Presentation outline

1. Part I - Introduction
    - Diversity to combat erasures and fadings.
    - The multiple antenna channel model.
    - Coding gain and diversity in MIMO channels.
2. Part II - Information Theory
    - Capacity when channel is unknown at transmitter.
    - Outage probability for non-ergodic channels.
3. Part III - Coding
    - Quick introduction to STBC.
    - Code design criteria for block fading channels.
    - Example of an LDPC code for MIMO channels.

# *Coding for erasure channels (1)*

The erasure channel is an extremal case of the Rayleigh fading channel (see Proakis 2000, Tse & Viswanath 2005). The erasure channel can also model an application layer where packets are lost due to a failure at the physical layer.

Let us consider the binary erasure channel (BEC). A codeword $c = (c_1, c_2, \dots, c_N)$ belonging to $\mathcal{C}[N, K, d_{min}]_2$ is transmitted on the BEC, where $\mathcal{C}$ is a linear binary code of length $N$, dimension $K$, and minimum Hamming distance $d_{min}$.

**The iid BEC**

- The channel is memoryless. If $y$ denotes the channel output then

$$p(y|c) = \prod_{i=1}^{N} p(y_i|c_i), \qquad p(y_i|c_i) = \begin{cases} 1-\epsilon, & y_i = c_i, \\ \epsilon, & y_i = X, \\ 0, & y_i = \overline{c_i}, \end{cases}$$

  where $c_i \in F_2$, $X$ represents an erasure, and $\epsilon \in [0,1]$.
- Binary elements are erased independently from each other. The output $y_i$ is equal to the input $c_i$ with probability $1-\epsilon$. No errors are encountered.

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# *Coding for erasure channels (2)*

## The non-ergodic BEC

- The channel has memory. Let us divide the codeword $c$ into $L$ blocks $C_\ell$, $\ell = 1 \dots L$, each block has length $N/L$ bits.
- Blocks are erased independently from each other, an erasure occurs with probability $\epsilon$. After writing $y = (Y_1, \dots, Y_L)$ and $c = (C_1, \dots, C_L)$, we get

$$p(y|c) = \prod_{\ell=1}^{L} p(Y_\ell|C_\ell), \qquad p(Y_\ell|C_\ell) = \begin{cases} 1-\epsilon, & Y_\ell = C_\ell, \\ \epsilon, & Y_\ell = X_1^L, \\ 0, & otherwise, \end{cases}$$

where $Y_\ell \in F_2^{N/L} \bigcup \{X_1^L\}$, and $X_1^L$ represents $L$ erased bits.

## Degrees of Freedom

The iid BEC has $N$ degrees of freedom whereas the non-ergodic BEC has only $L$ degrees of freedom. Exempli gratia, $N = 1000$ and $L = 3$.

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# Coding for erasure channels (3)

- Consider an iid BEC and a repetition code $\mathcal{C}(N, 1, N)_2$. The word error probability after decoding is $P_e = \epsilon^N$.
- Consider an iid BEC and a non-trivial code $\mathcal{C}(N, K, d_{min})_2$. A maximum likelihood decoder fails to decode an erasure pattern iff this pattern contains the support of a nonzero codeword (e.g. see Schwartz & Vardy 2005). Let $\Psi_{ML}(\omega)$ denote the number of such erasure patterns with weight $\omega$. Then,

$$P_e(ML) = \sum_{\omega=d_{min}}^{N} \Psi_{ML}(\omega)\epsilon^{\omega}(1-\epsilon)^{N-\omega}.$$

  A similar expression is obtained under iterative decoding using the notion of stopping sets (e.g. see Di, Proietti, Teletar, Richardson, & Urbanke 2002).
- For small $\epsilon$, the asymptotic behavior is

$$P_e(ML) \propto \epsilon^{d_{min}}.$$

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# *Coding for erasure channels (4)*

- Consider a non-ergodic BEC with parameter $L$ and a repetition code $\mathcal{C}(L,1,L)_2$ or a direct sum of $N/L$ versions of this code, i.e., $\mathcal{C} = (L,1,L) \oplus (L,1,L) \ldots \oplus (L,1,L)$. The error probability after decoding the repetition code is $P_e = \epsilon^L$.
- Consider a non-ergodic BEC and a non-trivial code $\mathcal{C}(N,K,d_{min})_2$. Then, the word error probability after decoding satisfies $P_e \geq \epsilon^L$.

**Definition 1: Diversity on erasure channels**

The diversity order $d$ attained by a code $\mathcal{C}$ is defined as

$$d = \lim_{\epsilon \longrightarrow 0} \frac{\log P_e}{\log \epsilon}.$$

Notice that diversity on iid BEC is upper bounded by $d_{min}$ ($d_{min} \leq N$) whereas diversity on non-ergodic BEC is upperbounded by $L$ ($L \leq N$). Non-ergodic channels are also referred to as limited diversity channels.

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# *Coding for erasure channels (5)*

Let $\omega_\ell$ be the Hamming weigh of the block $C_\ell$. The codeword weight $\omega(c)$ is the sum of partial weights, i.e., $\omega(c) = \sum_{\ell=1}^{L} \omega_\ell$.

**Theorem 2: Design criterion for non-ergodic BEC**

$\mathcal{C}$ is full diversity $(d = L)$ under ML decoding on a non-ergodic BEC if and only if, $\forall c \in \mathcal{C} \setminus \{0\}$, all partial Hamming weights are non zero, i.e., $\omega_\ell \neq 0, \forall \ell$.

**Example:** (Boutros, Guillén i Fàbregas, & Calvanese Strinati 2005)
The code is $\mathcal{C} = [8, 4, 4]_2$ and $L = 2$. There exist $8!/(4!)^2 = 70$ possibilities to define blocks $C_1$ and $C_2$. The 70 multiplexers are grouped into 2 different classes:

- 14 multiplexers with diversity 1 (no diversity) and weight enumerator

$$A(x, y) = \sum_{c \in \mathcal{C}} x^{\omega_1} y^{\omega_2} = 1 + x^4 + y^4 + 12x^2y^2 + x^4y^4.$$

- 56 multiplexers with full diversity $(d = L = 2)$ and weight enumerator

$$A(x, y) = 1 + 6x^2y^2 + 4x^3y + 4xy^3 + x^4y^4.$$

**Exercice:** Define $C_1 = (c_1, \ldots, c_{12})$ and $C_2 = (c_{13}, \ldots, c_{24})$. Find a full-diversity version of the [24, 12, 8] Golay code on $L = 2$ non-ergodic erasure channel.

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# *Coding for fading channels (1)*

The fading channel is defined by the input-output relation

$$y_i = h_i x_i + \eta_i,$$

where the fading coefficient $h_i$ is $\mathcal{CN}(0,1)$ (known at the receiver side) and the additive white noise $\eta_i$ is $\mathcal{CN}(0,2\sigma^2)$. The channel likelihood is

$$p(y_i|x_i,h_i) = \frac{1}{2\pi\sigma^2}\exp(-\frac{|y_i - h_i x_i|^2}{2\sigma^2}).$$

**Erasure and Fading**

By restricting $\alpha_i = |h_i|$ to $\{0,+\infty\}$, the fading channel becomes an erasure channel. $P(\alpha_i = 0) = \epsilon$ and $P(\alpha_i = +\infty) = 1-\epsilon$.

Usually $x_i = f(c_i)$, where $f : F_q \rightarrow \mathbb{Z}^2$ is a mapping that converts the finite field elements into complex symbols. This mapping is known as a QAM modulation.

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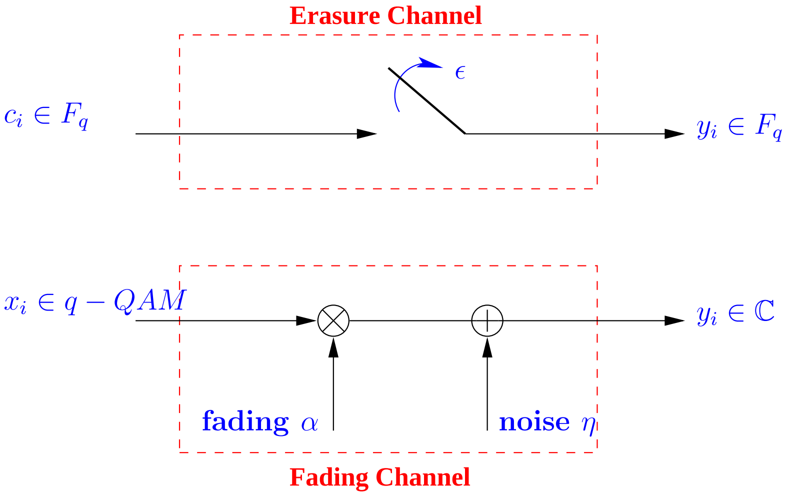
# *Coding for fading channels (2)*



# *Coding for fading channels (3)*

- The notions of iid and non-ergodic fading channels are directly derived from those defining the BEC given on slides 1 and 2.
- Definition 1 of diversity on BEC and Theorem 2 on the code design criterion for BEC are still valid on a Rayleigh fading channel. The erasure probability $\epsilon$ is replaced by the signal-to-noise ratio

$$\gamma = \frac{\mathcal{E}[|x_i|^2]}{\mathcal{E}[|\eta_i|^2]}.$$

- The word error probability at the decoder output is denoted by $P_e$. The full-diversity behavior $P_e \propto \epsilon^L$ becomes $P_e \propto 1/\gamma^L$.

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# *Coding for fading channels (4)*

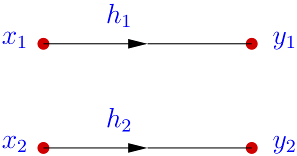
- Let $x = f(c)$ and $\hat{x} = f(\hat{c})$ be two codewords. The partial Hamming weights $\omega_\ell$ defined for the non-ergodic BEC can now be extended to the non-ergodic fading channel as follows:
  - Divide a codeword into $L$ blocks, each block containing $N/L$ components.
  - The quantity $\omega_\ell$ is the weight of the $\ell$th block in $x - \hat{x}$, i.e., the number of non-zero components in the difference.
- Example: $N = 4$, $q = 4$, and $L = 2$. Take $x = (+1, +1, -3, -3)$, if $\hat{x} = (-1, -1, -3, -3)$ then $\omega_1 = 2$ and $\omega_2 = 0$. If $\hat{x} = (-1, +1, +3, +3)$ then $\omega_1 = 1$ and $\omega_2 = 2$.

**Theorem 4: ML design criterion for non-ergodic fading channels**

$\mathcal{C}$ is full diversity $(d = L)$ on a non-ergodic fading channel iff, $\forall x, \hat{x} \in f(\mathcal{C})$, $x \neq \hat{x}$, all partial Hamming weights are non zero, i.e., $\omega_\ell \neq 0, \forall \ell$.

# The MIMO channel (1)

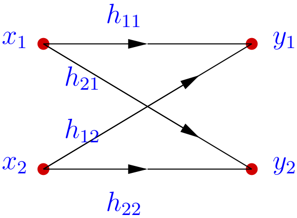
**Two parallel single antenna fading channels**



$$y = Hx + \eta$$

$$H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

**A multiple antenna (MIMO) fading channel**



$$y = Hx + \eta$$

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

# *The MIMO channel (3)*

## Mathematical (analytical) models

Physical modeling of a MIMO channel cannot lead to space-time coding design criteria. Mathematical modeling is necessary.

The simplest mathematical model for a $n_t \times n_r$ MIMO channel is

$$y = Hx + \eta,$$

where

- $H = [h_{ij}]$ is a $n_r \times n_t$ matrix with complex circularly symmetric iid gaussian entries of zero mean and unit variance, $h_{ij} \sim \mathcal{CN}(0,1)$.
- $x$ is a column vector including the $n_t$ transmitted symbols, $x_i \in q - QAM \subset \mathbb{Z}^2$.
- $\eta$ is a noise vector whose components are complex gaussian and iid, $\eta_i \sim \mathcal{CN}(0, 2\sigma^2)$.

# *The MIMO channel (5)*

- The diversity on a MIMO channel is also given by Definition 3, i.e.,

$$P_e \approx \frac{1}{(g\gamma)^d} \qquad \gamma \gg 1,$$

  where $g$ is referred to as the coding gain.

- The MIMO channel as defined in its simplest model on the previous slide has $n_t \times n_r$ degrees of freedom. For a static channel $H$, i.e., $H$ is constant within a codeword, we have

$$n_r \leq d \leq n_t \times n_r.$$

  The lower bound is attained in absence of coding. The ratio $d/n_r$ is known as the transmit diversity.

- The (receive) space dimension for one channel use is $n_r$. Hence, achieving the maximal diversity $n_t \times n_r$ must require $n_t$ channel transmissions at least. The expression "space-time coding" describes the spreading in both space and time of codes designed for MIMO channels.

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# *The MIMO channel (6)*

- The main objective of space-time coding is to build easily encodable and decodable codes that maximize both coding gain $g$ and diversity $d$ for a given information rate $R$.
- Let $R_c = K/N$ denotes the coding rate of $\mathcal{C}$. Then, the information rate expressed in bits per channel use (bpcu) is given by

$$R = n_t \times R_c \times \log_2(q) \qquad bpcu$$

- Distributing the components of a space-time code over the $n_t$ transmit antennas is referred to as spatial multiplexing.
- When compared to an uncoded single antenna system, the MIMO information rate is multiplied by a factor $\mu = n_t R_c$. This increase in information rate is called multiplexing gain.
- Another (asymptotic) information theoretical definition of $\mu$ is given by

$$\mu = \lim_{\gamma \to +\infty} \frac{R(\gamma)}{\log_2 \gamma},$$

which is equivalent to the assumption $R = \mu \log_2 \gamma + O(1)$.

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# *Capacity of MIMO channels (1)*

A single antenna (also single user) ideal channel without fading, known as AWGN channel, is described by

$$y = x + \eta \qquad x, y, \eta \in \mathbb{C}$$

Capacity is given by the famous formula (Shannon 1948)

$$C_{AWGN} = \log_2(1 + \gamma).$$

- Recall that it is possible to find a code $\mathcal{C}$ such that $P_e \to 0$ when $N \to +\infty$ iff $R < C$ (e.g., see Cover & Thomas 1993, Gallager 1968).

At high SNR, on a single antenna AWGN channel, doubling the transmitted energy increases the capacity by one bit only

$$C_{AWGN}(2\gamma) \approx \log_2(2\gamma) \approx 1 + C_{AWGN}(\gamma) \qquad bpcu.$$

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# *Capacity of MIMO channels (2)*

On a fading channel with $L$ parallel branches (diagonal MIMO), where the $L$ fading coefficients $H = \text{Diag}(h_1, \ldots, h_L)$ are only known at the receiver side and the SNR per branch is $\gamma_i = \gamma/L$, we have

$$C(H) = \sum_{i=1}^{L} \log_2(1+|h_i|^2\gamma_i) = \log_2 \left( \prod_{i=1}^{L} (1 + |h_i|^2 \frac{\gamma}{L}) \right) = \log_2 \det \left( I_L + \frac{\gamma}{L} H H^\dagger \right).$$

The above result is still valid for any $n_t \times n_r$ MIMO channel (Telatar 1995) where the exact proof is based on the fact that a circularly symmetric complex gaussian vector with covariance matrix $\Gamma$ yields a maximal differential entropy equal to $\log_2 \det(\pi e \Gamma)$.
For the MIMO channel $y = Hx + \eta$, $x \in \mathbb{C}^{n_t}, y \in \mathbb{C}^{n_r}$, the conditional capacity $C(H)$ is defined as the average mutual information $I(x; y)$ between $x$ and $y$ for a given channel matrix $H$. The capacity $C(H)$ is obtained by assuming that the input is gaussian with covariance matrix $Q = \frac{\gamma}{n_t} I_{n_t}$ (uniform gaussian input).

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# *Capacity of MIMO channels (3)*

Assume a uniform gaussian input and $H$ only known at the receiver side.

**Definition 5: Capacity of an ergodic iid Rayleigh MIMO channel**

The ergodic capacity of the $n_t \times n_r$ MIMO channel is

$$C(\gamma, n_t, n_r) = \mathcal{E}_H \left[ \log_2 \det \left( I_{n_r} + \frac{\gamma}{n_t} H H^\dagger \right) \right].$$

At high SNR ($\gamma \gg 1$), It can be shown that (Foschini 1996)

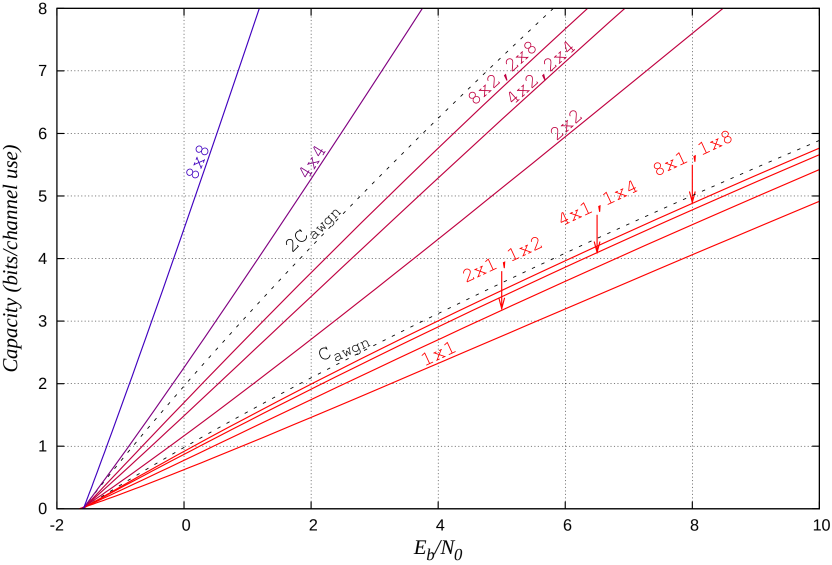
$$C(\gamma, n_t, n_r) = \min(n_t, n_r) \log_2(\gamma) + O(1).$$

Then, for a symmetric channel $n_t = n_r = n$, we have

$$C(2\gamma, n) \approx n \log_2(2\gamma) \approx n + C(\gamma, n)$$

Doubling the transmitted energy increases the capacity by $n$ bits. In the next slide, the ergodic capacity is plotted versus $E_b/N_0 = n_r\gamma/C(\gamma, n_t, n_r)$, recall that $Q = \mathcal{E}[xx^\dagger] = \frac{\gamma}{n_t} I_{n_t}$.

# Capacity of MIMO channels (4)



# *Outage probability (2)*

- For general non-ergodic fading channels, a key idea is to consider the mutual information $I(x;y|H)$ betweeen the channel input and output as a random variable. Each time we pick up a random instance of the channel $H$, it renders a new instantaneous value of $I(x;y|H)$. For a given information rate $R$ to be transmitted, an information theoretical limit on the word error probability is given by (Ozarow, Shamai, Wyner 1994, see also Biglieri, Proakis, Shamai 1998)

$$P_{out} = P\,(\ I(x;y|H)\ <\ R\ )$$

- A similar approach is used for non-ergodic MIMO channels (Telatar 1999, Foschini & Gans 1998). There is an outage each time $C(H,Q)$ is less than the targeted information rate. Here, the average mutual information between $x$ and $y = Hx + \eta$ is indexed by the MIMO channel matrix $H$ and the covariance $Q = \mathcal{E}[xx^{\dagger}]$. The outage probability is

$$P_{out} = P\left(\ \log\det\left(I_{n_r} + HQH^{\dagger}\right)\ <\ R\ \right)$$

It is conjectured that $P_{out}$ is minimized by using a uniform power allocation over a subset of the transmit antennas.

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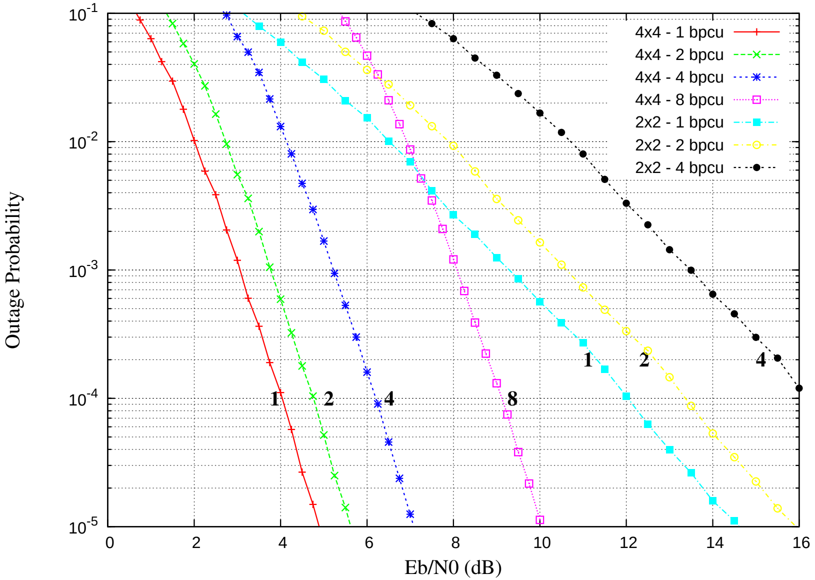
# *Outage probability (3)*



# *Coding for MIMO channels (1)*

## Coding for ergodic channels

• The problem of designing error-correcting codes for ergodic MIMO channels (fast fading) is not an issue.

• Any capacity-achieving code designed for the AWGN channel will do the job when transmitted on a channel with an infinite number of degrees of freedom.

## Coding for non-ergodic channels

• The problem of designing error-correcting codes for non-ergodic MIMO channels (slow fading) is a difficult issue.

• For example, an outage-approaching code should mix both randomness and determinism in its structure.

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# *Coding for MIMO channels (2)*

Unless otherwise stated, we restrict the rest of this lecture to static channels (non-ergodic) where a codeword undergoes a unique channel instance ($n_c = 1$). When $n_c > 1$, the diversity is multiplied accordingly and the code design is similar.

- As shown in the first part of this lecture, a rate 1/2 repetition code $\mathcal{C}[2,1]_q$ can achieve diversity 2 on a block erasure channel with two independent blocks per codeword, $P_e = \epsilon^2$.
- Let us start with a simple example on a 2 × 1 MIMO channel. What is the equivalent of $\mathcal{C}[2,1]_q$ on a MIMO channel?
  One simple solution: the Alamouti code.

# *Coding for MIMO channels (3)*

Consider the following codeword (Alamouti 1998) written in matrix format

$$x = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

where $x_i \in q - QAM \subset \mathbb{Z}^2$.

- Row 1 is transmitted on antenna 1. Row 2 is transmitted on antenna 2.
- Two time periods are needed to transmit $x$ on a $2 \times 1$ MIMO channel. The rate is $R = n_t \times R_c \times \log_2(q) = \log_2(q) \;\; bpcu$, where $n_t = 2$ and $R_c = 1/2$.
- The channel output is
  $$y = Hx + \eta$$
  where $H = [h_1 \; h_2]$ and $y, \eta \in \mathbb{C}^{1\times 2}$.

# Coding for MIMO channels (4)

- Develop the expression of the channel output when Alamouti code is transmitted.

$$\begin{aligned} y_1 &= h_1 x_1 + h_2 x_2 + \eta_1 \\ y_2 &= -h_1 x_2^* + h_2 x_1^* + \eta_2 \end{aligned}$$

- To decode, let us compute

$$\begin{aligned} h_1^* y_1 + h_2 y_2^* &= \left(|h_1|^2 + |h_2|^2\right) x_1 + \eta_1^{'} \\ h_2^* y_1 - h_1 y_2^* &= \left(|h_1|^2 + |h_2|^2\right) x_2 + \eta_2^{'} \end{aligned}$$

- Transmit diversity 2 is achieved since (see Tse & Viswanath 2005)

$$P\left((|h_1|^2 + |h_2|^2)\gamma \; < \; 1\right) \quad \propto \quad \frac{1}{\gamma^2}$$

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# Coding for MIMO channels (5)

- Alamouti code belongs to the family of Orthogonal Space-Time Block codes (OSTBC). The codewords satisfy

$$xx^{\dagger} \propto I_{n_t}$$

- Two examples of OSTBC for $n_t = 3$ and $n_t = 4$ antennas both with rate $R = \frac{3}{4}\log_2(q)$. Four time periods are needed to transmit a codeword.

$$\begin{bmatrix} x_1 & -x_2^* & x_3^* & 0 \\ x_2 & x_1^* & 0 & -x_3^* \\ x_3 & 0 & -x_1^* & x_2^* \end{bmatrix} \qquad \begin{bmatrix} x_1 & 0 & x_2 & -x_3 \\ 0 & x_1 & x_3^* & x_2^* \\ -x_2^* & -x_3 & x_1^* & 0 \\ x_3^* & -x_2 & 0 & x_1^* \end{bmatrix}$$

- OSTBC is an important subclass of linear STBC. For more information See the book by Larsson & Stoica 2003, or the book by Oestges & Clercks 2007.
- Main drawback: they suffer from a weak information rate $R$ (the equivalent embedded $R_c$ is too small).

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# *Coding for MIMO channels (6)*

- Consider a linear code $\mathcal{C}[Nn_t, K]_q$ of rate $R_c = Nn_t/K$. Write a codeword as

$$c = \begin{bmatrix} c_1^1 & c_2^1 & \dots & c_N^1 \\ \vdots & \vdots & & \vdots \\ c_1^{n_t} & c_2^{n_t} & \dots & c_N^{n_t} \end{bmatrix}$$

- Using the mapping $f : F_q \rightarrow \mathbb{Z}^2$, transmit $x = f(c)$ on $n_t \times n_r$ channel.
- It can be shown (e.g., see El Gamal & Hammons 2003) that the pairwise error probability is upper bounded as (ML decoder assumed)

$$P(c \rightarrow c') \quad \leq \quad \left( \frac{1}{\prod_{i=1}^{t}(1 + \lambda_i \gamma / 4n_t)} \right)^{n_r} \quad \leq \quad \left( \frac{g\gamma}{4n_t} \right)^{-tn_r}$$

where $t = rank(f(c) - f(c'))$, the coding gain is $g = (\lambda_1 \lambda_2 \cdots \lambda_t)^{1/t}$, and $\{\lambda_i\}$ are the eigen values of $[f(c) - f(c')][f(c) - f(c')]^\dagger$.

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# *Coding for MIMO channels (7)*

From the expression of the pairwise error probability, we can state (Guey, Fitz, Bell, & Kuo 1996, and Tarokh, Seshadri, & Calderbank 1998)

**Design criterion for static MIMO**

Under ML decoding, a space-time code should satisfy (over all pairs of distinct codewords $c$ and $c'$)

- Rank: Maximize the transmit diversity $t = rank(f(c) - f(c'))$.
- Product distance: Maximize the coding gain $g = (\lambda_1 \lambda_2 \cdots \lambda_t)^{1/t}$.

- The above design criterion cannot guarantee the construction of outage-achieving codes.
- The above design criterion cannot be used to build iteratively decodable graph codes (e.g., LDPC codes) for MIMO channels.
- Nevertheless, it has been used to successfully build space-time block codes (not including an error-correcting code $\mathcal{C}$) that guarantee excellent performance for uncoded $q$-QAM modulations.

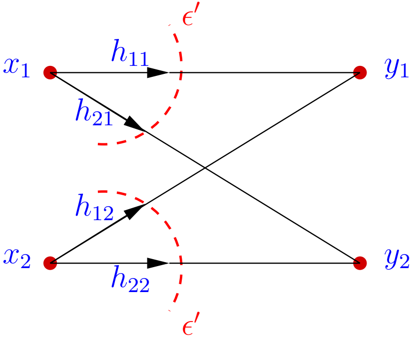
# *Coding for MIMO channels (8)*

Let us study an example of full-diversity LDPC coding for a 2 × 2 MIMO channel.

- Each transmit antenna behaves like a channel state. The state generates an erasure with a probability $\epsilon'$

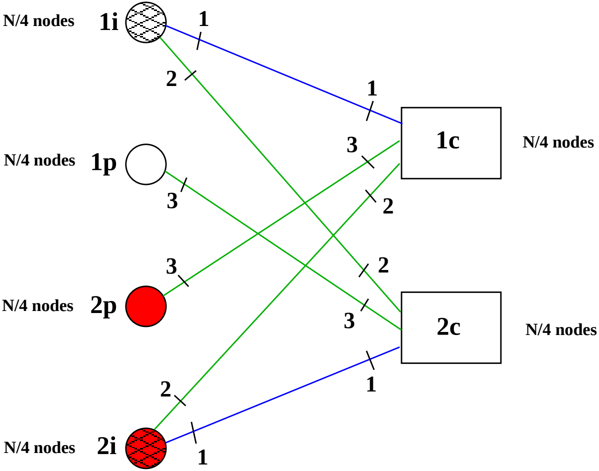
$$\epsilon' = P\left((|h_{11}|^2 + |h_{21}|^2)\gamma \ < \ 1\right) \quad \propto \quad \frac{1}{\gamma^2}$$

- Our aim is to achieve $(\epsilon')^2$, i.e., $P_e \propto \frac{1}{\gamma^4}$



# *Coding for MIMO channels (9)*

Rate-1/2 Full-diversity root-LDPC code for a 2-state block fading channel (Boutros, Guillén i Fàbregas, Biglieri, & Zémor 2007)



White-colored bits: antenna 1 Red-colored bits: antenna 2

# *Coding for MIMO channels (10)*

The parity-check matrix of the root-LDPC code has the following structure (mixture of randomness and determinism).

$$H = \begin{array}{c:c:c:c l} 1i & 1p & 2i & 2p & \\ \begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix} & \mathbf{0} & H_{2i} & H_{2p} & 1c \\ \hdashline H_{1i} & H_{1p} & \begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix} & \mathbf{0} & 2c \end{array}$$

**Theorem 6: threshold in absence of fading**

On a gaussian channel, under iterative decoding, a $(\lambda(x), \rho(x))$ root-LDPC code has the same decoding threshold as a random $(\lambda(x), \rho(x))$ LDPC code.

**Theorem 7: full-diversity for a rate-$1/2$ root-LDPC**

Consider a static $2 \times n_r$ MIMO channel. Under iterative decoding, a $(\lambda(x), \rho(x))$ root-LDPC code achieves full state diversity $(\epsilon')^2$, i.e., $P_e \propto 1/(\gamma)^{2n_r}$.

# *Summary*

- The erasure channel can be a starting point for the study of more complex channels such as the MIMO channel.
- The MIMO channel offers a higher capacity (higher data rates) than a single antenna medium.
- Several coding techniques are well established for space-time coding, practical applications are slowed down by the decoding complexity.