Parallel generation of ℓ–sequences

Cédric Lauradoux
Université de Princeton

Andrea Röck
Équipe SECRET, INRIA

Journées "Codage et Cryptographie" 2008

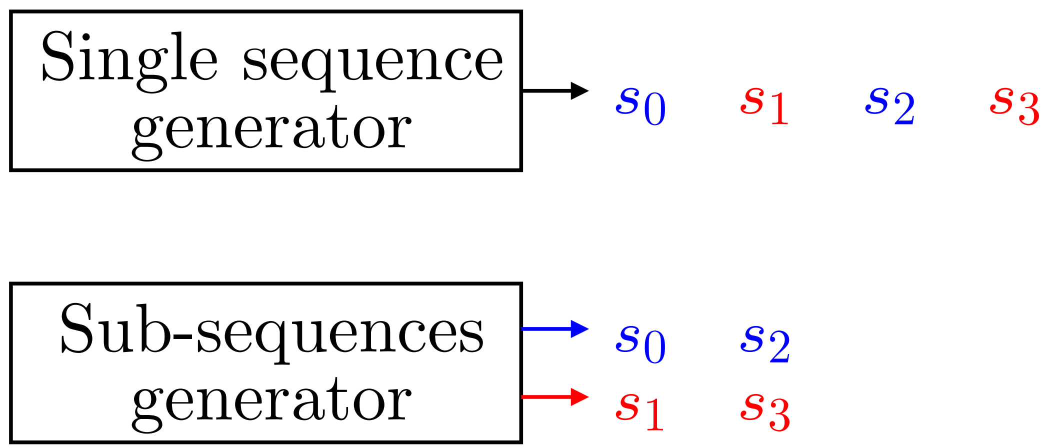
Outline

- Introduction
- LFSRs
  - Synthesis of sub-sequences
  - Multiple steps LFSR
- FCSRs
  - Synthesis of sub-sequences
  - Multiple steps FCSR
- Conclusion

Part 1

Introduction

Sub-sequences generator



- Goal: **parallelism**
  - better throughput
  - reduced power consumption

Notations

- $S = (s_0, s_1, s_2, \cdots)$: Binary sequence with period T.
- $S_d^i = (s_i, s_{i+d}, s_{i+2d}, \cdots)$: Decimated sequence, with $0 \leq i \leq d-1$.
  - $S_d^0 = (s_0, s_d, \cdots), \cdots, S_d^{d-1} = (s_{d-1}, s_{2d-1}, \cdots)$
- x_j: Memory cell.
- $(x_j)_t$: Content of the cell x_j.
- X_t: Entire internal state of the automaton.
- $next^d(x_j)$: Cell connected to the output of x_j.

LFSRs

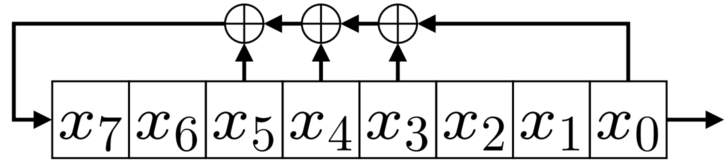
- Automaton with linear update function.
- Let $s(x) = \sum_{i=0}^{\infty} s_i x^i$ be the power series of $S = (s_0, s_1, s_2, \ldots)$. There exists two polynomials $p(x), q(x)$:

$$s(x) = \frac{p(x)}{q(x)}.$$

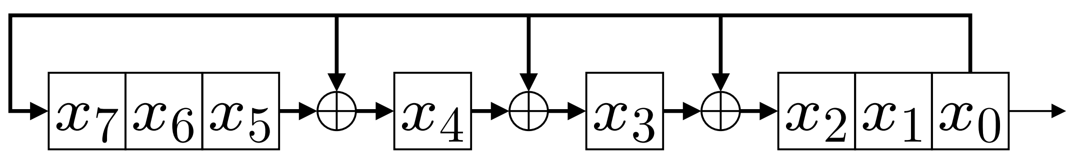
- $q(x)$: Connection polynomial of degree m.
- $Q(x) = x^m q(1/x)$: Characteristic polynomial.
- m–sequence: S has maximal period of $2^m - 1$. (*iff* $q(x)$ is a primitive polynomial)
- Linear complexity: Size of smallest LFSR which generates S.

Fibonacci/Galois LFSRs

Fibonacci setup.



Galois setup.



FCSRs

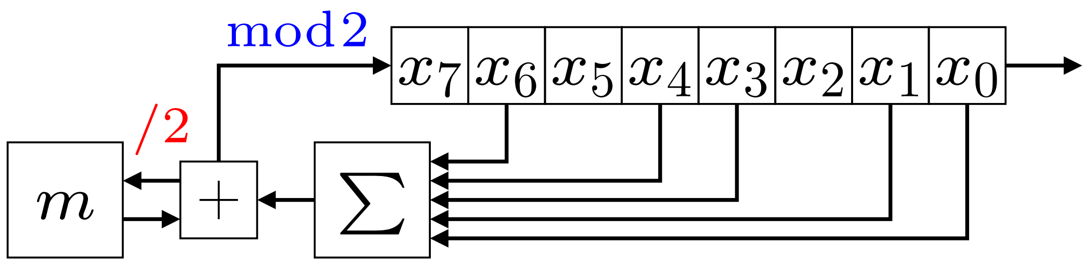
[Klapper Goresky 93]

- Instead of XOR, FCSRs use additions with carry.
  - Non-linear update function.
  - Additional memory to store the carry.
- S is the 2–adic expansion of the rational number: $\frac{h}{q} \leq 0$.
- Connection integer q: Determines the feedback positions.
- *ℓ–sequences*: S has maximal period $\varphi(q)$.
  (*iff* q is a prime power and $ord_q(2) = \varphi(q)$.)
- 2–adic complexity: size of the smallest FCSR which produces S.

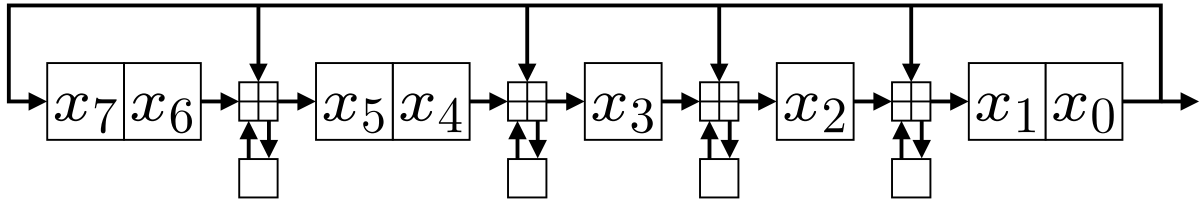
Fibonacci/Galois FCSRs

[Klapper Goresky 02]

Fibonacci setup.



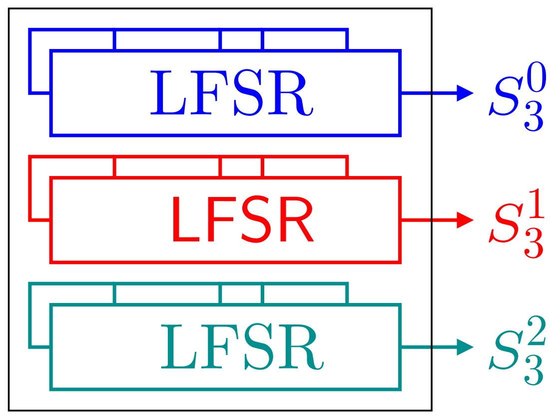
Galois setup.



Part 2

LFSRs

Synthesis of Sub-sequences (1)



- Use Berlekamp-Massey algorithm to find the smallest LFSR for each subsequence.
- All sub-sequences are generated using d LFSRs defined by $Q^{\star}(x)$ but initialized with different values.

Synthesis of Sub-sequences (2)

Theorem [Zierler 59]: Let S be produced by an LFSR whose characteristic polynomial $Q(x)$ is irreducible in $\mathbf{F}_2$ of degree m. Let α be a root of $Q(x)$ and let T be the period of S. For $0 \leq i < d$, S_d^i can be generated by an LFSR with the following properties:

- The minimum polynomial of α^d in $\mathbf{F}_{2^m}$ is the characteristic polynomial $Q^\star(x)$ of the new LFSR with:
- Period $T^\star = \frac{T}{gcd(d,T)}$.
- Degree $m^\star$ is the multiplicative order of 2 in $\mathbf{Z}_{T^\star}$.

Multiple steps LFSR
[Lempel Eastman 71]

- Clock d times the register in one cycle.
- Equivalent to partition the register into d sub-registers

$$x_i x_{i+d} \cdots x_{i+kd}$$

such that $0 \leq i < d$ and $i + kd < m$.

- Duplication of the feedback:

The sub-registers are linearly interconnected.

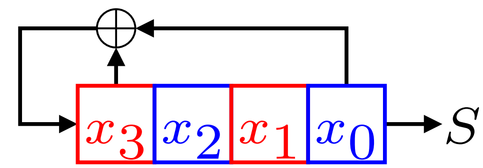
Fibonacci LFSR

$$next^1(x_0) = x_3$$
$$next^1(x_i) = x_{i-1} \text{ if } i \neq 0$$
$$(x_3)_{t+1} = (x_3)_t \oplus (x_0)_t$$
$$(x_i)_{t+1} = (x_{i-1})_t \text{ if } i \neq 3$$

$$next^2(x_0) = x_2$$
$$next^2(x_1) = x_3$$
$$next^2(x_i) = x_{i-2} \text{ if } i > 1$$

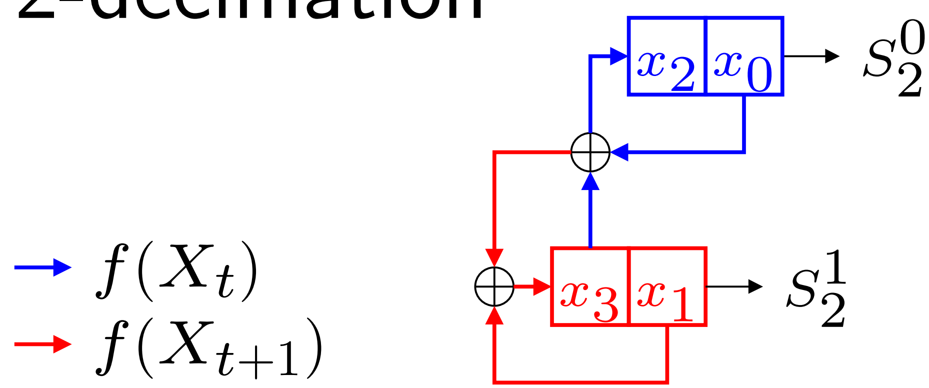
$$(x_i)_{t+2} = (x_{i-2})_t \text{ if } i < 2$$
$$(x_2)_{t+2} = (x_3)_t \oplus (x_0)_t$$
$$(x_3)_{t+2} = \underbrace{(x_3)_t \oplus (x_0)_t}_{(x_3)_{t+1}} \oplus (x_1)_t$$

1-decimation



$$\Downarrow$$

2-decimation



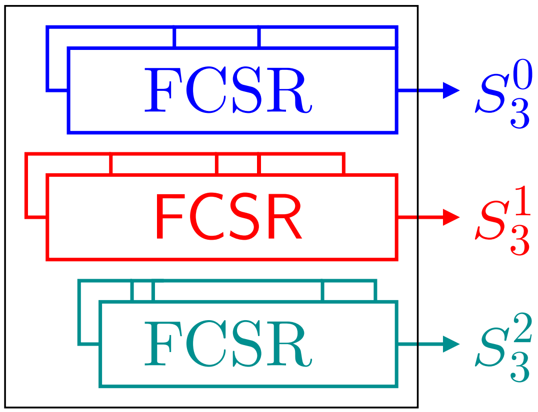
Comparison

Method	Memory cells	Logic Gates
LFSR synthesis	$d \times m^\star$	$d \times wt(Q^\star)$
Multiple steps LFSR	m	$d \times wt(Q)$

Part 3

FCSRs

Synthesis of Sub-sequences (1)



- We use an algorithm based on Euclid's algorithm **[Arnault Berger Necer 04]** to find the smallest FCSR for each sub-sequence.

- The sub-sequences do **not** have the same q.

Synthesis of Sub-sequences (2)

- A given S_d^i has period T^* and minimal connection integer q^*.

- **Period:** (True for all periodic sequences)
  - $T^* \Big| \frac{T}{\gcd(T,d)}$,
  - If $\gcd(T, d) = 1$ then $T^* = T$.

- If $\gcd(T, d) > 1$: T^* might depend on i!
  E.g. for $S = -1/19$ and $d = 3$: $T/gcd(T, d) = 6$.
  - S_3^0: The period $T^* = 2$.
  - S_3^1: The period $T^* = 6$.

Synthesis of Sub-sequences (3)

- **2-adic complexity [Goresky Klapper 97]:**
  - General case: $q^*|2^{T*}-1$.
  - $\gcd(T,d)=1$: $q^*|2^{T/2}+1$.
- **Conjecture [Goresky Klapper 97]:** Let S be an ℓ–sequence with connection integer $q=p^e$ and period T. Suppose p is prime and $q \notin \{5,9,11,13\}$. For any d_1, d_2 relatively prime to T and incongruent modulo T and any i,j:

  $S^i_{d_1}$ and $S^j_{d_2}$ are cyclically distinct.
- **Based on Conjecture:** Let q and p be prime and $T=q-1=2p$:

  If $1 \leq d < T$ and $d \neq p$ then $q^* > q$.

Multiple steps FCSR

- Clock d times the register in one cycle.
- Equivalent to partition the register into d sub-registers

$$x_i x_{i+d} \cdots x_{i+kd}$$

such that $0 \leq i < d$ and $i + kd < m$.

- Interconnection of the sub-registers.
- Propagation of the carry computation.

Fibonacci FCSR (1)

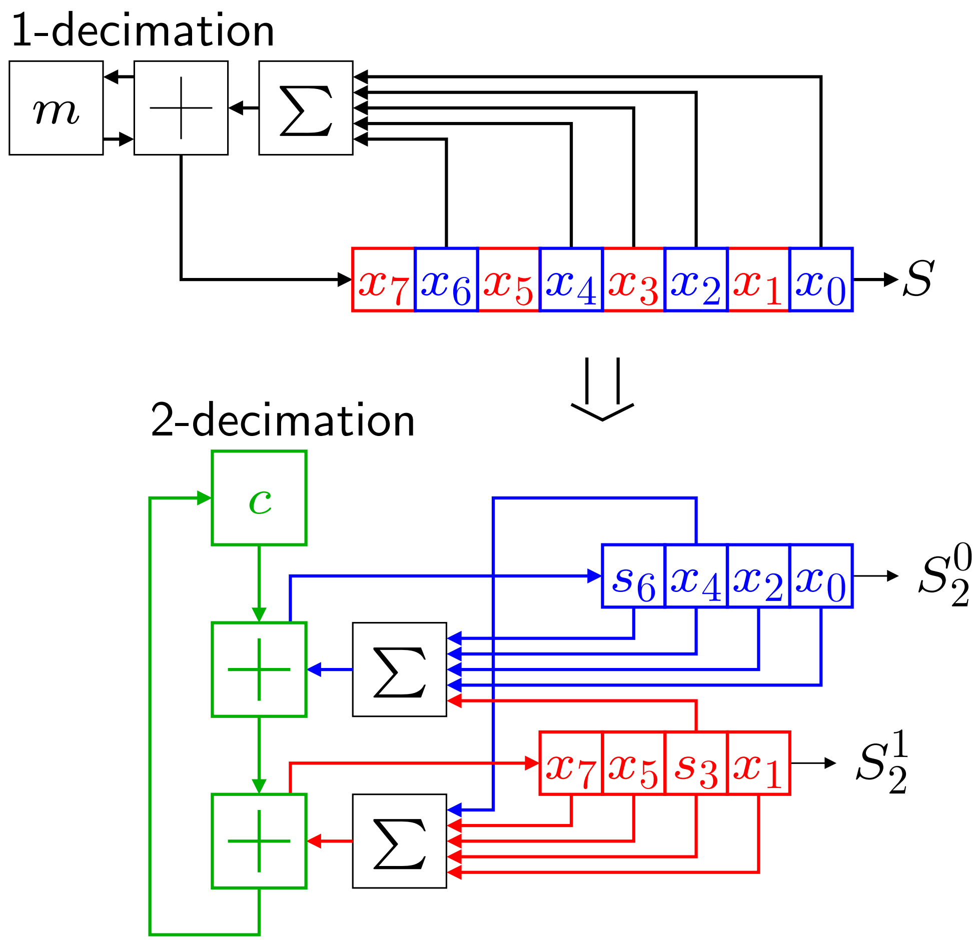
- Let the feedback function be defined by

$$g(X_t, c_t) = \sum_{j=0}^{m-1} (x_j)_t a_j + c_t$$

- We can use the following equations:

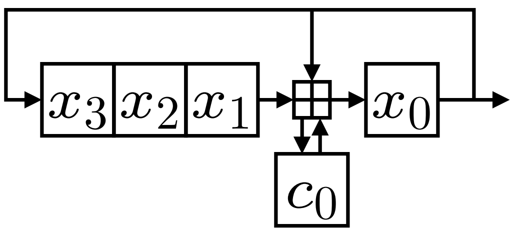
$$\begin{aligned} (x_i)_{t+d} &= \begin{cases} g\left(X_{t+d-m+i}, c_{t+d-m+i}\right) \bmod 2 & \text{if } m-d \leq i < m \\ (x_{i+d})_t & \text{if } i < m-d \end{cases} \\ c_{t+d} &= g(X_{t+d-m+i}, c_{t+d-m+i})/2 \end{aligned}$$

Fibonacci FCSR (2)



Galois FCSR (1)

- Example $q = -19$:



- Description at the bit-level:

$$\begin{cases} (x_0)_{t+1} & = (x_0)_t \oplus (x_1)_t \oplus (c_0)_t \\ (c_0)_{t+1} & = [(x_0)_t \oplus (x_1)_t]\,[(x_0)_t \oplus (c_0)_t] \oplus (x_0)_t \end{cases}$$

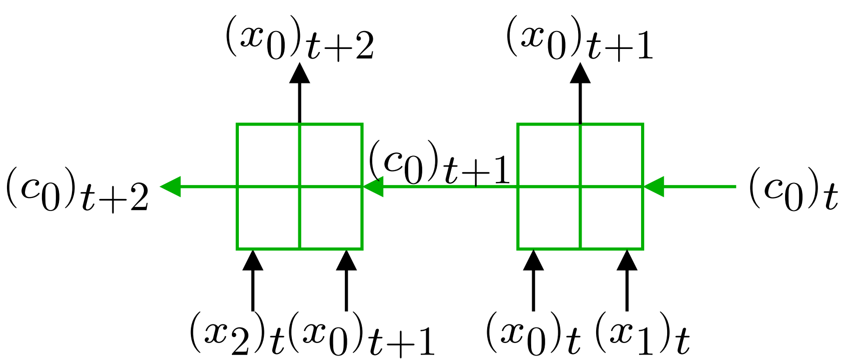
Galois FCSR (2)

- $d = 2$, description for the automaton at $t+1$ and $t+2$

$$t+1 \quad \begin{cases} (x_0)_{t+1} &= (x_0)_t \oplus (x_1)_t \oplus (c_0)_t \\ (c_0)_{t+1} &= [(x_0)_t \oplus (x_1)_t]\,[(x_0)_t \oplus (c_0)_t] \oplus (x_0)_t \end{cases}$$

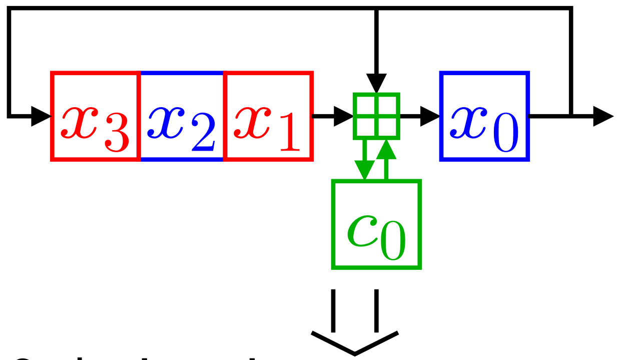
$$t+2 \quad \begin{cases} (x_0)_{t+2} &= (x_0)_{t+1} \oplus (x_2)_t \oplus (c_0)_{t+1} \\ (c_0)_{t+2} &= [(x_0)_{t+1} \oplus (x_2)_t]\,[(x_0)_{t+1} \oplus (c_0)_{t+1}] \oplus (x_0)_{t+1} \end{cases}$$

2-bit ripple carry adder

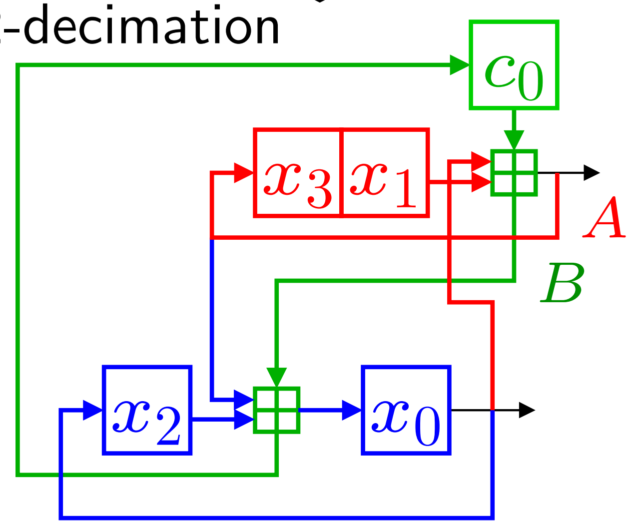


Galois FCSR (3)

1-decimation



2-decimation



$$A = \boxplus\left[(x_0)_t, (x_1)_t, (c_0)_t\right] \bmod 2$$

$$B = \boxplus\left[(x_0)_t, (x_1)_t, (c_0)_t\right] \div 2$$

$$(x_0)_{t+2} = \boxplus\left[A, B, (x_2)_t\right] \bmod 2$$

$$(c_0)_{t+2} = \boxplus\left[A, B, (x_2)_t\right] \div 2$$

$$(x_1)_{t+2} = (x_3)_t$$

$$(x_2)_{t+2} = (x_0)_t$$

$$(x_3)_{t+2} = A$$

Comparison

- **Synthesis of Sub-sequences:**
  - Period: If $gcd(T, d) > 1$ it might depend on i.
  - 2-adic complexity: q^* can be much bigger than q.

- **Multiple steps FCSR:**
  - Same memory size.
  - Propagation of carry by well-known arithmetic circuits.

Part 4

Conclusion

Conclusion

- The decimation of an ℓ–sequence can be used to increase the throughput or to reduce the power consumption.

- A separated FCSR for each sub–sequence is not satisfying.

  However, the multiple steps FCSR works fine.

- Sub-expressions simplification:
  - classical for LFSR.
  - new problem for FCSR.