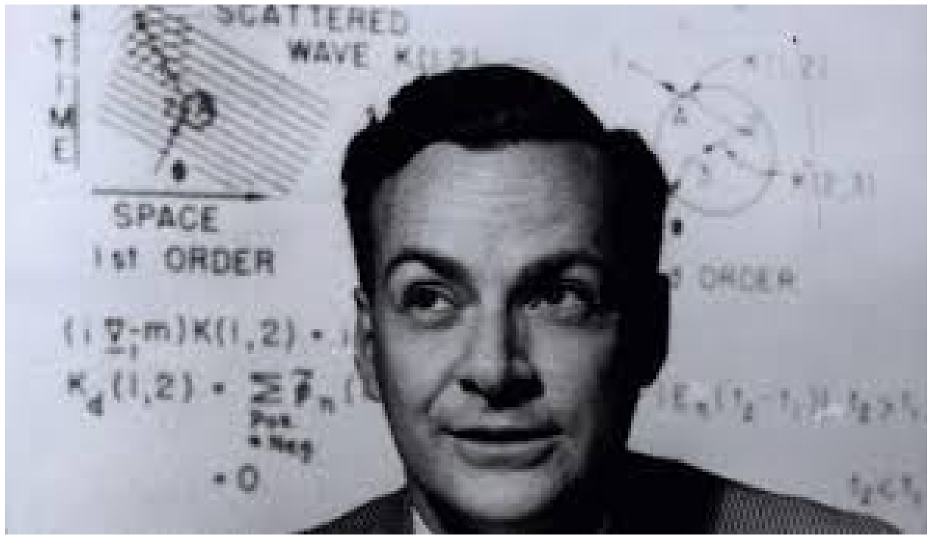
Lecture 7
Quantum simulation–
The HHL algorithm

February 27, 2020

Plan

1. Quantum simulation
2. The HHL algorithm

1. Quantum simulation; the dream of Feynman



Feynman (1982) "Can physics be simulated by a quantum computer? [. . .] the full description of quantum mechanics for a large system with R particles has too many variables, it cannot be simulated with a normal computer with a number of elements proportional to R [. . . but it can be simulated with] quantum computer elements. "

The Hamiltonian

- Dynamical behavior of quantum systems governed by Schrödinger's equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \mathbf{H}|\psi\rangle$$

- Key challenge in simulating quantum systems: exponential number of differential equations that have to be solved: a system of n qubits $\Rightarrow$ solving 2^n differential equations . . .

Applications of quantum simulation

- Condensed-matter physics
  - solving problems that are classically intractable: Hubbard model (simplest model of interactings particles on a lattice), spin systems..;
  - understanding phase transitions, disordered systems, high-temperature superconductivity,...
- High-energy physics
- Cosmology
- Atomic and nuclear physics

Application to quantum chemistry

- calculating the thermal rate constant
- obtaining the energy spectrum of a molecular system
- simulate the static and dynamical chemical properties of molecules
- simulate chemical reactions

Exponentiating the Hamiltonian

$$i\frac{d}{dt}|\psi\rangle = \mathbf{H}|\psi\rangle$$

$$\Downarrow$$

$$|\psi(t)\rangle = e^{-i\mathbf{H}t}|\psi(0)\rangle$$

Hamiltonian simulation

Problem 1. [Hamiltonian simulation]
Input: *Hamiltonian* $\mathbf{H}$ *acting on* n *qubits, time* $t \in \mathbb{R}^+$*, accuracy* $\varepsilon \in \mathbb{R}^+$
Output: *a quantum circuit/algorithm implementing a unitary* $\mathbf{U}$ *which is such that*

$$\left\| e^{it\mathbf{H}} - \mathbf{U} \right\| \leq \varepsilon$$

Cost: *the number of gates implementing* $\mathbf{U}$

- $\mathbf{H}$ is can be efficiently simulated if the quantum circuit consists of $\text{poly}(n, t, \frac{1}{\varepsilon})$ gates

Exercise: some simple principles

1. Show that if the unitary transform $\mathbf{U}$ can be efficiently implemented and the Hamiltonian $\mathbf{H}$ be efficiently simulated, then $\mathbf{U}\mathbf{H}\mathbf{U}^*$ can be efficiently simulated
2. if $\mathbf{H}$ is diagonalizable in a basis corresponding to a unitary $\mathbf{U}$ that can be efficiently implemented and if its eigenvalues can be efficiently computed, show that such $\mathbf{H}$ can be efficiently simulated

Solution

1.

$$\begin{aligned} e^{it\mathbf{U}\mathbf{H}\mathbf{U}^*} &= \sum_{i=0}^{\infty} \frac{(it\mathbf{U}\mathbf{H}\mathbf{U}^*)^i}{i!} \\ &= \sum_{i=0}^{\infty} \mathbf{U}\frac{(it\mathbf{H})^i}{i!}\mathbf{U}^* \\ &= \mathbf{U}e^{i\mathbf{H}t}\mathbf{U}^* \end{aligned}$$

2. Assume first that $\mathbf{H}$ is diagonalizable in the computational basis. Then $\mathbf{H}$ can be efficiently simulated by performing the following steps

$$\begin{aligned} |a,0\rangle &\mapsto |a,\lambda(a)\rangle \\ &\mapsto e^{it\lambda(a)}|a,\lambda(a)\rangle \\ &\mapsto e^{it\lambda(a)}|a,0\rangle \\ &= e^{it\mathbf{H}}|a\rangle|0\rangle \end{aligned}$$

The general case is handled by using the previous principle

The Hamiltonian encountered in practice

$$\mathbf{H} = \sum_{j=1}^{m} \mathbf{H}_i$$

where the $\mathbf{H}_i$ only involve a few qubits

Method 1: the Lie-Suzuki-Trotter method

- If $\mathbf{H}_1$ and $\mathbf{H}_2$ can be efficiently simulated, then $\mathbf{H}_1$ and $\mathbf{H}_2$ can also be efficiently simulated

Theorem 1.

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}} + O\left(\|A\| \cdot \|B\|\right)$$

Zassenhaus formula

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}e^{-\frac{1}{2}[\mathbf{A},\mathbf{B}]}\dots$$

The method

$$
\begin{aligned}
e^{i\mathbf{H}t} &= (e^{i\mathbf{H}t/r})^r \\
&= \left(e^{i\mathbf{H}_1 t/r + i\mathbf{H}_2 t/r}\right)^r \\
&= \left(e^{i\mathbf{H}_1 t/r} e^{i\mathbf{H}_2 t/r} + E\right)^r \\
&= \left(e^{i\mathbf{H}_1 t/r} e^{i\mathbf{H}_2 t/r}\right)^r + O(r\,\|E\|) \\
\|E\| &= O\left(\|i\mathbf{H}_1 t/r\| \cdot \|i\mathbf{H}_2 t/r\|\right) \\
&= O\left(\|\mathbf{H}_1\| \cdot \|\mathbf{H}_2\| \frac{t^2}{r^2}\right) \\
\text{choose } r &= O\left(\frac{t^2}{\varepsilon\,\|\mathbf{H}_1\| \cdot \|\mathbf{H}_2\|}\right) \\
\Rightarrow e^{i\mathbf{H}t} &= \left(e^{i\mathbf{H}_1 t/r} e^{i\mathbf{H}_2 t/r}\right)^r + O(\varepsilon)
\end{aligned}
$$

- In general for a sum of m hamiltonians : $O(mt^2/\varepsilon)$ simulations of individual hamiltonians

Method 2: quantum walk approach

- Dependency in t of the previous approach $O(t^2)$
- Optimal dependency in t : $O(t)$
- Can be obtained by a quantum walk approach

Block encoding

- A recent and flexible approach with a logarithmic dependency in ε
- Assume that $\mathbf{A}$ acts on n qubits with operator norm $\|\mathbf{A}\| \leq 1$ and we know how to implement an $(n+a)$-qubit unitary operator

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Definition 1. [block-encoding] $\mathbf{U}$ *is said to be an a-block encoding of $\mathbf{A}$. If*

$$\mathbf{U} = \begin{pmatrix} \frac{\tilde{\mathbf{A}}}{s} & \cdot \\ \cdot & \cdot \end{pmatrix} \quad \textit{with} \quad \left\| \mathbf{A} - \tilde{\mathbf{A}} \right\| \leq \varepsilon$$

then $\mathbf{U}$ *is an (s, a, ε) approximate block encoding of* $\mathbf{A}$

Sparse access

- We assume that $\mathbf{A}$ is s-sparse and have sparse access to $\mathbf{A}$

$$\begin{aligned} O_A : |i,j\rangle \left|0^b\right\rangle &\mapsto |i,j\rangle\,|A_{ij}\rangle \\ O_r : |i,\ell\rangle &\mapsto |i, r(i,\ell)\rangle \\ O_c : |\ell,j\rangle &\mapsto |c(\ell,j),j\rangle \end{aligned}$$

where

- A_{ij} is a b-bit description of A_{ij},
- $r(i,\ell)$ denotes the location of the ℓ-th nonzero entry of the i-th row of $\mathbf{A}$,
- $c(\ell,j)$ denotes the location of the ℓ-th nonzero entry of the j-th column of $\mathbf{A}$

Exercise

Let

$$
\begin{aligned}
W_1 : |0\rangle\,|0^n\rangle\,|j\rangle &\mapsto \frac{1}{\sqrt{s}}\,|0\rangle \sum_{k:A_{kj}\neq 0} |k,j\rangle \\
W_2 : |0\rangle\,|k,j\rangle\,\left|0^b\right\rangle &\mapsto \left(A_{kj}\,|0\rangle + \sqrt{1-|A_{kj}^2}\,|1\rangle\right)|k,j\rangle\,\left|0^b\right\rangle \\
W_3 : |0\rangle\,|0^n\rangle\,|i\rangle &\mapsto \frac{1}{\sqrt{s}}\,|0\rangle \sum_{\ell:A_{i\ell}\neq 0} |i,\ell\rangle
\end{aligned}
$$

We assume $s = 2^m$.

1. Show how to implement W_1 and W_3 using an O_c and O_r queries and a few other A-independent gates.
2. Show how to implement W_2 using an O_A-query, an O_A^{-1}-query, and a few other A-independent gates.
3. Show that the $(0^{n+1}i, 0^{n+1}j)$-entry of $W_3^{-1}W_1$ is exactly $1/s$ if $A_{ij} \neq 0$, and is 0 if $A_{ij} = 0$
4. Show that the $(0^{n+1}i, 0^{n+1}j)$-entry of $W_3^{-1}W_2W_1$ is exactly A_{ij}/s

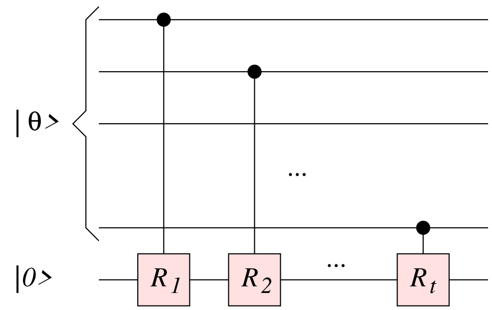
Solution

1.

$$
\begin{aligned}
|0\rangle\,|0^n\rangle\,|j\rangle \quad & \xrightarrow{\mathbf{Id}\otimes\mathbf{H}^{\otimes m}\otimes\mathbf{Id}} \quad \sum_{\ell} \frac{1}{\sqrt{s}}\,|0\rangle\,|\ell\rangle\,|j\rangle \\
& \xrightarrow{\mathbf{Id}\otimes 0_C} \quad \sum_{\ell} \frac{1}{\sqrt{s}}\,|0\rangle\,|c(\ell,j)\rangle\,|j\rangle \\
& = \quad \frac{1}{\sqrt{s}}\,|0\rangle \sum_{k:A_{kj}\neq 0} |k,j\rangle
\end{aligned}
$$

2.

$$
\begin{aligned}
R(\theta) & \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
\theta & = \cos^{-1}(A_{kj}) \\
R_j & \stackrel{\text{def}}{=} R(2\pi/2^j)
\end{aligned}
$$



$$
\begin{aligned}
|0\rangle\,|k,j\rangle\,\left|0^b\right\rangle \quad &\xrightarrow{\mathbf{Id}\otimes O_A} \quad |0\rangle\,|k,j\rangle\,|A_{kj}\rangle \\
&\xrightarrow{\mathbf{Id}\otimes\cos^{-1}} \quad |0\rangle\,|k,j\rangle\,|\theta_{kj}\rangle \\
&\xrightarrow{\text{c-}R} \quad (\cos\theta_{kj}\,|0\rangle + \sin\theta_{kj}\,|1\rangle)\,|k,j\rangle\,|\theta_{kj}\rangle \\
&\xrightarrow{\mathbf{Id}\otimes\cos} \quad \left(A_{kj}\,|0\rangle + \sqrt{1-A_{kj}^2}\,|1\rangle\right)|k,j\rangle\,|A_{kj}\rangle \\
&\xrightarrow{\mathbf{Id}\otimes O_A^{-1}} \quad \left(A_{kj}\,|0\rangle + \sqrt{1-A_{kj}^2}\,|1\rangle\right)|k,j\rangle\,\left|0^b\right\rangle
\end{aligned}
$$

3.

$$\left\langle 0^{n+1}\right|\langle i|\, W_3^* W_1 \left|0^{n+1}\right\rangle |j\rangle = \frac{1}{s} \text{ if } A_{ij} \neq 0 \text{ and } 0 \text{ otherwise}$$

4.

$$\left\langle 0^{n+1}\right|\langle i|\, W_3^* W_2 W_1 \left|0^{n+1}\right\rangle |j\rangle = \frac{A_{ij}}{s}$$

Low-degree approximation

- We want to implement

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

given a block encoding of $\mathbf{A}$,

$$U : |0^a\rangle |\psi\rangle \mapsto |0^a\rangle \mathbf{A} |\psi\rangle + \sum_{j>0} |j\rangle |\psi_j\rangle$$

examples of interest:
  - $f(x) = e^{ixt}$
  - $f(x) = \frac{1}{x}$
- We have a degree polynomial P approximating f

Theorem 2. *Let $P : [-1,1] \to \{z \in \mathbb{C} : |z| \leq 1/4\}$ be a degree-d polynomial and let U an (s, a, ε) approximate block encoding of $\mathbf{A}$. We can implement an $(1, a+2, 4d\sqrt{\varepsilon/s})$ approximate block encoding of $P(\mathbf{A}/s)$ with d applications of $\mathbf{U}$ and $\mathbf{U}^*$, a single application of c-$\mathbf{U}$ and $O(ad)$ other 1 and 2-qubit gates*

Exercise

Show that a sufficiently large constant c can be chosen such that for all hermitian $\mathbf{A}$ with operator norm $\|\mathbf{A}\| \leq 1$, we have

$$\left\| e^{it\mathbf{A}} - \sum_{k=0}^{c(t+\log(1/\varepsilon))-1} \frac{(it\mathbf{A})^k}{k!} \right\| \leq \varepsilon$$

Solution

$$
\begin{aligned}
\left\| e^{it\mathbf{A}} - \sum_{k=0}^{c(t+\log(1/\varepsilon))-1} \frac{(it\mathbf{A})^k}{k!} \right\| &= \left\| \sum_{k=c(t+\log(1/\varepsilon))}^{\infty} \frac{(it\mathbf{A})^k}{k!} \right\| \\
&\leq \sum_{k=c(t+\log(1/\varepsilon))}^{\infty} \left(\frac{te}{k} \right)^k \\
&\leq \sum_{k=c(t+\log(1/\varepsilon))}^{\infty} \left(\frac{e}{c} \right)^k \\
&\leq \frac{\left(\frac{e}{c}\right)^{c(t+\log(1/\varepsilon))}}{1 - \frac{e}{c}}
\end{aligned}
$$

Let $c = e^2$

$$\left\| e^{it\mathbf{A}} - \sum_{k=0}^{c(t+\log(1/\varepsilon))-1} \frac{(it\mathbf{A})^k}{k!} \right\| \leq \frac{\left(\varepsilon \frac{1}{e^{e^2 t}}\right)}{1 - \frac{1}{e}}$$

$$\leq \varepsilon$$

Hamiltonian simulation via transforming block-encoded matrices

- Approximate $f(x) = e^{ixt}$ with a degree $d = O(t + \log(1/\varepsilon))$ polynomial P
- If $\mathbf{H}$ is s-sparse $\Rightarrow$ block encoding $\mathbf{U}$ of $\mathbf{H}/s$ using $O(1)$ queries to $\mathbf{H}$ and $O(n)$ other gates evolving $\mathbf{H}$ for time t = evolving $\mathbf{H}/s$ for time st
- Previous theorem $\Rightarrow$ block-encoding $\mathbf{V}$ of $P(x) \approx \frac{e^{it\mathbf{H}}}{4}$ by invoking $\mathbf{U}$ an $\mathbf{U}^{-1}$ $O(st + \log(1/\varepsilon))$ times and mapping

$$\mathbf{V} : |0\rangle |\psi\rangle \mapsto |0\rangle P(\mathbf{H}) |\psi\rangle + |\phi\rangle$$

  where $|\phi\rangle$ has no support on basis states starting with $|0\rangle$
- Complexity of ε-precise Hamiltonian simulation of s-sparse $\mathbf{H}$ is $O(st + \log(1/\varepsilon))$ queries to $\mathbf{H}$ and $O(n(st + \log(1/\varepsilon)))$ 2-qubit gates

The HHL algorithm

$$\begin{aligned} N &\stackrel{\text{def}}{=} 2^n \\ \mathbf{A} &\in \mathbb{C}^{N\times N} \\ \mathbf{b} &\in \mathbb{C}^{N} \\ \text{Find } \mathbf{x} \text{ s.t. } \mathbf{Ax} &= \mathbf{b} \end{aligned}$$

Problem 2. **[quantum linear system problem (QLSP)]** *Find an n-qubit state $|\hat{\mathrm{x}}\rangle$ such that*

$$\begin{aligned} (i) \quad \| |\mathbf{x}\rangle - |\hat{\mathbf{x}}\rangle \| &\leq \varepsilon \\ (ii) \quad \mathbf{Ax} &= \mathbf{b} \end{aligned}$$

Assumptions

1. $\mathbf{A}$ is non-singular
2. $|b\rangle$ can be prepared using a circuit of B 2-gates
3. $\mathbf{A}$ is s-sparse
4. $\lambda_i \in (0, 1]$ for all i where $\lambda_1, \cdots, \lambda_N$ are the singular values of $\mathbf{A}$

Complexity

$$\kappa \stackrel{\text{def}}{=} \frac{\max_j \mu_j}{\min_j \mu_j}$$

Problem	Algorithm	Complexity
LSP	Conjugate Gradient	$O(Ns\kappa \log(1/\varepsilon))$
QLSP	HHL 2009	$O\left(\frac{s^2\kappa^2 \log N}{\varepsilon}\right)$
QLSP	VTAA-HHL (Ambainis 2010)	$O\left(\frac{s^2\kappa \log N}{\varepsilon}\right)$
QLSP	Childs et al 2017	$O\left(s\kappa \text{polylog}(s\kappa/\varepsilon)\right)$
QLSP	QLSA 2018	$O\left(\frac{\kappa^2 \text{polylog}(n)\sqrt{\mathbf{Tr}(\mathbf{AA}^*)}}{\varepsilon}\right)$

Application: recommendation system

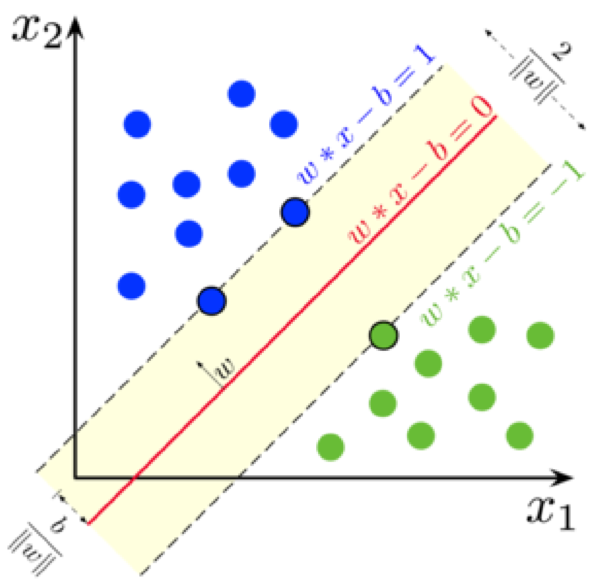
Problem 3. [Recommendation system] *An unknown (hidden)* $m \times n$ *binary matrix* $\mathbf{P}$ *modelling customers preferences and* $\mathbf{P}$ *is of low rank* k. *For a customer* i *one should output columns* j *such that it is likely that* $P_{ij} = 1$.

- Quantum algorithm based on HHL that is of complexity $O(\text{poly}(k)\text{polylog}(mn))$ (we do not use all the entries of $\mathbf{P}$!)

Application: support vector machine

Problem 4. [Support Vector Machine]
Input: M *training data points of the form* $\{(\mathbf{x}_i, y_i),\ \mathbf{x}_i \in \mathbb{R}^N, y_i = \pm 1\}_{i=1\cdots M}$
Output: $\mathbf{w} \in \mathbb{R}^N$ *and* b *that minimizes* $\|\mathbf{w}\|$ *under the constraint* $y_i(\mathbf{w} \cdot \mathbf{x}_i + b) \geq 1$

- best classical algorithm takes poly(M, N) whereas quantum complexity $O(\log(MN))$



Exercise: solving a Boolean system

1. Solve the following system over $\mathbb{F}_2$:

$$\begin{cases} x_1x_2 + x_1x_3 & = & 0 \\ x_1x_3 + x_2x_3 + x_2x_4 & = & 1 \\ x_1x_2 + x_2x_3 & = & 0 \\ x_1x_2 + x_2x_4 & = & 1 \end{cases}$$

2. Outline a strategy for solving a polynomial system involving the multiplication of the polynomial equations by all monomials of degree $\leq D$
3. Can you associate to a polynomial system over $\mathbb{F}_2$ a polynomial system over $\mathbb{C}$ that has as only solutions the solutions of the previous system ?
4. What happens if you apply HHL to this system over $\mathbb{C}$?

Exercise : $\mathbf{A}$ hermitian ?

1. Give a $2N \times 2N$ hermitian matrix $\mathbf{A}'$ and $\mathbf{b}' \in \mathbb{C}^{2N}$ such that a solution $\mathbf{x}$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be read off from a solution $\mathbf{x}'$ to $\mathbf{A}'\mathbf{x} = \mathbf{b}'$
2. Relation between the condition number of $\mathbf{A}$ and $\mathbf{A}'$?

Solution

1.

$$\mathbf{A}' = \begin{pmatrix} 0 & \mathbf{A}^\intercal \\ \mathbf{A} & 0 \end{pmatrix}$$

$$\mathbf{b}' = \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

$$\mathbf{x}' = \begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}$$

2.

$$\mathbf{A}'^2 = \begin{pmatrix} \mathbf{A}^\intercal \mathbf{A} & 0 \\ 0 & \mathbf{A}\mathbf{A}^\intercal \end{pmatrix}$$

$$\Downarrow$$

$$\kappa(\mathbf{A}) = \kappa(\mathbf{A}')$$

Approach

- We assume that

$$
\begin{aligned}
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{A} &= \sum_j \lambda_j \mathbf{u}_j \mathbf{u}_j^\intercal \\
\mathbf{A}^{-1}\mathbf{u}_j &\mapsto \frac{1}{\lambda_j}\mathbf{u}_j \\
\mathbf{b} &= \sum_j \beta_j \mathbf{u}_j \\
\mathbf{A}^{-1}\mathbf{b} &= \sum_j \frac{\beta_j}{\lambda_j}\mathbf{u}_j
\end{aligned}
$$

- Problem: $\mathbf{A}^{-1}$ is not unitary in general
- $\mathbf{A}$ is hermitian $\Rightarrow$ $e^{i\mathbf{A}}$ is unitary

Phase estimation

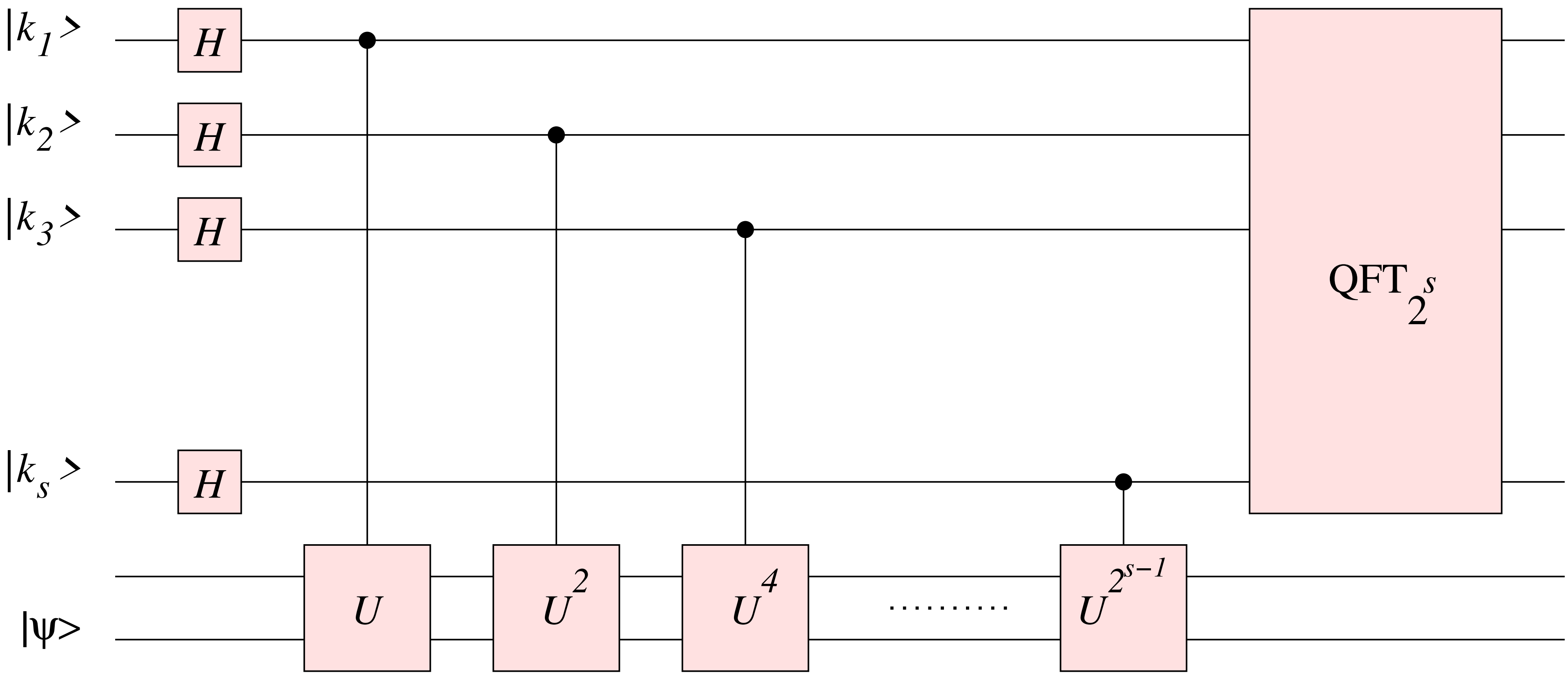
Theorem 3. *For every unitary operator U acting on m qubits, there exists a quantum circuit $\mathbf{PE}(U)$ acting on $m+s$ qubits satisfying the following properties*

1. *the circuit $\mathbf{PE}(U)$ uses $2s$ Hadamard gates, $O(s^2)$ controlled phase rotations and makes 2^{s+1} calls to c-U*
2. *maps with probability $1 - 1/\text{poly}(n)$*

$$\sum_j \alpha_j |\psi_j\rangle |0\rangle \mapsto \sum_j \alpha_j |\psi_j\rangle \left|\tilde{\theta}_j\right\rangle$$

where $|\psi_j\rangle$ are the eigenvectors of $\mathbf{U}$, $e^{i\theta_j}$ is the associated eigenvalue and $|\theta_j - \tilde{\theta}_j| \leq 2^{-m}$

The circuit of phase estimation



Approach (II)

- With Hamiltonian simulation we can implement $e^{i\mathbf{A}}$

$$
\begin{aligned}
|0\rangle\,|0\rangle\,|v_i\rangle &\overset{\text{PE}}{\mapsto} |0\rangle\,|\lambda_j\rangle\,|v_j\rangle \\
&\mapsto \left(\frac{1}{\kappa\lambda_j}|0\rangle + \sqrt{1-\frac{1}{(\kappa\lambda_j)^2}}\,|1\rangle\right)|\lambda_j\rangle\,|v_j\rangle \\
&\overset{\text{PE}^{-1}}{\mapsto} \left(\frac{1}{\kappa\lambda_j}|0\rangle + \sqrt{1-\frac{1}{(\kappa\lambda_j)^2}}\,|1\rangle\right)|0\rangle\,|v_j\rangle \\
V : |v_i\rangle\,|0\rangle\,|0\rangle &\mapsto |v_j\rangle\,|0\rangle\left(\frac{1}{\kappa\lambda_j}|0\rangle + \sqrt{1-\frac{1}{(\kappa\lambda_j)^2}}\,|1\rangle\right)
\end{aligned}
$$

Approach (III)

$$
\begin{aligned}
|0\rangle\,|b\rangle \quad &= \quad |0\rangle \sum_j \beta_j\,|v_j\rangle \\
&\overset{\mathbf{V}}{\mapsto} \quad \left(\frac{1}{\kappa\lambda_j}\,|0\rangle + \sqrt{1-\frac{1}{(\kappa\lambda_j)^2}}\,|1\rangle\right)\sum_j \beta_j\,|v_j\rangle \\
&\qquad = K\,|0\rangle\,|x\rangle + |1\rangle\,|\phi\rangle
\end{aligned}
$$

Exercise

1. Show that the probability of measuring $|0\rangle$ in the first register is $\geq \frac{1}{\kappa^2}$
2. How can this probability be improved ?

Solution

1. Let $p = \mathbf{Prob}(\text{meas. } 0)$, then

$$p = \sum_i \frac{|\beta_i|^2}{\lambda_i^2 \kappa^2} \geq \sum_i \frac{|\beta_i|^2}{\kappa^2} \geq \frac{1}{\kappa^2}$$

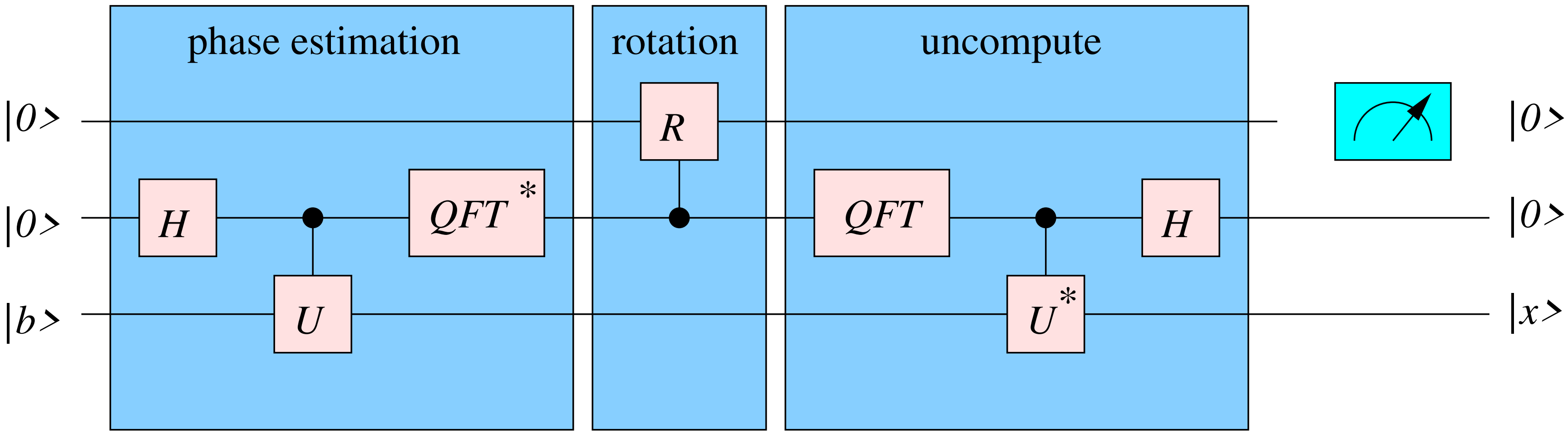
2. amplitude amplification $\Rightarrow O(\kappa)$ calls of the algorithm for having a probability of success $\Omega(1)$

Exercise

Give a quantum circuit performing

$$|0\rangle|\lambda_j\rangle \mapsto \left(\frac{1}{\kappa\lambda_j}|0\rangle + \sqrt{1-\frac{1}{(\kappa\lambda_j)^2}}|1\rangle\right)|\lambda_j\rangle$$

Circuit



Complexity

- Leads to an algorithm that produces a state $|\tilde{x}\rangle$ that is ε-close to $|x\rangle$ using $\kappa^2 s/\varepsilon$ queries to $\mathbf{A}$ and roughly $\kappa s(\kappa n/\varepsilon + B)$ other 2-qubit gates

Improving the efficiency of HHL

- Use the block encoding method of quantum simulation to perform $f(\mathbf{A})$ with

$$\begin{aligned} f(x) &\stackrel{\text{def}}{=} \frac{1-(1-x^2)^b}{x} \\ b &= \kappa^2 \ln(\kappa/\varepsilon) \end{aligned}$$

- Complexity: $O(\kappa^2 s \log(\kappa/\varepsilon))$ queries to $\mathbf{A}$ and $O(\kappa s(\kappa n \log(\kappa/\varepsilon) + B))$ 2-qubit gates

Exercise

1. Let $I = [-1, -1/\kappa] \cup [1/\kappa, 1]$. Give an upperbound on $|f(x) - \frac{1}{x}|$ on I.
2. Show that the polynomial $p(x) = f(x)/(4(\kappa + \varepsilon))$ meets the conditions of Theorem 1

Solution