Lecture 4 : Adaptive source coding algorithms

January 31, 2020

Outline

1. Motivation;
2. adaptive Huffman encoding;
3. Gallager and Knuth's method;
4. Dictionary methods : Lempel-Ziv 78, Lempel-Ziv-Welsh, Lempel-Ziv 77.

1. Motivation

Huffman/arithmetic encoding needs two passes

1. first pass to compute the source statistics
2. second pass : Huffman/arithmetic encoding

Moreover additional information is needed for the decoder

- either the statistics are known;
- or the encoding table is known.

Universal source coding

Universal source coding : no assumption on the source.

Idea: Compute at each time a dynamic source model that could produce the observed text and use this model to compress the text which has been observed so far.

Illustration:

- adaptive Huffman algorithm (memoryless source model),
- adaptive arithmetic coding algorithm,
- Lempel-Ziv algorithm and its variants (using the dependencies between the symbols).

2. Adaptive Huffman – Outline

Assume that n letters have been read so far, and that they correspond to K distinct letters.

- Let X_n be the source over $K+1$ symbols formed by the K letters observed so far with probability proportional to their number of occurrences in the text + void symbold which has probability 0.
- Compute the Huffman code tree T_n of this source,
- the $n+1$-th letter is read and encoded
  - with its codeword when it exists,
  - with the $K+1$-th codeword + ascii code of the letter otherwise.

Adaptive Huffman – Coding

The initial Huffman tree has a single leaf corresponding to the void symbol. Each time a new letter x is read

- if already seen
  - print its codeword,
  - update the Huffman tree,
- else
  - print the codeword of the void symbol followed by an unencoded version of x (ascii code for instance),
  - add a leaf to the Huffman tree,
  - update the Huffman tree.

Adaptive Huffman – Decoding

The initial tree is formed by a single leaf corresponding to the void symbol. Until all the encoded text is read, perform a walk in the tree by going down left when '0' is read and going down right when '1' is read until a leaf is reached.

- if the leaf is not the void symbol
  - print the letter,
  - update the tree,
- else
  - print the 8 next bits to write the ascii code of the letter
  - add a leaf to the tree,
  - update the tree.

Adaptive Huffman – preliminary definitions

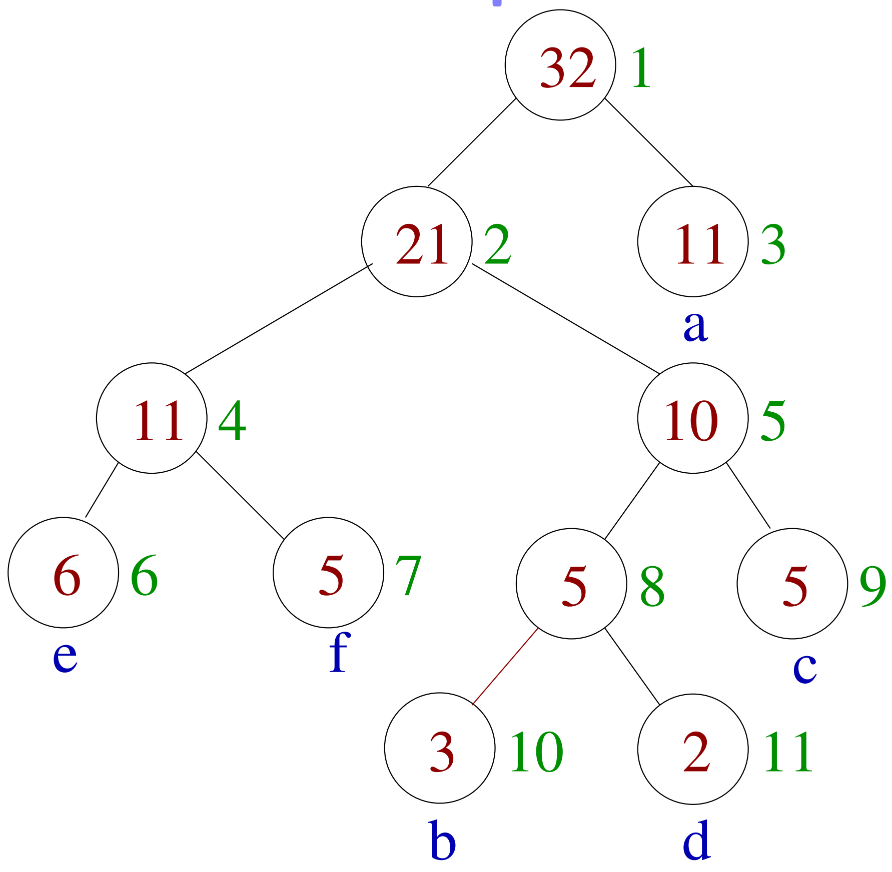
Prefix code of a d.s. X: binary tree with $|X|$ leaves $=$ letters of X.
Definition Consider a prefix code.

- the *leaf weight* is the probability of the corresponding letter
- the *node weight* is the sum of the weights of its children.

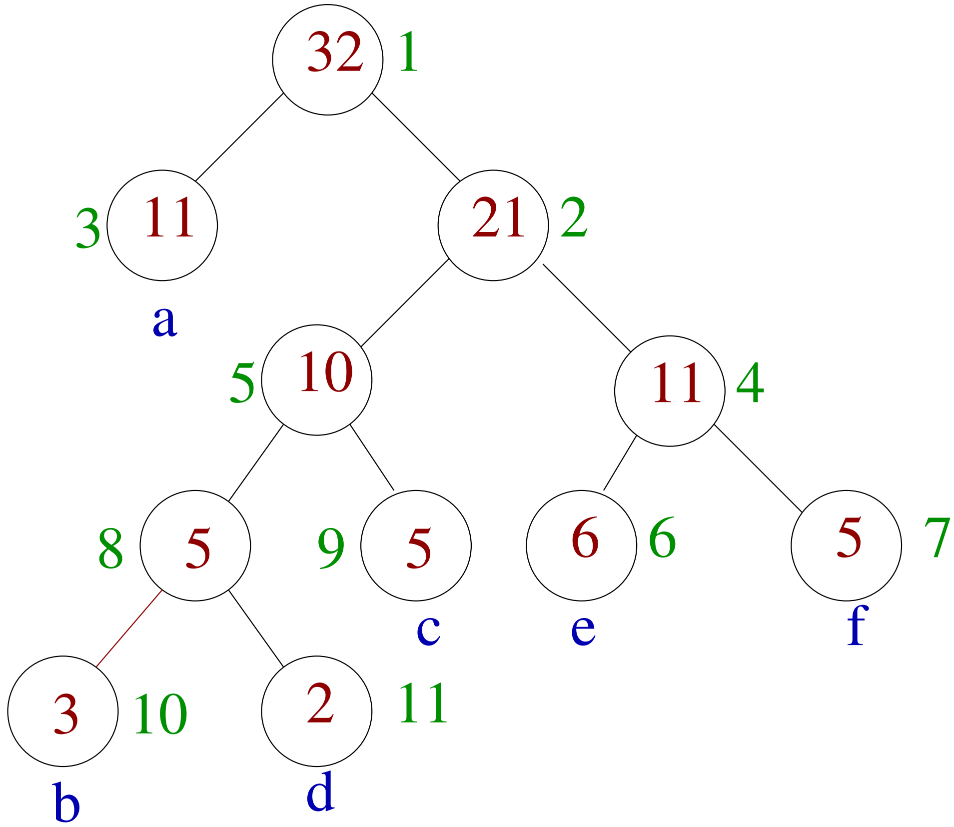
Definition A *Gallager order* $u_1, \ldots, u_{2K-1}$ on the nodes of an irreducible code (of a source of cardinality K) verifies

1. the weights of u_i are decreasing,
2. u_{2i} and u_{2i+1} are brothers for all i such that $1 \leq i < K$.

Example



Example



Adaptive Huffman – Properties

Theorem 1. *[Gallager] Let T be a binary tree corresponding to a prefix code of a source X. T is a Huffman tree of X iff there exists a Gallager order on the nodes of T.*

Proof

T Huffman $\implies$ T admits a Gallager order.

The two codewords of minimum weight are brothers $\Rightarrow$ remove them and keep only their common parent.

The obtained tree is a Huffman tree which admits a Gallager order (induction) $u'_1 \geq \cdots \geq u'_{2K-3}$.

The parent appears somewhere in this sequence. Take its two children and put them at the end. This gives a Gallager order.

$$u'_1 \geq \cdots \geq u'_{2K-3} \geq u_{2K-2} \geq u_{2K-1}$$

Proof

T admits a Gallager order $\implies$ T Huffman.

T has a Gallager order $\implies$ nodes are ordered as

$$u_1 \geq \cdots \geq u_{2K-3} \geq u_{2K-2} \geq u_{2K-1}$$

where u_{2K-2} and u_{2K-1} are brothers, leaves and are of minimum weight.

Let T' be the tree corresponding to $u_1, \dots, u_{2K-3}$. It has the Gallager order $u_1 \geq \cdots \geq u_{2K-3}$. It is a Huffman tree (induction).

By using Huffman's algorithm, we know that the binary tree corresponding to $u_1, \dots, u_{2K-1}$ is a Huffman tree, since once of its nodes is the merge of u_{2K-2} and of u_{2K-1}.

Update

Proposition 1. *Let X_n be the source corresponding to the n-th step and let T_n be the corresponding Huffman tree.*

Let x be the $n+1$-th letter and let $u_1, \ldots, u_{2K-1}$ be the Gallager order on the nodes of T_n.

If $x \in X_n$ and if all the nodes $u_{i_1}, u_{i_2}, \ldots, u_{i_\ell}$ that are on the path between the root and x are the first ones in the Gallager order with this weight, then T_n is a Huffman tree for X_{n+1}.

Proof Take the same Gallager order.

Adaptive Huffman – Updating the tree

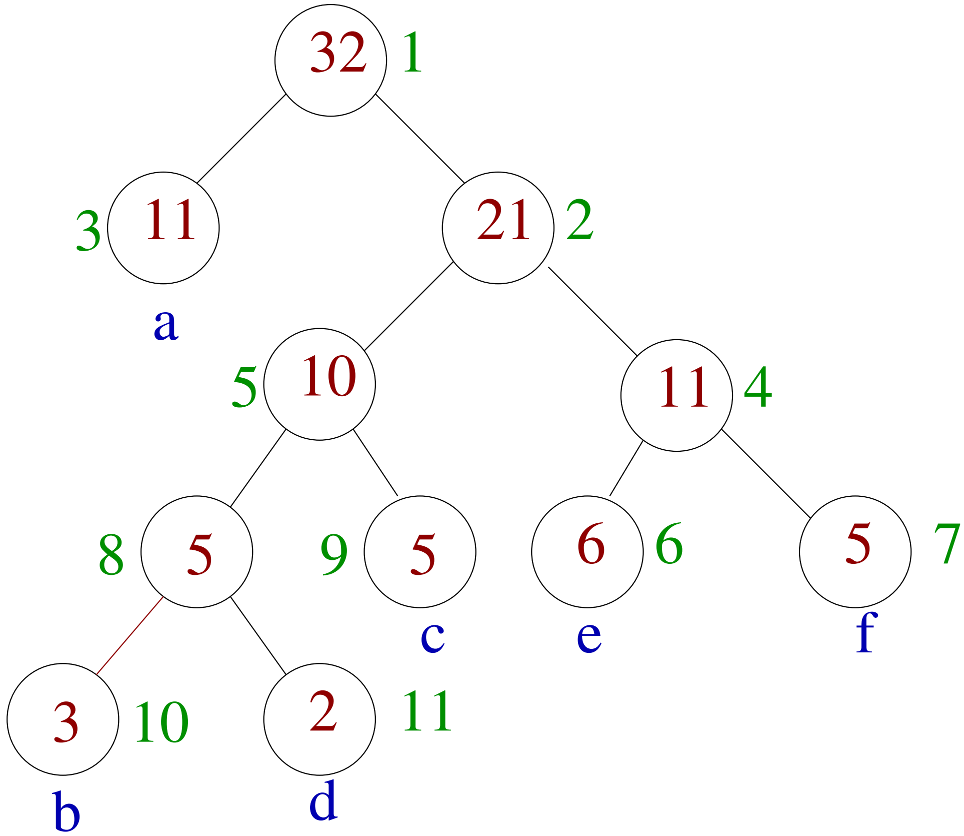
Let T_n be the Huffman tree at Step n and let $u_1, \ldots, u_{2K-1}$ be its corresponding Gallager order.
Assumption : $x \in T_n$ (at node u).

repeat until u is not the root

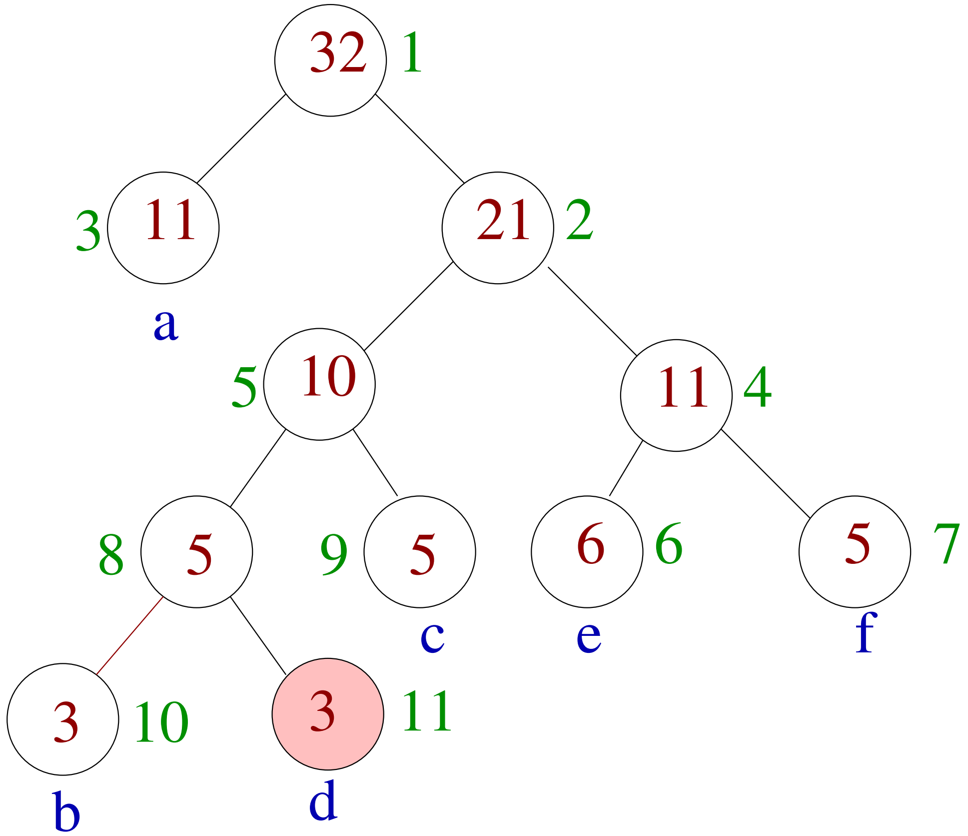
- let $\tilde{u}$ be the first node in the Gallager order of the same weight as u,
- exchange u and $\tilde{u}$,
- exchange u and $\tilde{u}$ in the Gallager order,
- Increment the weight of u *(weight = nb occurrences)*
- $u \leftarrow$ parent of u

This algorithm is due to D. Knuth.

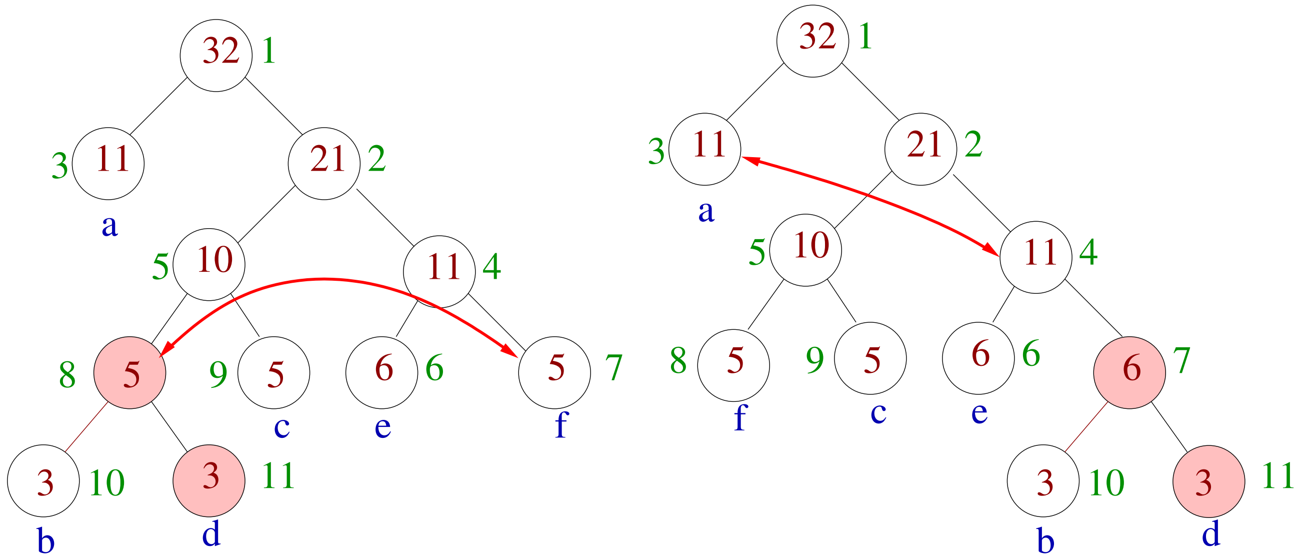
Example



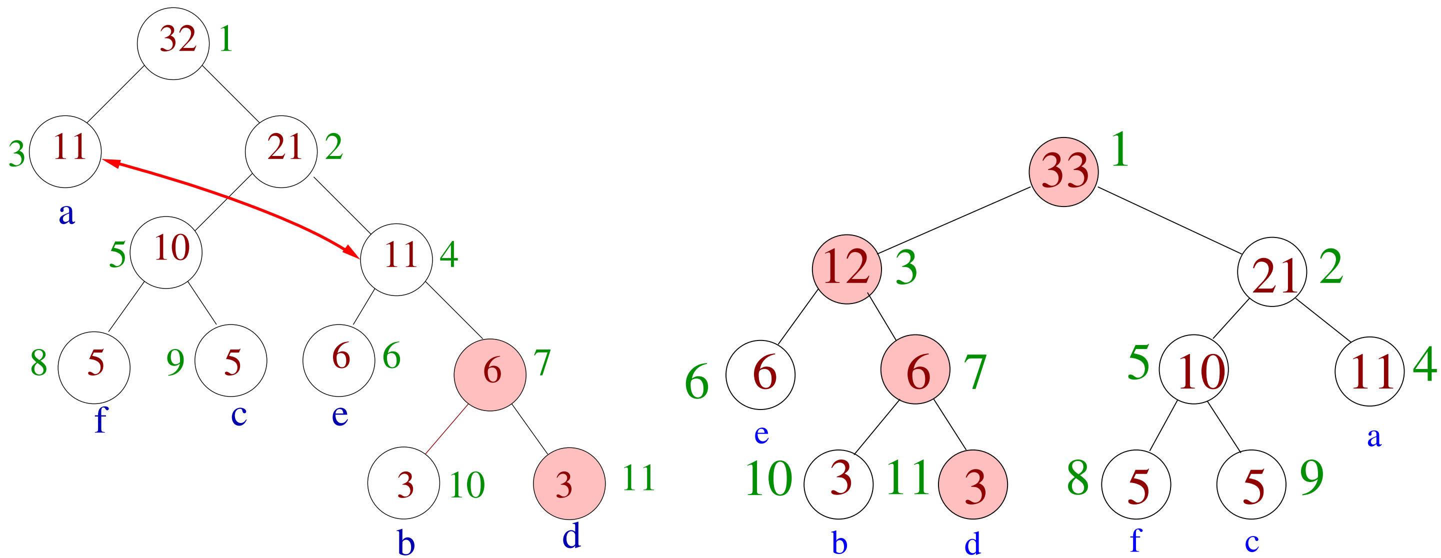
Example



Example

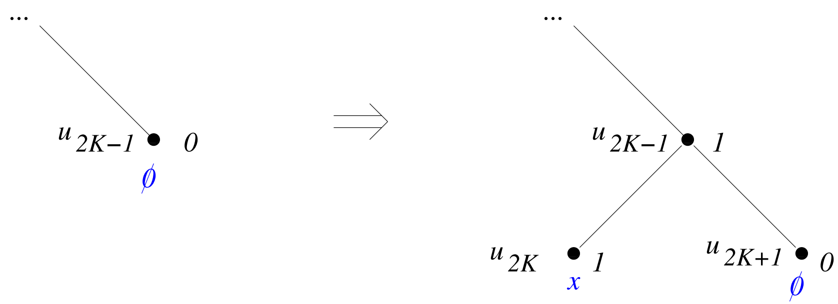

32
1
11
3
a
21
2
10
5
11
4
5
8
5
9
c
6
6
e
5
7
f
3
10
b
3
11
d
32
1
11
3
a
21
2
10
5
11
4
5
8
f
5
9
c
6
6
e
6
7
3
10
b
3
11
d

Example


32 1
3 11 a
21 2
5 10
11 4
8 5 f
9 5 c
6 e 6
6 7
3 10 b
3 11 d
33 1
12 3
21 2
6 6 e
6 7
5 10
11 4 a
10 3 b
11 3 d
8 5 f
5 9 c

Adaptive Huffman – Adding a Leaf

When the current letter x does not belong to the tree, the update uses the void symbol. It is replaced by a tree with two leaves, one for the void symbol and one for x.



The two new nodes are added at the end of the sequence (u_i).

Methods based on dictionaries

Idea : maintaining a dictionary (key,string).

The keys are written on the output, rather than the string.

The hope is that the keys are shorter than the strings.

Lempel-Ziv 78 – Outline

The Lempel-Ziv algorithm (1978) reads a text composed of symbols from an alphabet $\mathcal{A}$. Assume that N symbols have been read and that a dictionary of the words which have been seen has been constructed.

- Read the text by starting from the $(N+1)$-th symbol until a word of length n which is not in the dictionary is found, print the index of the last seen word (it is of length $n-1$) together with the last symbol.
- Add the new word (of length n) to the dictionary and start again at the $(N+n+1)$-th symbol.

Lempel-Ziv 78 – Data Structure

We need an efficient way of representing the dictionary. Useful property: when a word is in the dictionary all its prefixes are also in it.

$\Rightarrow$ the dictionaries that we want to represent are $|\mathcal{A}|$-ary trees. Such a representation gives a simple and efficient implementation for the functions which are needed, namely

- check if a word is in the tree,
- add a new word

Lempel-Ziv 78 – Coding

The dictionary is empty initially. Its size is $K = 1$ (empty word). Repeat the following by starting at the root until this is not possible anymore

- walk on the tree by reading the text letters until this is not possible anymore

  Let $b_1, \ldots, b_n, b_{n+1}$ be the read symbols and let i, $0 \leq i < K$ (K being the size of the dictionary), be the index of the word $(b_1, \ldots, b_n)$ in the dictionary,

- $(b_1, \ldots, b_n, b_{n+1})$ is added to the dictionary with index K,

- print the binary representation of i with $\lceil \log_2 K \rceil$ bits followed by the symbol b_{n+1}.

Lempel-Ziv 78 – Example

1	0	0 1	0 1 1	1 0	0 0	1 1	1 0 0

Dictionary		pair	
indices	word	(index,symbol)	codeword
0	ε		
1	**1**	(0,**1**)	**1**
2	**0**	(0,**0**)	**00**
3	**01**	(2,**1**)	**10 1**
4	**011**	(3,**1**)	**11 1**
5	**10**	(1,**0**)	**001 0**
6	**00**	(2,**0**)	**010 0**
7	**11**	(1,**1**)	**001 1**
8	**100**	(5,**0**)	**101 0**

Lempel-Ziv 78 – Decoding

The dictionary is now a table which contains only the empty word $M_0 = \emptyset$ and $K = 1$. Repeat until the whole encoded text is read

- read the $\lceil \log_2 K \rceil$ first bits of the encoded text to obtain index i. Let M_i be the word of index i in the dictionary
- read the next symbol b,
- add a K-th entry to the table $M_K \leftarrow M_i \parallel b$,
- print M_K.

The Welsh variant

- Initially, all words of length 1 are in the dictionary.
- Instead of printing the pair (i, b) print only i.
- Add (i, b) to the dictionary.
- Start reading again from symbol b.

$\Rightarrow$ slightly more efficient.

used in the unix `compress` command, or for GIF87.

In practice, English text is compressed by a factor of 2.

Lempel-Ziv-Welsh – Example

1 0 0 1 0 1 1 1 0 0 0 1 1 1 0 0 1 0 1 . . .

Dictionary		Word		
indices	words	word	index	codeword
0	**0**			
1	**1**			
2	**10**	**1**	1	**1**
3	**00**	**0**	0	**00**
4	**01**	**0**	0	**00**
5	**101**	**10**	2	**010**
6	**11**	**1**	1	**001**
7	**110**	**11**	6	**110**
8	**000**	**00**	3	**011**
9	**011**	**01**	4	**0100**
10	**1100**	**110**	7	**0111**
11	**010**	**01**	5	**0101**

Lempel-Ziv-Welsh, encoding

1. Read the text until finding m followed by a letter a such that m is in the dictionary, but not $m||a$,
2. print the index of m in the dictionary,
3. add $m||a$ to the dictionary,
4. Continue by starting from letter a.

Lempel-Ziv-Welsh, decoding

1. read index i,

2. print the word m of index i,

3. add at the end of the dictionary $m||a$ where a is the first letter of the following word (a will be known the next time a word is formed).

Lempel-Ziv 77

This variant appeared before the previous one. More difficult to implement.

Assume that N bits have been read.

Read by starting from the $N+1$-th bit the longest word (of length n) which is in the previously read text (=dictionary) and print (i, n, b), where b is the next bit.

In practice, implemented by a sliding window of fixed size, the dictionary is the set of words in this sliding window.

Used in `gzip`, `zip`.

Example

[0]	0	1	0	1	1	1	0	0	0	1	1	1	0	0	1
(0,0,0)															

Example

0	0 1	0	1	1	1	0	0	0	1	1	1	0	0	1
(0,0,0)	(1,1,1)													

Example

0	0 1	0 1 1	1	0	0	0	1	1	1	0	0	1
(0,0,0)	(1,1,1)	(2,2,1)										

Example

0	0 1	0 1 1	1 0 0	0	1	1	1	0	0	1
(0,0,0)	(1,1,1)	(2,2,1)	(3,2,0)							

Example

0	0 1	0 1 1	1 0 0	0 1 1 1 0 0 1
(0,0,0)	(1,1,1)	(2,2,1)	(3,2,0)	(4,6,1)

Stationnary Source

Definition A source is *stationary* if its behavior does not change with a time shift. For all nonnegative integers n and j, and for all $(x_1, \ldots, x_n) \in \mathcal{X}^n$

$$p_{X_1 \ldots X_n}(x_1, \ldots, x_n) = p_{X_{1+j} \ldots X_{n+j}}(x_1, \ldots, x_n)$$

Theorem 2. *For all stationary sources the following limits exist and are equal*

$$H(\mathcal{X}) = \lim_{n \to \infty} \frac{1}{n} H(X_1, \ldots, X_n) = \lim_{n \to \infty} H(X_n \mid X_1, \ldots, X_{n-1}).$$

the quantity $H(\mathcal{X})$ is called the entropy per symbol.

The fundamental property of Lempel-Ziv

Theorem 3. *For all stationary and ergodic sources $\mathcal{X}$ the compression rate goes to $H(\mathcal{X})$ with prob. 1 when the text size goes to ∞.*