Abstract—We propose here a new construction of spatially
coupled quantum LDPC codes using a small amount of entangled
qubit pairs shared between the encoder and the decoder which
improves quite significantly all other constructions of quantum
LDPC codes or turbo-codes with the same rate.

I. INTRODUCTION

Quantum codes suitable for iterative decoding. Turbo-codes
[1] and LDPC codes [2] and their variants are one of the
most satisfying answers to the problem of devising codes
promised by Shannon’s theorem. They display outstanding
performances for a large class of error models with a decoding
algorithm of reasonable complexity. Generalizing these codes
to the quantum setting seems a promising way to efficiently
approach the quantum capacity, and quantum generalizations
of LDPC codes have indeed been proposed in [3].

However, it has turned out that the design of high per-
formance quantum LDPC codes is much more complicated
than in the classical setting. In particular, most constructions
suggested in the literature [4], [3], [5], [6], [7], [8], [9], [10],
[11] suffer from having either a bounded minimum distance or
a vanishing rate. There are only a few exceptions, namely [12],
[13], [14], [15]. However in all these constructions, unlike in
the classical setting, there are issues with the decoder: 4-cycles
in their Tanner graph if decoding is performed over $\mathbb{F}_4$, code
degeneracy which impairs the decoder [16].

On the other hand, generalizing turbo-codes to the quantum
setting has first been achieved in [17]. However this construc-
tion had rather poor performance under iterative decoding.
In [18] it was shown that it was possible to come up with
quantum turbo-codes with good performance under iterative
decoding. However, the families of codes constructed in this
article have bounded minimum distance and the performance
of these codes degrades for large blocklength. It was even
proved there that it is not possible to obtain quantum serial
turbo-codes with unbounded minimum distance and with an
iterative decoding algorithm which converges. This is due to
the fact that it can be proved that quantum convolutional
encoders which are at the same time non-catastrophic and
recursive do not exist [18].

Spatially coupled quantum LDPC codes. Spatially coupled
LDPC have been introduced in [19] (they were named ter-
minated convolutional LDPC codes there). They might be
viewed in the following way, take several several instances
of a certain LDPC code family, arrange them in a row and
then mix the edges of the codes randomly among neighboring
layers. Moreover fix the bits of the first and last layers to zero.
It has soon been found out that iterative decoding behaves
much better for this code than for the original LDPC code.
A breakthrough occurred when it was proved that for the
binary erasure channel, the noise threshold (that is the maximal
probability of erasure which can be sustained by a code
of infinite length) under iterative decoding of the spatially
coupled ensemble coincides with the noise threshold under
Maximum A Priori (MAP) decoding of the underlying LDPC
code (which consists in taking the optimal decision for each
bit) [20]. This has some dramatic consequences.

• The MAP threshold can be significantly better than the
iterative decoding threshold and this especially when the
degrees of the LDPC code get large.

• The MAP threshold can be already quite close to capacity
for small regular degrees (for instance for a (4,8) regular
LDPC code) and converges to capacity quickly as the
degrees increase. This allows to use for instance regular
degrees in the construction and we do not need to
choose well optimized LDPC code ensembles with a large
amount of degree 2 nodes which give codes of minimum
distance at most logarithmic in the code length [21],
[22]. LDPC codes with linear minimum distance in the
blocklength can be chosen in the spatially coupled con-
struction. This simplifies considerably the design of long
codes with very low error floor and excellent performance
under iterative decoding.

• This nice behavior does not only hold for the erasure
channel, it actually holds for all binary input memo-
ryless output-symmetric channels (BIMS) [23] and this
universally: the authors construct there a single spatially
coupled ensemble which attains a desired probability of
error after decoding at a certain desired gap from capacity
for all BIMS channels.

All these nice features of classical spatially coupled LDPC
codes suggest to study whether they have a quantum analogue.
The fact that spatially coupled LDPC codes may afford to have
large degrees and still perform well under iterative decoding
would be quite interesting in the quantum setting, since by the
very nature of the quantum construction of stabilizer codes the
rows of the parity-check matrix of the quantum code have to
belong to the code which is decoded by the iterative decoder.
This implies that we should have rather large row weights to
avoid severe error-floor phenomena and/or oscillatory behavior
of iterative decoding which degrades significantly its perform-
ance [16]. A first step in this direction was achieved in [24]
where a certain family of quantum spatially coupled LDPC
codes was suggested. They showed a family of (quantum)
rate $\frac{1}{2}$ codes which correct for a length of 181,000 qubits a depolarizing error of 0.03 for a probability of error after decoding of about $10^{-4}$. This improves the underlying LDPC code construction since codes of this class which are 2 times longer only have this error probability after decoding for a depolarizing error probability of 0.026. This should be put in perspective with the hashing bound capacity of the depolarizing channel (which is a lower bound for the true capacity of this channel) which says that there are quantum codes of rate $\frac{1}{2}$ which can operate successfully up to a depolarizing error probability of 0.075. The authors stayed in the classical stabilizer formalism and this complicates matters significantly, since there is no way with this construction alone to have a satisfactory quantum analogue of bits fixed to zero.

We choose here another route which assumes some additional resource consisting of shared entangled qubits between the transmitter and the receiver and which are noiseless on the receiver side [25]. This is called entanglement-assisted quantum error correction. In particular, the orthogonality constraints are less stringent than in the stabilizer formalism. It can also be viewed as stabilizer codes where some qubits (namely the halves of the maximally entangled qubits which are on the receiver side) are noise free. This is exactly what is needed to have an equivalent in the quantum world of information bits set to zero. Entanglement is used here in order to have qubits participating to the quantum code without having to sending them and therefore allowing them to be error free.

Our construction can now be described as follows. We start by giving a spatially coupled version of a construction of quantum LDPC codes suggested by [6] based on a couple of orthogonal (classical) LDPC codes obtained from low density generator matrix (LDGM) codes. This gives a stabilizer code of rate $\frac{1}{2}$ and a few first layers and a few last layers of the spatially coupled construction are error-free because these outermost layers are formed by the qubits of the receiver side which are not sent and are therefore noiseless. We use therefore only a very moderate amount of this resource in our construction. Despite this fact, we obtain a tremendous performance improvement over other families of codes of rate $\frac{1}{4}$. In our case, the probability of error after decoding drops down sharply after $p = 0.102$. This is not a real threshold since these codes are LDGM codes and have therefore a constant minimum distance, but no deterioration of the “threshold” could be observed experimentally when the length increases and this even for the quite large lengths (up to 76800) which were considered. This should be compared to the hashing bound capacity for codes of rate $\frac{1}{2}$ which corresponds to a depolarizing noise of $p_c \approx 0.1269$. Moreover our construction belongs to the family of CSS codes [26], [27] and from the way our codes are decoded, we decode namely two binary codes of rate $\frac{1}{2}$ affected by a binary symmetric channel of crossover probability $p' = \frac{2p}{3}$, we can not expect that this kind of strategy would be able to operate successfully for depolarizing noise above $p_0 \approx 0.1087$. Finally, we also demonstrate that our scheme is able to tolerate some moderate error noise on the qubits of the outermost layers without suffering severe performance loss. This is in strong contrast with catalytic error correction [28] or other coding strategies making use entangled qubits such as quantum polar coding [29] which can not tolerate any amount of noise on these qubits.

II. ENTANGLEMENT ASSISTED STABILIZER CODES

We review in this section the entanglement assisted stabilizer code formalism [25]. The style of presentation we adopt here is to suit a readership familiar with classical codes but not with quantum information theory. Let us recall that the trace hermitian inner product between $E = (E_i)_{1 \leq i \leq n}$ and $F = (F_i)_{1 \leq i \leq n}$ in $\mathbb{F}_4^n$ is given by:

$$E \ast F \triangleq \sum \text{Tr} E_i F_i,$$

where $\bar{u} = u^2$ and $\text{Tr} u = u + \bar{u}$ for $u$ in $\mathbb{F}_4$.

An entanglement assisted stabilizer code of type $[(n, k; c)]$ using $c$ entangled qubit pairs shared between the transmitter and the receiver is defined by

1. The rows of $H$ are independent over $\mathbb{F}_2$ and orthogonal with respect to the trace hermitian inner product,
2. If we let $H'$ be the submatrix of $H$ formed by erasing the aforementioned $c$ columns in $H$ and if we denote by $H_i'$ the $i$-th row of $H'$, then the matrix $M' \triangleq (H_i' \ast H_j')_{1 \leq i, j \leq n+c-k}$ has rank $c$.

$H$ is called a parity-check matrix for the stabilizer code. When $c = 0$, the code is a called a stabilizer code.

Remarks

1) This definition is not completely standard, generally such a code is specified by the subgroup generated by the rows $H_i'$ which satisfy Condition (ii), but it is better suited to our construction (we start with a construction of stabilizer codes) and specify the $c$ columns later on.

2) An equivalent way of expressing Condition (ii) which can be proved by symplectic geometry arguments (see for instance [30, Lemma 2]) is the fact that the group generated by the $H_i'$s can be generated by $2c + (n-k-c)$ elements of $\mathbb{F}_4^n$ $u_1, u_2, \ldots, u_c, v_1, \ldots, v_c, w_1, \ldots, w_{n-k-c}$ which are all orthogonal to each other (meaning that $u_i \ast u_j = 0$ for instance) with the exception of $u_i \ast v_i \neq 1$ for all $i$ in $\{1, \ldots, c\}$.

3) Such a choice of matrix amounts to a particular syndrome measurement.

Error model and information available for decoding. We consider Pauli channels and in this case the errors which occur are elements of $\mathbb{F}_4^{n+c}$. A very important channel error of this kind is the depolarizing channel model. It is given by the following definition.

Definition 2 (Depolarizing channel): The depolarizing channel on $n$ qubits of error probability $p$ picks up an
element $E \in \mathbb{F}_4^n$ by choosing randomly the coordinates $E_i$ of $E$ independently of each other according to $P(E_i = 0) = 1 - p, P(E_i = 1) = P(E_i = \omega) = P(E_i = \omega^2) = \frac{p}{3}$.

The entanglement assisted setting consists in encoding $k$ information qubits together with $n - k - c$ ancilla qubits and $c$ maximally entangled pairs of qubits (making up for a total of $n + c$ qubits), the transmitter and the receiver holding each one qubit of these pairs, and encoding takes place only on the transmitter side (the $c$ qubits held by the receiver do not participate in the encoding). The $n$ qubits held by the transmitter are sent through a noisy channel whereas the $c$ qubits on the receiver side are noiseless. We assume that the noisy channel is a Pauli channel meaning that the errors change the quantum state belonging to an entanglement assisted stabilizer code.

For a classical linear code the minimum distance of the code is equal to the minimum weight of a nonzero error of zero orthogonality conditions there is another fundamental differ-
ence between the rows of $H_1$ and $H_\omega$: the rows of $H_1$ have to be orthogonal to the rows of $H_\omega$, or what is the same, if we let $c_1$ be the code with parity-check matrix $H_1$ and $c_\omega$ be the code with parity-check matrix $H_\omega$, then we should have $c_\omega^\perp \subset c_1$.

Condition (ii) has also a simple expression in terms of $H_1$ and $H_\omega$. Let $I_1$ be the set of indices of the rows of $H_1$ which belong to $H_1$ and $I_\omega$ be the set of indices of the rows of $H_\omega$ which belong to $H_\omega$. If we let $H'_1$ and $H'_\omega$ be the submatrices of $H_1$ and $H_\omega$ formed by the columns which are not in $I$ and if we denote by $H'_1(i)$ the row of index $i$ in $H_1$ and $H'_\omega(i)$ the row of index $i$ in $H_\omega$, then Condition (ii) is equivalent to the fact that the matrix $M' = \langle H'_1(i), H'_\omega(j) \rangle_{i \in I_1, j \in I_\omega}$ has rank $c$ where $\langle x, y \rangle = \sum x_i y_i$ is the standard inner product between $x = (x_i)_i$ and $y = (y_i)_i$ which are vectors in $\mathbb{F}_2^n$. There is a simple way to check Condition (ii) which is given by the following proposition.

**Proposition 1:** Let $H'_1$ and $H'_\omega$ be the submatrices of $H_1$ formed by the $c$ columns which belong to $I$. A necessary and sufficient for Condition (ii) to hold is that $H'_1$ and $H'_\omega$ are both of rank $c$.

Moreover a suboptimal decoding of the quantum code can be performed by decoding $c_1$ and $c_\omega$, since the error $E$ can be written in a unique way as $E = E_1 + \omega E_\omega$ with $E_1$ and $E_\omega$ in $\mathbb{F}_2^n$, and by noticing that the only entries of $s(E_1)$ which are non zero correspond to the rows of $H_1$ which belong to $H_\omega$. Take only these rows for the syndrome $s_1$ of $E_1$, and we obtain

$$s_1(E_1) = \langle \omega H_\omega(i) * E_1 \rangle_{i \in I_\omega} = \langle H_\omega(i), E_1 \rangle_{i \in I_\omega}.$$  

Similarly we have

$$s_\omega(E_\omega) = \langle H_1(i) * \omega E_\omega \rangle_{i \in I_1} = \langle H_1(i), E_\omega \rangle_{i \in I_1}.$$  

A depolarizing channel model of probability of error $p$ translates into a binary symmetric channel error model of probability of error $\frac{p}{2}$ for $E_1$ and $E_\omega$.

**III. OUR CONSTRUCTION**

Basically the idea of our construction is to exploit an idea due to [31], [6] which begins with the observation that the dual of a low density generator matrix code is a low density parity-check code. This can be exploited to yield a CSS code at the expense of a constant minimum distance. However, if the weights of the rows of the low density generator matrix are chosen to be large enough, this is not necessarily a problem.
Our observation is now just that the dual of a spatially coupled low density generator matrix code is (essentially) a spatially coupled low density parity-check code.

A. Overview of the construction of [6]

Let us present the construction of a low density generator matrix code given in [6]. A Tanner graph used for decoding this code is depicted in Figure 1. The length of the code is \(n\) and half of the variable nodes are of degree 1 (they correspond to \(u_1\)) in the Tanner graph, while the other half (which corresponds to \(u_2\)) is of some constant degree \(d\). There is a first set of check nodes, corresponding to \(c_1\), all of degree \(d+1\) which form a bipartite subgraph of degree \(d\) with the variable nodes of degree \(d\). There is a matching of these check nodes with \(n/2\) state nodes (corresponding to \(r_1\) in the figure) and there are two matchings between the \(n/2\) check nodes of the second level (corresponding to \(c_2\)) and the variable nodes of \(u_1\) and \(r_1\) respectively. Then there is a last matching between the \(n/2\) check nodes of \(c_2\) and the \(n/2\) state nodes of \(r_2\). Finally the subgraph of the Tanner graph formed by the state nodes of \(r_2\) and the last level of check nodes corresponding to \(c_3\) has three type of nodes:

- \(s_1\) check nodes of degree 1 (this implies that the associated state node of \(r_2\) should be equal to 0,
- \(s_2\) check nodes of some constant degree \(x\),
- all the state nodes of \(r_2\) are of some constant degree \(y\) in the subgraph.

The purpose of the check nodes of degree 1 of the last level is to ensure that iterative decoding does not get stuck at the initial stage (it corresponds to some kind of doping of the last level of state nodes corresponding to \(r_2\)). If we denote this code by \(C_1\), then it is proved in [6] that there exists another code \(C_\omega\) with the same Tanner graph structure which satisfies \(C_\omega \subset C_1\). This is basically a consequence of the fact that \(C_1\) is an LDGM code whose dual is an LDPC code and of the particular form of the low density generator matrix of \(C_1\). The LDGM structure implies however that these codes have constant minimum distance, the point is here that the weight of the rows of the low density generator matrix can be chosen to be large enough so that this does not deteriorate iterative decoding performances.

B. The associated spatially coupled construction

There is a spatially coupled version of these codes which can be described as follows. Take \(L + 2\delta\) such codes, number them with \(0, 1, \ldots, L + 2\delta - 1\) and consider the associated Tanner graphs (and say that the \(t\)-th Tanner graph corresponds to level \(t\)). For each \(i \in \{-\delta, \ldots, \delta\}\), and \(t \in \{0, 1, \ldots, L + 2\delta - 1\}\), we swap a fraction \(\frac{1}{L+2\delta}\) of the edges which link a variable node with a check node at level \(t\) with an edge which links a variable node to a check node at level \(t + i \mod (L + 2\delta)\) such that the variable node at level \(t\) is now adjacent to the check node at level \(t + i \mod (L + 2\delta)\) and vice versa. We do not swap the edges which link the state nodes to variable nodes on the other hand. The variable nodes and the check nodes which are of degree 1 have their corresponding edge which stays at the same level. The variable node positions at positions \(\{0, \ldots, 2\delta - 1\}\) will correspond to the set \(I\) of the entanglement assisted code (and we also check that Condition of Proposition 1 holds). In other words, these variable nodes are set to zero when we perform iterative decoding. We used the window decoder described in [32] to reduce its complexity.

We have performed the computations for the following parameters

<table>
<thead>
<tr>
<th>type</th>
<th>(L)</th>
<th>(\delta)</th>
<th>(d)</th>
<th>(x)</th>
<th>(y)</th>
<th>(s_1)</th>
<th>(s_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>type 1</td>
<td>20</td>
<td>2</td>
<td>20</td>
<td>9</td>
<td>3</td>
<td>\frac{25}{25}</td>
<td>\frac{20}{20}</td>
</tr>
<tr>
<td>type 2</td>
<td>20</td>
<td>2</td>
<td>25</td>
<td>9</td>
<td>3</td>
<td>\frac{24}{24}</td>
<td>\frac{25}{25}</td>
</tr>
</tbody>
</table>

IV. Results

The \(x\)-axis and the \(y\)-axis of the following curves give respectively the depolarizing error probability and the probability of error after decoding.

As shown in Fig. 2, the spatially coupled codes clearly outperform significantly the previously known LDPC code constructions as well as the quantum turbo-code constructions. The type 1 codes (SC1A of length \(N = 19200\), SC 1B, \(N = 38400\) and SC 1C \(N = 76800\)) are slightly better than the type 2 codes (SC2A of length \(N = 19200\), SC2B, SC2C and SC2D) of depolarizing before decoding. The \(2\delta\) levels which are fixed to 0 (they belong to \(I\)) as shown by Fig. 4 which shows various depolarizing noise levels \(p\) on these positions for a type 1 code of length 38400.

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Comparison with other codes of rate $\frac{1}{2}$.

- SC1C: spatially coupled code of type 1 and length 76800
- Garcia-Liu: [6]
- Codes of length $n = 785700$, $n = 242500$ from [15]
- turbo codes: [33]
- MacKay: [3]
- Lou-Garcia: [31]
- Camara-Ollivier-Tillich: [5]

REFERENCES