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Edge isoperimetric inequalities for product graphs ☆

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Abstract

It is well known that there is a simple equivalence between isoperimetric inequalities and certain analytic inequalities in Riemannian manifolds (see Rothaus, J. Funct. Anal. 64 (1985) 296–313). We generalize these results to graphs, and use them to derive isoperimetric inequalities for product graphs. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Our main concern in this article is to get useful tools to obtain a 'good' *edge-isoperimetric inequality* of a graph G with vertex set V and edge set E, namely a function $\mathscr{F}$ such that for every nonempty subset of vertices Ω

$$|\partial\Omega| \geqslant \mathscr{F}(|\Omega|)$$

where $\partial\Omega$ (the *boundary* of Ω) denotes the set of edges of the graph connecting vertices of Ω with vertices of its complement $\bar{\Omega} = V\backslash\Omega$. We aim at obtaining isoperimetric inequalities which can be sharp for certain subsets of vertices. These subsets are in this case *isoperimetric sets*, i.e. sets of given size which have the smallest edge-boundary.

Sharp isoperimetric inequalities are known for only a few classes of graphs. Basically, the best isoperimetric function (i.e. $\mathscr{F}(k) = \min_{|\Omega|=k} |\partial\Omega|$) is known for some trivial examples like

The complete graph K_n $|\partial\Omega| = (n - |\Omega|)|\Omega|$.

The cycle C_n $|\partial\Omega| \geqslant 2$ (for $|\Omega| \neq n$).

The infinite d-regular tree $|\partial\Omega| \geqslant (d-2)|\Omega| + 2$, (the isoperimetric sets are balls of some radius l with possibly additional vertices on the sphere with the same centre and

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radius $l+1$, and it is straightforward to check in this case that for these sets we have $|\partial\Omega| = (d-2)|\Omega| + 2$).

For these examples getting the best isoperimetric function (or the isoperimetric sets) follows from straightforward combinatorial arguments. Much more is involved in order to get the best isoperimetric function for more complicated graphs. Basically, all the other (nontrivial!) examples where the best isoperimetric function is known are some families of cartesian products of graphs (an exception being the Johnson graph [27]). The cartesian product of two graphs G and H is a graph whose vertex set is the product $V(G) \times V(H)$ and two vertices (g,h) and (g',h') are adjacent if and only if $h=h'$ and gg' is an edge of G, or if $g=g'$ and hh' is an edge of H. Here are a few examples where good isoperimetric inequalities are known:

The n-cube Q_n $|\partial\Omega| \geqslant \log_2(|V|/|\Omega|)|\Omega|$.
The grid $[k]^n$, $|\partial\Omega| \geqslant \min\{d|\Omega|^{(d-1)/d}k^{n/d-1} | d=1,2,\ldots,n\}$, when $|\Omega| \leqslant k^n/2$.
The lattice $\mathbb{Z}^n$ $|\partial\Omega| \geqslant 2n|\Omega|^{(n-1)/n}$.

The result on the n-cube follows from the work of Bernstein, Harper, Hart and Lindsey [5,25,26,29], (we refer to these articles for a much sharper statement, namely the determination of the isoperimetric sets; indeed the inequality given above is only tight when Ω is a subcube, and is a good approximation for the other sets which are not of size 2^l). The result on the lattice can be found in [1]; again the inequality above is only an approximation of the best isoperimetric inequality which is tight when Ω is an n-dimensional cube. The result on the grid (i.e. the cartesian product of n paths of length k) follows from the work of Bollobas and Leader ([11]) (see also [10] for a related result on a torus, and [1,7]). They make use of the idea of compressing a set Ω to obtain a new set Ω' with the same cardinality and $|\partial\Omega'| \leqslant |\partial\Omega|$. For more information about isoperimetric problems we refer to the surveys [6,8] which are devoted to edge-isoperimetric problems.

Besides these combinatorial methods, there are eigenvalue techniques which give in general a good isoperimetric inequality. For instance (see [2]), if we denote by λ the second smallest eigenvalue of the Laplacian of G (which is a square matrix $(\triangle_{i,j})_{i,j\in V}$ such that $\triangle_{ii}$ is the degree of vertex i, $\triangle_{ij} = -1$ for $i \neq j$ iff there is an edge between i and j, and $\triangle_{ij} = 0$ otherwise) then we have

$$|\partial\Omega| \geqslant \lambda \frac{|\Omega||\bar{\Omega}|}{|V|}.$$

For a random d-regular graph it follows from the work of Friedman [20] and Bollobas [12] that this is a good approximation of the best isoperimetric function. Furthermore, this bound is quite practical, in the sense that λ can be estimated within an arbitrary precision ε in polynomial time. Nevertheless for a given family of graphs the above inequality can be far from the best bound (for example for the n-cube $\lambda = 2$, the bound is tight only if $|\Omega| = 2^{n-1}$ and is quite weak in comparison with the bound given previously for small sets Ω).

To enlarge the set of tools for obtaining isoperimetric inequalities for a graph, we are going to mimick the tools which were quite successful to study isoperimetric problems in $\mathbb{R}^n$ (or even more generally for Riemannian manifolds). Perhaps the most fundamental tool in this setting is the equivalence observed by Federer and Fleming (see [19]) between certain Sobolev inequalities and isoperimetric inequalities. This has been generalized for Riemannian manifolds and for a much wider range of inequalities in [33] for example. We aim here at generalizing these results to graphs, and deduce from this equivalence isoperimetric inequalities for product graphs. It should be noted that this kind of approach has been used before (for example [13,16,17,35]) in order to study Markov chains or isoperimetric numbers of graphs. For instance in [35] the author generalizes the equivalence of Sobolev inequalities with a certain kind of isoperimetric inequality for Riemannian manifolds to Markov chains, in order to get information on their rate of convergence.

It might be insightful to review some of the results obtained by Federer and Fleming. For example, they were able to prove that if M is some subdomain of $\mathbb{R}^n$, then if we denote by:

- V the (n-dimensional) volume and A the area (i.e. the $(n-1)$-dimensional volume) in R^n,
- $i(M)$ the isoperimetric constant of M which is defined as

$$\inf_{\Omega \subset M, V(\Omega) \leqslant V(M)/2} \frac{A(\partial\Omega)}{V(\Omega)},$$

then

$$i(M) = \inf_{f \in C_c^\infty(M)} \frac{\int_M |\nabla f|}{\inf_a \int_M |f - a|},$$

where $C_c^\infty(M)$ denotes the set of compactly supported C^∞ functions on M.

It has been noted several times (basically already in [35] and then in [13,15,16,22]) that this can be generalized to a graph G with vertex set V and edge set E with

$$i(G) = \inf_f \frac{\sum_{xy \in E} |f(x) - f(y)|}{\inf_a \sum_{x \in V} |f(x) - a|},$$

where

$$i(G) = \inf_{\Omega \subset V, |\Omega| \leqslant |V|/2} \frac{|\partial\Omega|}{|\Omega|}.$$

We will see in what follows, that this phenomenon can be generalized to other ways of defining the isoperimetric number of a graph and leads to interesting results about isoperimetric inequalities of graph products.

Moreover we will show that by using this equivalence between analytic inequalities and isoperimetric inequalities, and by finding analytic inequalities which display a 'nice' behaviour with respect to the cartesian product, isoperimetric inequalities can be derived for a cartesian product of graphs for which we already have a certain kind of isoperimetric inequality. As an application of our results we will rederive the isoperimetric inequalities obtained for the n-cube and the lattice $\mathbb{Z}^n$, respectively, and show that they simply follow, respectively, from $|\partial\Omega| \geqslant |\Omega| \log_2(|V|/|\Omega|)$ which holds for the 2-point graph, and $|\partial\Omega| \geqslant 2$ which holds on $\mathbb{Z}$.

One of our results obtained by this approach, and which seems to be the most useful in general for product graphs is the following.

Theorem 1. *For a graph G with vertex set V we denote by $l(G)$ the minimum of $|\partial\Omega|/|\Omega| \ln(|V|/|\Omega|)$ over all nonempty subsets of vertices Ω of G. Then for any cartesian power G^n we have*

$$l(G) = l(G^n).$$

Moreover, let Ω be a subset of vertices of G for which $|\partial\Omega|/|\Omega| \ln(|V|/|\Omega|)$ is minimum; then Ω is clearly an isoperimetric set for G, and all the sets $\Omega^{(k,n)} = \Omega^k \times V^{n-k}$ for $0 \leqslant k \leqslant n$ are isoperimetric sets for G^n.

We use here the convention that $\Omega^k \times V^{n-k}$ is equal to Ω^n when $k = n$, and to V^n when $k = 0$. This theorem is proved in Section 4. We will see in Section 4 that we can obtain with this theorem good isoperimetric inequalities for product graphs, and that we can also find some isoperimetric sets with it.

Notation. Throughout this paper we will deal with the cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2, and with functions on two variables $f(x_1, x_2)$ defined on the vertex set $V_1 \times V_2$ of the cartesian product (V_i being the vertex set of G_i for $i \in \{1, 2\}$). We denote by $f(\cdot, x_2)$ the function defined on V_1 which associates with x_1 the number $f(x_1, x_2)$. $f(x_1, \cdot)$ is defined on V_2 and associates $f(x_1, x_2)$ with x_2.

From now on we are going to note $\int_E |\nabla f|^p$ the sum $\sum_{xy \in E} |f(x) - f(y)|^p$. We use this slightly unconventional notation for two purposes:

- first, we want with this notation to put the stress on the fact that there is for all the inequalities in this article involving $\sum_{xy \in E} |f(x) - f(y)|^p$ a corresponding inequality in R^n (or more generally in Riemannian geometry) involving $\int_M |\nabla f|^p$.
- second, by viewing a graph G as a metric space $\mathscr{G}$, i.e. as a set of vertices glued together with edges considered as intervals of length 1 (see [21,22]) it is readily checked that if we extend f to a function $\tilde{f}$ defined over the vertices and the edges, then $\int_{\mathscr{G}} |\nabla \tilde{f}|^p \geqslant \sum_{xy \in E} |f(x) - f(y)|^p$ with equality iff $\tilde{f}$ is an edgewise linear function. Since we will be only interested in lower bounds on $\sum_{xy \in E} |f(x) - f(y)|^p$ we can look instead for lower bounds on $\int_{\mathscr{G}} |\nabla \tilde{f}|^p$ involving only the values of $\tilde{f}$ on the vertices, that is f. In a sense, we can always assume in what follows that $\int_E |\nabla f|^p$ coincides with a continuous integration of f on the set of edges of the graph. It appears that by doing so, several tools in Riemannian geometry are more easily generalized (see [22]).

We will use several times for a function f defined on the set of vertices of the graph, its average or expected value, that is $(1/|V|)\sum_{x\in V} f(x)$; this value is denoted by $E(f)$.

2. The equivalence of a certain kind of analytic inequality with an isoperimetric inequality

Assume now that the following inequality holds for every function f defined on the vertex set V of a given graph with edge set E:

$$\int_E |\nabla f| \geqslant C\mathscr{F}(f), \tag{1}$$

where $\mathscr{F}$ is some function of f. We will prove that with mild conditions on $\mathscr{F}$ such an analytic inequality is indeed equivalent to the isoperimetric inequality

$$|\partial\Omega| \geqslant C|\Omega|_{\mathscr{F}}, \tag{2}$$

where the size $|\Omega|_{\mathscr{F}}$ of Ω is measured by $\mathscr{F}(\chi_\Omega)$, χ_Ω is the characteristic function of Ω.

Remark. (1) The example we have seen in the introduction falls in this category: we let $\mathscr{F}(f) = \inf_a \sum_{x\in V} |f(x)-a|$. It is easy to check that $|\Omega|_{\mathscr{F}}$ is, in fact, equal to $\min\{|\Omega|, |\bar{\Omega}|\}$, and that the largest constant C for which (2) holds is equal to the isoperimetric number $i(G)$ of the graph.

(2) It is straightforward to check that

$$\int_E |\nabla \chi_\Omega| = |\partial\Omega|$$

which holds for every subset of vertices. This implies, with the hypotheses on $\mathscr{F}$ we have in mind, that the minimum in (1) is actually attained when f is equal to some characteristic function χ_Ω.

The proof of this equivalence relies on a summation trick which is quite standard in the setting of Riemannian manifolds, and which is called the co-area formula (see [14, Chapter IV, Section 1]). For graphs this trick has been used, for example, to derive Cheeger-like inequalities (see [18] for example)

Lemma 1 (Co-area formula). *Let f be a nonnegative function acting on the vertices of a graph with edge set E, and $\Omega_t = \{x \in V; f(x) > t\}$.*

$$\int_E |\nabla f| = \int_0^\infty |\partial\Omega_t|\,\mathrm{d}t = \int_0^\infty \int_E |\nabla \chi_{\Omega_t}|\,\mathrm{d}t.$$

Proof. There are two ways of proving the co-area formula:

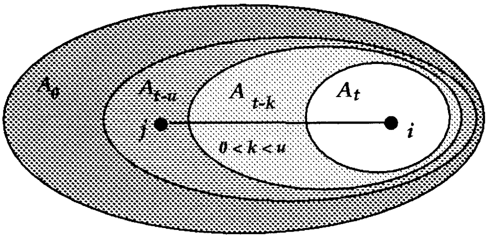

Fig. 1.

- either we use the co-area formula for Riemannian manifolds, and use the fact that $\int_E |\nabla f|$ has a 'continuous' interpretation, as $\int_{\mathscr{G}} |\nabla \tilde{f}|$ where f is extended to an edgewise linear function $\tilde{f}$ over the edges (see [22]), or we use the standard combinatorial way of proving it, which goes as follows. Let $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_N$ be the sequence of all values of f, and A_t the set of vertices i such that $f(i) \geqslant \alpha_t$.

$$\int_E |\nabla f| = \sum_{t=1}^{N} \sum_{f(i)=\alpha_t} \sum_{j\sim i, f(j)<\alpha_t} f(i) - f(j). \tag{3}$$

Assume that $f(i) = \alpha_t$ and $f(j) = \alpha_{t-u}$, for some $u \in \{1, 2, \dots, t\}$. Then $f(i) - f(j) = (\alpha_t - \alpha_{t-1}) + \cdots + (\alpha_{t-u+1} - \alpha_{t-u})$. Substituting into (3), and noting that the edge ij is in all subsets ∂A_k ($k \in \{t-u+1, \dots, t\}$) (see Fig. 1 above)we obtain

$$\int_E |\nabla f| = \sum_{t=1}^{N} |\partial A_t|(\alpha_t - \alpha_{t-1}) = \int_0^{\infty} |\partial \Omega_t| \, \mathrm{d}t.$$

The second part of the lemma follows from the second remark above. □

From now on $\mathscr{F}$ will denote a functional which meets at least one the following properties:

- $\mathscr{F}(f)$ is a *semi-norm*.
- $\mathscr{F}(f)$ can be expressed as

$$\text{(QL)} \quad \mathscr{F}(f) = \sup_{(u,v)\in\mathscr{V}} \left\{ \sum_{x\in V} f^+(x)u(x) + \sum_{x\in V} f^-(x)v(x) \right\},$$

where $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$ and $\mathscr{V}$ is a set of pairs of functions (u, v) acting on the set of vertices of G. We say that $\mathscr{F}$ admits a (QL) (i.e. =quasi-linear) representation. The last expression may seem little artificial and is a generalization of the functionals $\mathscr{F}$ which can be written as $\mathscr{F}(f) = \sup_{u\in U} \sum_{x\in V} u(x)f(x)$. When a functional admits such a representation, it is called the quasi-linearized form of $\mathscr{F}$ and many interesting inequalities involving $\mathscr{F}$ can be derived from this form (see Chapter 1, Section 19 in [4]).

Theorem 2. *Assume that* $\mathscr{F}$ *is either a semi-norm or has a* (QL) *representation. The necessary and sufficient condition that* $\int_E |\nabla f| \geqslant C\mathscr{F}(f)$ for all functions f is that $|\partial\Omega| \geqslant C|\Omega|_{\mathscr{F}}$ for all subsets Ω of the graph.

Proof. That the semi-norm property implies the theorem has been proved in [22]. The proof that the second kind of property on $\mathscr{F}$ implies the theorem is rather similar to the proof where $\mathscr{F}$ is a semi-norm, and we will only prove this second case here, i.e. we assume

$$\mathscr{F}(f) = \sup_{(u,v)\in\mathscr{V}} \left\{ \sum_{x\in V} f^+(x)u(x) + \sum_{x\in V} f^-(x)v(x) \right\}.$$

If $\int_E |\nabla f| \geqslant C\mathscr{F}(f)$ for all functions f then clearly $|\partial\Omega| \geqslant C|\Omega|_{\mathscr{F}}$ for all subsets Ω of the graph (put $f = \chi_\Omega$ and note that $|\partial\Omega| = \int |\nabla\chi_\Omega|$). Let us prove the converse. First, let us notice that

$$\int_E |\nabla f| = \int_E |\nabla f^+| + \int_E |\nabla f^-|.$$

Note that in order to prove this equality, it is sufficient to prove it when the graph has only one edge; this is readily verified. We use the co-area formula with f^+ and f^- to obtain

$$\int_E |\nabla f^+| = \int_0^\infty |\partial\Omega_t^+| \, \mathrm{d}t,$$

$$\int_E |\nabla f^-| = \int_0^\infty |\partial\Omega_t^-| \, \mathrm{d}t.$$

Here $\Omega_t^\pm = \{x \in V \mid f^\pm(x) > t\}$. Therefore for every $(u,v) \in \mathscr{V}$ we have

$$\begin{aligned}
\int_E |\nabla f| &= \int_0^\infty |\partial\Omega_t^+| \, \mathrm{d}t + \int_0^\infty |\partial\Omega_t^-| \, \mathrm{d}t \\
&\geqslant C \left\{ \int_0^\infty |\Omega_t^+|_{\mathscr{F}} \, \mathrm{d}t + \int_0^\infty |\Omega_t^-|_{\mathscr{F}} \, \mathrm{d}t \right\} \\
&\geqslant C \int_0^\infty \left\{ \sum_{x\in\Omega_t^+} u(x) + \sum_{x\in\Omega_t^-} v(x) \right\} \mathrm{d}t \\
&\geqslant C \left\{ \sum_{x\in V} f^+(x)u(x) + \sum_{x\in V} f^-(x)v(x) \right\}.
\end{aligned}$$

Hence,

$$\int_E |\nabla f| \geqslant C\mathscr{F}(f). \qquad \square$$

There are quite a few functionals $\mathscr{F}$ which meet this property and which are strongly related to known definitions of the isoperimetric number of a graph.

Besides $\mathscr{F}(f)=\inf_a \sum_{x\in V} |f(x)-a|$ that we have already seen, let us mention the following examples

Example 1. $\mathscr{F}(f)=\sum_{x\in V} |f(x)-E(f)|$ with $E(f)=(1/|V|)\sum_{x\in V} f(x)$. This functional is clearly a semi-norm, but it can also be noticed that $\mathscr{F}$ admits a (QL) representation, $\mathscr{F}(f)=\sup_{u\in U}\sum_{x\in V} f(x)u(x)$ where U is the set of functions satisfying $E(u)=0$ and $\sup u - \inf u \leqslant 2$ (see the appendix Section A). Here $|\Omega|_{\mathscr{F}}=2|\Omega||\bar{\Omega}|/|V|$, and therefore,

$$\forall f, \quad \int_E |\nabla f| \geqslant C \sum_{x\in V} |f(x)-E(f)| \Leftrightarrow \forall \Omega, \quad |\partial\Omega| \geqslant \frac{2C|\Omega||\bar{\Omega}|}{|V|}.$$

Example 2. $\mathscr{F}(f)=(1/|V|)\sum_{x,y\in V} |f(x)-f(y)|$. This functional is also a semi-norm. Let us note that $|\Omega|_{\mathscr{F}}=2|\Omega||\bar{\Omega}|/|V|$ coincides with the previous definition and yields the equivalence

$$\forall f, \quad \int_E |\nabla f| \geqslant C \frac{1}{|V|} \sum_{x,y\in V} |f(x)-f(y)| \Leftrightarrow \forall \Omega, \quad |\partial\Omega| \geqslant \frac{2C|\Omega||\bar{\Omega}|}{|V|}.$$

Example 3. $\mathscr{F}(f)=\{\sum_{x\in V} |f(x)|^p\}^{1/p}$ with $1\leqslant p \leqslant \infty$. This functional arises in isoperimetric inequalities in infinite graphs (see Section 5). This functional is a norm, but it can also be noted that $\mathscr{F}(f)=\sup_{u\in U}\sum_{x\in V} f(x)u(x)$ where U is the set of functions satisfying $\sum_{x\in V} |u(x)|^q \leqslant 1$, where q is the dual exponent (i.e. $1/p+1/q=1$), see the appendix for a proof. Here we have the equivalence:

$$\forall f, \int_E |\nabla f| \geqslant C \left\{ \sum_{x\in V} |f(x)|^p \right\}^{1/p} \Leftrightarrow \forall \Omega, \quad |\partial\Omega| \geqslant C|\Omega|^{1/p}.$$

Example 4. $\mathscr{F}(f)=\inf_a\{\sum_{x\in V} |f(x)-a|^p\}^{1/p}$ with $1\leqslant p \leqslant \infty$. This is a generalization of the aforementioned functional $\mathscr{F}(f)=\inf_a \sum_{x\in V} |f(x)-a|$ which corresponds to the usual definition of the isoperimetric number, and has been used in [17] for instance. This functional is a semi-norm, but it can also be noted that $\mathscr{F}(f)=\sup_{u\in U}\sum_{x\in V} f(x)u(x)$ where U is the set of functions satisfying $E(u)=0$ and $\sum_{x\in V} |u(x)|^q \leqslant 1$, where q is the dual exponent (i.e. $1/p+1/q=1$). The proof of this fact is in the appendix. It can be verified (see [22] for instance) that $\inf_a\{\sum_{x\in V} |f(x)-a|^p\}^{1/p}$ is attained at any value a such that $\sum_{x\in V} \mathrm{sign}(f(x)-a)|f(x)-a|^{p-1}=0$. Hence $|\Omega|_{\mathscr{F}}=(|\Omega|^{1-q}+|V\backslash\Omega|^{1-q})^{-1/q}$. Here we have the equivalence

$$\forall f, \quad \int_E |\nabla f| \geqslant C \inf_a \left\{ \sum_{x\in V} |f(x)-a|^p \right\}^{1/p} \Leftrightarrow \forall \Omega,$$

$$|\partial\Omega| \geqslant C(|\Omega|^{1-q}+|V\backslash\Omega|^{1-q})^{-1/q}.$$

Example 5.

$$\mathscr{F}(f)=\sum_{x\in V}|f(x)|\ln\frac{|f(x)|}{\sum_{x\in V}|f(x)|}+\ln|V|\sum_{x\in V}|f(x)|=\sum_{x\in V}|f(x)|\ln\frac{|f(x)|}{E(|f|)}.$$

This functional has a (QL) representation since $\mathscr{F}(f)=\sup_{u\in U}\sum_{x\in V}|f(x)|u(x)$ where U is the set of functions satisfying $\sum_{x\in V}\mathrm{e}^{u(x)}\leqslant|V|$ (we can even choose the set of functions such that $\sum_{x\in V}\mathrm{e}^{u(x)}=|V|$, that is such that $E(e^u)=1$). The proof of this statement is also in the appendix. Here $|\Omega|_{\mathscr{F}}=|\Omega|\ln(|V|/|\Omega|)$ and we obtain the following equivalence:

$$\forall f,\quad \int_E|\nabla f|\geqslant C\sum_{x\in V}|f(x)|\ln\frac{|f(x)|}{E(|f|)}\ \Leftrightarrow\ \forall\Omega,\quad |\partial\Omega|\geqslant C|\Omega|\ln\left(\frac{|V|}{|\Omega|}\right).$$

3. Analytic inequalities which are 'stable' with respect to graph products

3.1. Motivation

It has been observed several times (see [16,30,34] for example), that the isoperimetric number $i(G)$ of a product of graphs $G=G_1\times G_2\times G_k$ is strongly related to the isoperimetric numbers $i(G_1)$, $i(G_2),\ldots,i(G_k)$. For instance, when the isoperimetric number of a graph with vertex set V is defined by $\min_{\Omega\subset V;|\Omega|\leqslant|V|/2}|\partial\Omega|/|\Omega|$, we have the straightforward inequality (see for example [30])

$$i(G)\leqslant\min\{i(G_1),i(G_2),\ldots,i(G_k)\}$$

(it suffices to consider subsets of vertices of G of the form $X^{(i)}=V_1\times V_2\times V_{i-1}\times X_i\times V_{i+1}\cdots V_k$ where the V_j's are the vertex sets of the G_j's and X_i is a subset of vertices of G_i such that $i(G_i)=|\partial X_i|/|X_i|$ and to notice that $|\partial X^{(i)}|=|\partial X_i|\Pi_{j\neq i}|V_j|=(|\partial X_i|/|X_i|)|X^{(i)}|=i(G_i)|X^{(i)}|$).

Unfortunately, there are graphs G and H for which $i(G\times H)<\min\{i(G),i(H)\}$, see for instance [30]. Nevertheless, a recent result of Chung and Tetali [16] (improving upon an earlier result of Houdré and Tetali, see [28] where these authors give a slightly weaker statement with a very interesting proof) shows that the difference between both terms cannot be too large, more precisely if we denote by G the graph product $G_1\times G_2\times\cdots\times G_k$ then

$$\tfrac{1}{2}\min_i i(G_i)\leqslant i(G)\leqslant\min_i i(G_i).$$

We will show now how to rederive this result (actually with a slight improvement on the first inequality) by choosing another definition of the isoperimetric number of a graph G with vertex set V, namely

$$i'(G)=\min_{\Omega}\left\{\frac{|\partial\Omega||V|}{2|\Omega||\bar{\Omega}|}\right\}.$$

The attractive feature of this isoperimetric number is:

Proposition 1.

$$i'(G \times H) = \min\{i'(G), i'(H)\}.$$

From the observation

$$\frac{i(G)}{2} < i'(G) \leqslant i(G),$$

we obtain

$$\tfrac{1}{2} \min_i i(G_i) < i(G) \leqslant \min_i i(G_i),$$

which improves little on Chung and Tetali's result.

To prove this proposition we will proceed by using the relationship between analytic and isoperimetric inequalities of Section 2. More generally, we use the following technique (which has also been used by Chung and Tetali in [16]):

1. We first use the results of Section 2 to find which kind of analytic inequality is equivalent, or nearly equivalent, to the isoperimetric inequality we consider.
2. Show that when we have such an analytic inequality for two graphs G and H, we can deduce an analytic inequality of the same kind for $G \times H$. This is not always the case but happens for a very large class of functionals (see the next subsection).
3. Use the results of Section 2 again, to translate the analytic inequality obtained for $G \times H$ in terms of an isoperimetric inequality for $G \times H$.

We will show that some interesting results about the isoperimetric numbers of graph products can be obtained from this technique.

Our first task now is to find a way to derive an analytic inequality on the product $G = G_1 \times G_2$, when we have two analytic inequalities for the graphs G_1 and G_2. Let us bring in a few notations. Let V, V_1, V_2 be the vertex sets of G, G_1, G_2, and E, E_1, E_2 the set of edges of these graphs. An analytic inequality for the graph G involves a function f on V, such a function can be considered as a function of two variables $f(x_1, x_2)$ where $x_1 \in V_1$ and $x_2 \in V_2$. It will be quite useful in what follows to consider the functions of a single variable $f(x_1, \cdot)$, which associates with a vertex y in V_2 the number $f(x_1, y)$, and $f(\cdot, x_2)$, which associates with a vertex y in V_1 the number $f(y, x_2)$. First let us note that for any $p > 0$:

Lemma 2.

$$\int_E |\nabla f|^p = \sum_{x_1 \in V_1} \int_{E_2} |\nabla f(x_1, \cdot)|^p + \sum_{x_2 \in V_2} \int_{E_1} |\nabla f|(\cdot, x_2)|^p.$$

Proof. E can be partitioned into $E_1' \cup E_2'$ where E_1' is the set of edges of the form $(x_1, y) - (y_1, y)$, and E_2' the set of edges of the type $(y, x_2) - (y, y_2)$. The equality above follows from

$$\int_{E_2'} |\nabla f|^p = \sum_{x_1\in V_1} \int_{E_2} |\nabla f(x_1,\cdot)|^p,$$

$$\int_{E_1'} |\nabla f|^p = \sum_{x_2\in V_2} \int_{E_1} |\nabla f(\cdot,x_2)|^p. \qquad \square$$

A straightforward utilization of the previous lemma gives:

Lemma 3. *If there exist two constants C_1 and C_2, such that we have for every f_1 and f_2, acting on the set of vertices V_1 and V_2, respectively,*

$$\int_{E_1} |\nabla f_1| \geqslant C_1 \frac{1}{|V_1|} \sum_{x,y\in V_1} |f_1(x) - f_1(y)|,$$

$$\int_{E_2} |\nabla f_2| \geqslant C_2 \frac{1}{|V_2|} \sum_{x,y\in V_2} |f_2(x) - f_2(y)|,$$

then for every function f on $V = V_1 \times V_2$ we have

$$\int_{E} |\nabla f| \geqslant \min\{C_1, C_2\} \frac{1}{|V|} \sum_{x,y\in V} |f(x) - f(y)|.$$

Proof. From Lemma 2 we know that

$$\int_{E} |\nabla f| = \sum_{x_1\in V_1} \int_{E_2} |\nabla f(x_1,\cdot)| + \sum_{x_2\in V_2} \int_{E_1} |\nabla f(\cdot,x_2)|.$$

By using the first hypothesis we have

$$\begin{aligned}\sum_{x_1\in V_1} \int_{E_2} |\nabla f(x_1,\cdot)| &\geqslant \frac{C_2}{|V_2|} \sum_{x_1\in V_1} \sum_{x_2,y_2} |f(x_1,x_2) - f(x_1,y_2)| \\ &\geqslant \frac{C_2}{|V|} \sum_{x_1,y_1\in V_1} \sum_{x_2,y_2} |f(x_1,x_2) - f(x_1,y_2)|.\end{aligned}$$

Similarly,

$$\sum_{x_2\in V_2} \int_{E_1} |\nabla f(\cdot,x_2)| \geqslant \frac{C_1}{|V|} \sum_{x_2,y_2\in V_2} \sum_{x_1,y_1} |f(x_1,y_2) - f(y_1,y_2)|.$$

Therefore by summing these two inequalities, and by using the triangle inequality

$$\begin{aligned}\int_{E} |\nabla f| &\geqslant \frac{\min(C_1,C_2)}{|V|} \sum_{x_1,y_1\in V_1;x_2,y_2\in V_2} |f(x_1,x_2) - f(x_1,y_2)| \\ &\quad + |f(x_1,y_2) - f(y_1,y_2)| \\ &\geqslant \frac{\min(C_1,C_2)}{|V|} \sum_{x_1,y_1\in V_1;x_2,y_2\in V_2} |f(x_1,x_2) - f(y_1,y_2)|. \qquad \square\end{aligned}$$

We are ready now to prove Proposition 1.

Proof of Proposition 1. From Example 2 given in Section 2, and the previous lemma we deduce that

$$i'(G) \geqslant \min\{i'(G_1), i'(G_2)\}.$$

The converse follows by considering a set of vertices of the form $X = X_1 \times V_2$ where $i'(G_1) = |V_1||\partial X_1|/2|X_1||V_1 \backslash X_1|$. It is readily seen that

$$\frac{|V_1||\partial X_1|}{2|X_1||V_1 \backslash X_1|} = \frac{|V||\partial X|}{2|X||V \backslash X|} \geqslant i'(G).$$

Similarly $i'(G_2) \geqslant i'(G)$. □

There is a corollary to Proposition 1, namely:

Corollary 1. *Let $V_G, V_H, V_{G\times H}$ be the vertex sets of $G, H, G\times H$. If $i'(G) \leqslant i'(H)$ and if Ω is a set such that $i'(G) = |V_G||\partial\Omega|/2|\Omega||V_G \backslash \Omega|$, then*

$$i'(G \times H) = \frac{|V_{G\times H}||\partial(\Omega \times V_H)|}{2|\Omega \times V_H||V_{G\times H} \backslash (\Omega \times V_H)|}.$$

$\Omega \times V_H$ is therefore an isoperimetric set of $G \times H$.

This is a consequence of Proposition 1 and the calculation at the end of the proof of Proposition 1.

3.2. Some other analytic inequalities which are stable with respect to the cartesian product

It can be argued now that the reason for $i'(G \times H) = \min\{i'(G), i'(H)\}$ is actually the fact that the inequality $\int_E |\nabla f| \geqslant C(1/|V|) \sum_{x\neq y} |f(x) - f(y)|$ is stable with respect to the usual cartesian product of graphs (SWRCP in short). By this we mean an inequality

$$\int_E |\nabla f| \geqslant \mathscr{F}(f)$$

such that if it holds for all functions defined on the vertex set of a graph G_1 and all functions defined on the vertex set of a graph G_2, then it also holds for all functions defined on the vertex set of $G = G_1 \times G_2$. One might wonder whether there are other analytic inequalities of the kind $\int_E |\nabla f|^p \geqslant \mathscr{F}(f)$ which display the same behaviour, and draw some consequences on the associated isoperimetric inequalities.

We now give a sufficient condition for such an inequality to be SWRCP.

Proposition 2. *The inequality* $\int_E |\nabla f|^p \geqslant \mathscr{F}(f)$ *is SWRCP if for every function* $f(x_1,x_2)$ *defined on* $V = V_1 \times V_2$, *a product of vertex sets of two graphs* G_1 *and* G_2, *we have*

$$\text{(P)}\quad \mathscr{F}(f) \leqslant \sum_{x_1\in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2\in V_2} \mathscr{F}(f(\cdot,x_2)).$$

We say in this case that $\mathscr{F}$ *satisfies property* (P).

Proof. Let us denote by E_i the edge set of the graph G_i ($i \in \{1,2\}$), and let f be a function defined on V. Assume that we have $\int_{E_i} |\nabla f_i|^p \geqslant \mathscr{F}(f_i)$ for every function f_i n V_i. From Lemma 2 we deduce

$$\begin{aligned} \int_E |\nabla f|^p &= \sum_{x_1\in V_1} \int_{E_2} |\nabla f(x_1,\cdot)|^p + \sum_{x_2\in V_2} \int_{E_1} |\nabla f(\cdot,x_2)|^p \\ &\geqslant \sum_{x_1\in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2\in V_2} \mathscr{F}(f(\cdot,x_2)) \\ &\geqslant \mathscr{F}(f). \qquad \square \end{aligned}$$

We have seen in the previous subsection an example of a functional which meets this property which has the form $(1/|V|)\sum_{x,y\in V} |f(x)-f(y)|$. Actually, many functionals of the type $\mathscr{F}(f) = (1/|V|)\sum_{x,y\in V} \psi(f(x),f(y))$ meet this property, this is shown by the next proposition (as soon as the function in two variables ψ satisfies a certain condition). In order to simplify the statement of this proposition we will write $\psi(x,y)$ instead of $\psi(f(x),f(y))$. Without loss of generality, we may assume that ψ is a symmetric function, i.e. $\psi(x,y) = \psi(y,x)$. Indeed if we bring in $\phi(x,y) = \frac{1}{2}(\psi(x,y) + \psi(y,x))$ we just have to note that $\sum_{x,y\in V} \psi(x,y) = \sum_{x,y\in V} \phi(x,y)$.

We now give a sufficient condition on ψ (when ψ is symmetric) which implies property (P) of Proposition 2. Let $V = V_1 \times V_2$. To describe this condition we need to define the notion of a *rectangle*. We say that four vertices x,y,z,t of V form a rectangle with diagonals xy and zt iff there exist x_i and y_i in V_i ($i \in \{1,2\}$) such that $x = (x_1,x_2)$, $y = (y_1,y_2)$, $z = (x_1,y_2)$, $t = (y_1,x_2)$.

The sufficient condition on ψ is the 'diagonal inequality' (DI) which has to hold for every rectangle $xzyt$ with diagonals xy and zt (see Fig. 2):

$$\text{(DI)}\quad \psi(x,y) + \psi(z,t) \leqslant \psi(x,z) + \psi(z,y) + \psi(y,t) + \psi(t,x).$$

Proposition 3. *If* ψ *satisfies* (DI) *then the associated functional* $\mathscr{F}(f) = (1/|V|) \sum_{x,y\in V} \psi(x,y)$ *satisfies* (P) *and therefore the inequality* $\int_E |\nabla f|^p \geqslant \mathscr{F}(f)$ *is SWRCP for every* $p > 0$.

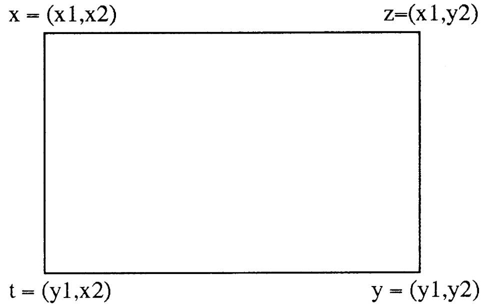

Fig. 2.

Proof. Let $V = V_1 \times V_2$, and f a function on $V = V_1 \times V_2$. Let us check that property (P) holds for $\mathscr{F}$.

$$
\begin{aligned}
&\sum_{x_1 \in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2 \in V_2} \mathscr{F}(f(\cdot,x_2)) \\
&= \frac{1}{|V_2|} \sum_{\substack{x_1 \in V_1 \\ x_2,y_2 \in V_2}} \psi(x_1x_2,x_1y_2) + \frac{1}{|V_1|} \sum_{\substack{x_1,y_1 \in V_1 \\ x_2 \in V_2}} \psi(x_1x_2,y_1x_2) \\
&= \frac{1}{|V|} \left\{ \sum_{\substack{x_1,y_1 \in V_1 \\ x_2,y_2 \in V_2}} \psi(x_1x_2,x_1y_2) + \psi(x_1x_2,y_1x_2) \right\} \\
&= \frac{1}{|V|} \sum_{R=xzyt} a(R) \{\psi(xz) + \psi(zy) + \psi(yt) + \psi(tx)\}.
\end{aligned}
$$

The last sum ranges over all possible rectangles $R = xzyt$, and $a(R)$ is equal to the number of vertices of the rectangle divided by two (for instance $a(R) = 1$ when the rectangle is formed only by two distinct points). Since ψ satisfies (DI) we have

$$
\begin{aligned}
\sum_{x_1 \in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2 \in V_2} \mathscr{F}(f(\cdot,x_2)) &\geqslant \frac{1}{|V|} \sum_{R=xzyt} a(R)(\psi(xy) + \psi(zt)) \\
&\geqslant \frac{1}{|V|} \sum_{a,b \in V} \psi(a,b) \\
&\geqslant \mathscr{F}(f). \qquad \square
\end{aligned}
$$

If ψ is a distance function, it clearly meets citerion (DI). This is the case for $\psi(x,y) = C|f(x) - f(y)|$ that we have already met. Another interesting example is given by $\psi(x,y) = (f(x) - f(y))^2$ ((DI) is easily checked for this example). In other words:

Proposition 4. *The inequality*

$$\int_E |\nabla f|^2 \geqslant \frac{C}{|V|} \sum_{x,y \in V} |f(x) - f(y)|^2$$

is SWRCP.

This inequality is known as a *Poincaré inequality*, and can also be written as

$$\int_E |\nabla f|^2 \geqslant 2C \sum_{x \in V} |f(x) - E(f))|^2.$$

This inequality is well known for being SWRCP, and this is generally proved by using the fact that $\int_E |\nabla f|^2 / \sum_{x\in V} |f(x) - E(f))|^2$ is equal to the smallest nontrivial eigenvalue of the Laplacian of the graph.

Another class of functionals which are well known for being SWRCP (at least in the context of Markov chains and Riemannian manifolds) are *log-Sobolev inequalities* ([23,24]) which have the form for a Riemannian manifold M of finite volume:

$$\int_M |\nabla f|^2 \geqslant C \int_M f^2 \ln\left(\frac{f^2}{E(f^2)}\right).$$

We will show that much more is true for finite graphs namely that not only for $p = 2$ but for any $p \neq 0$ the following inequalities are SWRCP:

$$\int |\nabla f|^p \geqslant C \sum_{x \in V} |f(x)|^p \ln\left(\frac{|f(x)|^p}{E(|f|^p)}\right).$$

Actually, we will show that a more general class of inequalities is SWRCP, and they are built from functionals of the type

$$\mathscr{F}(f) = \sup_{u \in U} \sum_{x \in V} (u(x) - E_\phi(u))\psi(f(x)).$$

Here U is a subset of functions acting on V which depends on the graph G but not on f — we will write U_G to emphasize this dependance. ϕ is just any one-to-one mapping, and ψ any real function, and $E_\phi(u) = \phi^{-1}(E(\phi(u)))$. Not every functional of this type meets property (P) (and therefore does not necessarily give an inequality $\int |\nabla f|^p \geqslant \mathscr{F}(f)$ which is SWRCP). However this will be the case as soon as U satisfies the following two conditions:

for all graphs G_1, G_2, and every $x_1 \in V_1$ (V_1 is the vertex set of G_1), whenever $u \in U_{G_1 \times G_2}$ (we consider u as a function of two variables $u(\cdot,\cdot)$, the first being in V_1, the second in V_2)) we have

(i) $u(x_1, \cdot) \in U_{G_2}$.

(ii) $v : x_1 \to E_\phi(u(x_1, \cdot)) \in U_{G_1}$

Proposition 5. *If U satisfies the two conditions given above, then*

$$\mathscr{F}(f) = \sup_{u \in U} \sum_{x \in V} (u(x) - E_\phi(u))\psi(f(x))$$

satisfies property (P).

Proof. To check that $\mathscr{F}$ satisfies property (P), we prove that for any graphs G_1 and G_2 with vertex sets V_1 and V_2, any function f defined on $V_1 \times V_2$ we have

$$\mathscr{F}(f) \leqslant \sum_{x_1 \in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2 \in V_2} \mathscr{F}(f(\cdot,x_2)).$$

For this purpose we just have to check that for any function $u(\cdot,\cdot) \in U_{G_1,G_2}$ we have

$$\sum_{\substack{x_1 \in V_1 \\ x_2 \in V_2}} (u(x_1,x_2) - E_\phi(u))\psi(f(x_1,x_2)) \leqslant \sum_{x_1 \in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2 \in V_2} \mathscr{F}(f(\cdot,x_2)).$$

In order to do so, we observe that since $u(x_1,\cdot) \in U_{G_2}$ we have

$$\mathscr{F}(f(x_1,\cdot)) \geqslant \sum_{x_2 \in V_2} (u(x_1,x_2) - E_\phi(u(x_1,\cdot)))\psi(f(x_1,x_2)).$$

Since $v(x_1) = E_\phi(u(x_1,\cdot)) \in U_{G_1}$ we also have

$$\mathscr{F}(f(\cdot,x_2)) \geqslant \sum_{x_1 \in V_1} (v(x_1) - E_\phi(v))\psi(f(x_1,x_2)).$$

We sum these two inequalities with respect to x_1 and x_2, respectively, add them together to get

$$\sum_{x_1 \in V_1} \mathscr{F}(f(x_1,\cdot)) + \sum_{x_2 \in V_2} \mathscr{F}(f(\cdot,x_2)) \geqslant \sum_{\substack{x_1 \in V_1 \\ x_2 \in V_2}} (u(x_1,x_2) - E_\phi(u))\psi(f(x_1,x_2)).$$

We have used the fact here that the two terms involving $\sum_{x_1 \in V_1, x_2 \in V_2} v(x_1)\psi(f(x_1,x_2))$ and $-\sum_{x_1 \in V_i} E_\phi(u(x_1,\cdot))\psi(f(x_1,x_2))$ cancel and that $E_\phi(u) = E_\phi(v)$. $\square$

A consequence of this last proposition is that the following analytic inequalities are SWRCP.

Proposition 6. *The following inequalities are SWRCP*:

(i)

$$\int_E |\nabla f| \geqslant C \sum_{x \in V} |f(x) - E(f)|.$$

(ii)

$$\int_E |\nabla f|^p \geqslant C \sum_{x \in V} |f(x)|^p \ln\left(\frac{|f(x)|^p}{E(|f|^p)}\right) \quad \textit{for } p > 0.$$

Proof. To prove (i) we recall (see Example 1 in Section 2) that $\mathscr{F}(f) = \sum_{x \in V} |f(x) - E(f)| = \sup_{u \in U} \sum_{x \in V} (u(x) - E(u)) f(x)$ where $U = \{u | \sup u - \inf u \leqslant 2\}$. It is readily checked that U satisfies both conditions given just before Proposition 5. Hence $\mathscr{F}(f)$ satisfies (P) and we conclude by using Proposition 2.

To prove (ii) we use the characterization of $\mathscr{F}(f)=\sum_{x\in V}|f(x)|^p\ln(|f(x)|^p/E(|f|^p))$ given in Example 5 of Section 2, i.e

$$\mathscr{F}(f)=\sup_{u\in U'}\sum_{x\in V}u(x)|f(x)|^p=\sup_{u\in U}\sum_{x\in V}(u(x)-E_{\exp}(u))|f(x)|^p,$$

where U' is the set of functions such that $E(e^u)=1$ and U is the set of all functions defined on the vertex set of the graph we consider (this follows from the calculation $E(\mathrm{e}^{u-E_{\exp}(u)})=E(\mathrm{e}^{u-\ln(E(\mathrm{e}^u))})=1$). $\square$

Remark. Basically it has already been proved in [16] that the first inequality is SWRCP.

4. Isoperimetric inequalities for product graphs

4.1. A general bound and the proof of Theorem 1

We have already seen in the previous section that with the 'right' definition of the isoperimetric number, the isoperimetric number of a product of graphs $G_1,G_2,\dots,G_n$ can be deduced from the isoperimetric numbers of the G_i's. When all the G_i's are equal, we obtain the following isoperimetric inequality for $G^n=\underbrace{G\times G\times\cdots\times G}_{n}$ (V denotes in what follows the vertex set of G):

$$|\partial\Omega|\geqslant i'(G)\frac{2|\Omega||\bar{\Omega}|}{|V|^n}.$$

This kind of isoperimetric inequality is quite weak when $|\Omega|=1$ for large n. We may expect a much better isoperimetric inequality of the form

$$|\partial\Omega|\geqslant C|\Omega|\ln\left(\frac{|V|^n}{|\Omega|}\right).$$

Let us explain why. Specifying Ω to be a set of the form $A^k\times V^{n-k}$ (with A a subset of vertices of V), we would have $|\partial\Omega|=k|\partial A||A|^{k-1}|V|^{n-k}=C(A)|\Omega|\log(|V|^n/|\Omega|)$, where $C(A)=|\partial A|/|A|\log|V|/|A|$. Indeed, when we look at the n-cube the isoperimetric sets of size 2^k are of the kind $\{0\}^{n-k}\times\{0,1\}^k$ and we have a very good approximation of the best isoperimetric inequality with

$$|\partial\Omega|\geqslant|\Omega|\log_2\left(\frac{|V|^n}{|\Omega|}\right).$$

We are going to show now that this is a general phenomenon, i.e. that for every graph G we can expect a isoperimetric inequality which holds for every power G^n, which has the form

$$|\partial\Omega|\geqslant C|\Omega|\ln\left(\frac{|V|^n}{|\Omega|}\right)$$

with a constant C which depends on G but not on n.

This is obtained by the following proposition:

Proposition 7. *Assume that for some $p>0$ we have for every function f on the vertex set of G:*

$$\int_E |\nabla f|^p \geqslant C \sum_{x\in V} |f(x)|^p \ln \frac{|f(x)|^p}{E(|f|^p)}.$$

Then for every $n\geqslant 1$ and all subsets Ω of G^n we have

$$|\partial\Omega| \geqslant C|\Omega| \ln \frac{|V|^n}{|\Omega|}.$$

Proof. From Proposition 6 we have for every function f acting on the set of vertices V^n of $G^n(V^n,E^n)$:

$$\int_{E^n} |\nabla f|^p \geqslant C \sum_{x\in V^n} |f(x)|^p \ln \frac{|f(x)|^p}{E(|f|^p)}.$$

We just have to plug in $f=\chi_\Omega$ and note that $\int_{E^n} |\nabla \chi_\Omega|^p = |\partial\Omega|$ to prove the proposition. □

We now have to compute (at least for one value of p)

$$C_p = \inf_f \frac{\int_E |\nabla f|^p}{\sum_{x\in V} |f(x)|^p \ln |f(x)|^p / E(|f|^p)}.$$

When $p=1$ we can use Theorem 2, and notice that the minimum is attained for a characteristic function of a subset of vertices Ω, and is therefore equal to

$$\min_{\Omega\in V} \frac{|\partial\Omega|}{|\Omega| \ln |V|/|\Omega|}.$$

We have to find in this case the minimum among a large (but finite!) number of possibilities. It might also be interesting to look for the minimum when $p=2$; this problem is related to finding the best log-sobolev constant. There are many results in this direction for Riemannian manifolds and Markov chains, see [24,31,32]. The relevance of the case p=2 does not lie in improving the constant C in the corresponding isoperimetric inequality over G^n:

$$|\partial\Omega| \geqslant C|\Omega| \ln \frac{|V|^n}{|\Omega|}$$

(since the largest constant C which can be put into this inequality is indeed C_1), but lies in the fact that C_2 can by characterized by other means (for instance, hypercontractivity, see [24]). Therefore if the computation of

$$\min_{\Omega} \frac{|\partial\Omega|}{|\Omega| \ln(|V|/|\Omega|)}$$

is intractable (for instance if G is too large) there might be another way of obtaining an isoperimetric inequality of the kind

$$|\partial\Omega| \geqslant C|\Omega| \ln \frac{|V|^n}{|\Omega|}$$

by estimating C_2.

The calculation at the beginning of this section shows that whenever we have a subset Ω which attains the minimum in

$$\min_{\Omega\in V} \frac{|\partial\Omega|}{|\Omega| \ln|V|/|\Omega|},$$

then all the subsets $\Omega^{(k,n)} = \Omega^k \times V^{n-k}$ are isoperimetric sets of G^n for every $0 \leqslant k \leqslant n$, and satisfy

$$\frac{|\partial\Omega|}{|\Omega| \ln|V|/|\Omega|} = \frac{|\partial\Omega^{(k,n)}|}{|\Omega^{(k,n)}| \ln|V^n|/|\Omega^{(k,n)}|}.$$

This concludes the proof of Theorem 1. $\square$

4.2. *Applications to certain families of graphs*

4.2.1. *Powers of a complete graph*

It is straightforward to check that for the complete graph K_p on p vertices the set Ω which minimizes

$$\min_{\Omega\in V} \frac{|\partial\Omega|}{|\Omega| \ln|V|/|\Omega|}$$

is just a single vertex. The minimum is equal to $(p-1)/\ln p$.

By using Theorem 1 we have the following isoperimetric inequality for K_p^n:

$$|\partial\Omega| \geqslant (p-1)|\Omega| \log_p(p^n/|\Omega|).$$

This gives for $p=2$ the sharp isoperimetric inequality for the n-cube given in the introduction.

4.2.2. *Powers of a path*

Whereas a complete solution of the edge-isoperimetric problem is known for products of complete graphs (see [29]), it is not the case for products of paths (see [1]). There are only partial results like the isoperimetric inequality obtained by Bollobas and Leader for the nth cartesian power $[k]^n$ of a path of length k.

For instance, for a path $[k]$ of length k we clearly have for any subset of vertices Ω:

$$\min_{\Omega} \frac{|\partial\Omega|}{|\Omega| \ln(k/|\Omega|)} = \min_{1\leqslant u\leqslant k-1} \frac{1}{u \ln(k/u)} \geqslant \min_{x\in[0,1]} \frac{1}{kx \ln(1/x)} \geqslant \frac{e}{k}.$$

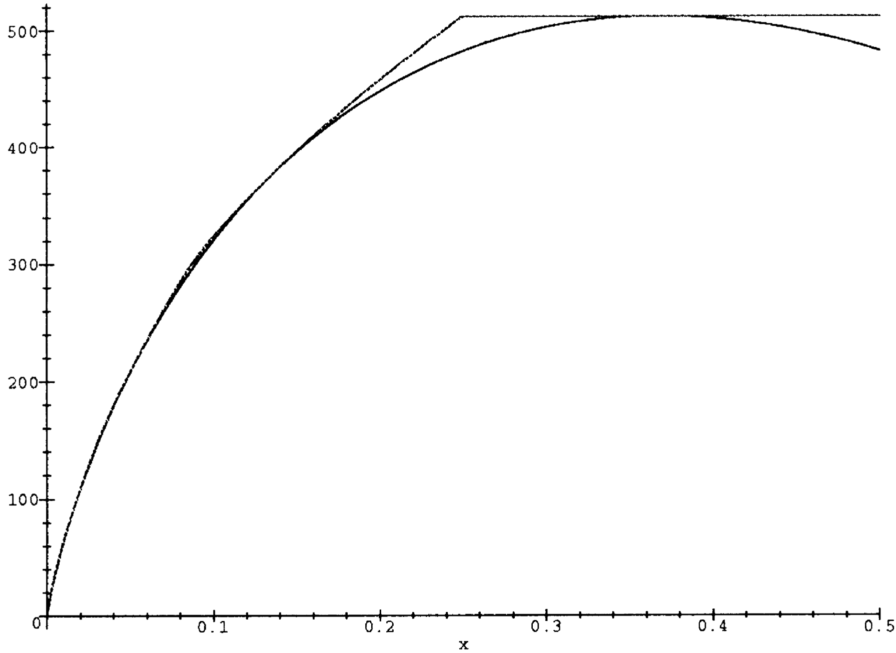

Fig. 3.

Therefore by using Theorem 1 we obtain that for any subset of vertices in $[k]^n$:

$$|\partial\Omega| \geqslant \frac{e}{k}|\Omega| \ln\left(\frac{k^n}{|\Omega|}\right).$$

It can be checked that this isoperimetric inequality is about as sharp as the isoperimetric inequality for $[k]^n$ given at the beginning of the introduction. We have plotted these two isoperimetric functions for $k=8$ and $n=4$, the smooth curve (see Fig. 3) is our isoperimetric function, and the other one which is a sequence of broken lines is the isoperimetric function of Bollobas and Leader given in the introduction. x denotes the ratio $|\Omega|/k^n$.

4.2.3. Powers of the Petersen graph

Recall that this graph is given by Fig. 4, and that the best isoperimetric function is not known for powers of this graph. However we have a partial answer, and we can deduce at least some isoperimetric sets from Theorem 1 and Corollary 1.

We denote by V the vertex set of the Petersen graph, and by P the Petersen graph itself. We observe that a subset of vertices which attains the minimum of

$$\frac{|\partial\Omega||V|}{2|\Omega||\bar{\Omega}|}$$

is for example $\{1,2,3,4,5\}$. Therefore, from Corollary 1 we know that $\{1,2,3,4,5\} \times V^{n-1}$ is an isoperimetric set for P^n, for every $n \geqslant 1$. This minimum is equal to 1 and

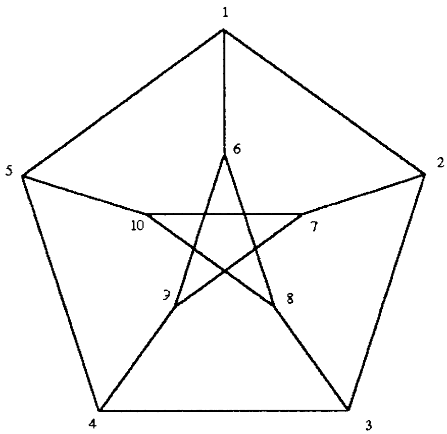

Fig. 4.

by using Proposition 1, we therefore have for any power of the Petersen graph the isoperimetric inequality

$$|\partial\Omega| \geqslant \frac{2|\Omega|\ |V^n \backslash \Omega|}{|V^n|}.$$

Another isoperimetric inequality can be obtained by using Theorem 1 by finding the subset of vertices of P which attains the minimum of

$$\frac{\partial\Omega|}{|\Omega|\ \ln|V|/|\Omega|}.$$

It is straightforward to check that the minimum is attained for $\Omega=\{1,2\}$ and that the corresponding ratio is equal to $2/\ln 5$. Hence for any power of the Petersen graph the following isoperimetric inequality holds:

$$|\partial\Omega| \geqslant 2|\Omega| \log_5|V^n|/|\Omega|.$$

Moreover, any set of the form $\{1,2\}^k \times V^{n-k}$ is an isoperimetric set of P^n.

In Fig. 5 we have plotted the two isoperimetric functions divided by $|V^n|$, namely $2x(1-x)$ and $2x\log_5(1/x)$ in terms of $x=|\Omega|/|V^n|$. It can be observed that the second isoperimetric inequality is better for sets of size $\leqslant \alpha|V^n|$, where $\alpha \approx 0.353$, whereas the first one is better for sets of size s which satisfy $\alpha|V^n| \leqslant s \leqslant 0.5|V^n|$.

5. Infinite graphs

The results given in the previous sections apply to finite graphs and do not explain the form of the isoperimetric inequality given for $\mathbb{Z}^n$ in the introduction. Our aim in this section is to give tools which enable us to recover this almost optimal isoperimetric inequality for $\mathbb{Z}^n$ with analytic tools, and which is useful in general to obtain good isoperimetric inequalities for cartesian product of infinite graphs. These results

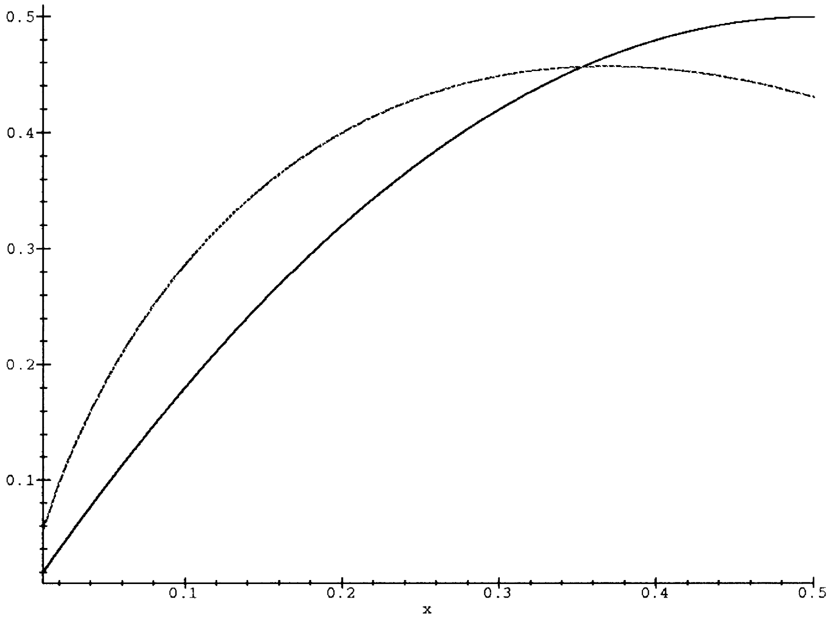

Fig. 5.

generalize tools for obtaining isoperimetric inequalities of products of non-compact Riemanannian manifolds (see pp. 306–307 in [33]). We point out that the tools we have obtained here for graphs are somehow sharper than the corresponding results obtained for Riemannian manifolds (see next section for an explanation of this phenomenon). For instance, whereas the isoperimetric inequalities obtained in this way for $\mathbb{R}^n$ from the optimal isoperimetric inequality on $\mathbb{R}$ is not tight (see [33, Example 2, p. 303 and Lemma 5, p. 307]), the sharp inequality on $\mathbb{Z}^n$: $|\partial\Omega| \geqslant 2.n|\Omega|^{(n-1)/n}$ will be seen to follow from the simple inequality $|\partial\Omega| \geqslant 2$ which holds for subsets of $\mathbb{Z}$. This is a consequence of the more general:

Proposition 8.

$$\frac{c_{n_1+n_2}(G_1 \times G_2)}{n_1+n_2} \geqslant \left(\frac{c_{n_1}(G_1)}{n_1}\right)^{n_1/(n_1+n_2)} \left(\frac{c_{n_2}(G_2)}{n_2}\right)^{n_2/(n_1+n_2)},$$

$$\frac{c_{nk}G^{(k)}}{nk} \geqslant \frac{c_n(G)}{n}.$$

$c_n(G)$ denotes the n-dimensional isoperimetric constant of G, that is the infimum over all finite subsets Ω of vertices of G of the ratio $|\partial\Omega|/|\Omega|^{(n-1)/n}$. This somehow mimicks the usual isoperimetric constant of $\mathbb{R}^n$ which is defined to be the infimum over all subsets $\Omega \subset \mathbb{R}^n$ of finite volume of the ratio $\sigma(\partial\Omega)/\mu(\Omega)^{(n-1)/n}$, where μ is the n-dimensional volume, and $\sigma(\partial\Omega)$ the $(n-1)$-dimensional volume of its boundary.

This last constant is known to be $n\omega_n^{1/n}$ where ω_n is the volume of the unit ball (see [3]).

Clearly, the second part of the proposition can be proved from the first part by induction on k, and we use this second part in the obvious way to get from the optimal isoperimetric $|\partial\Omega|\geqslant 2$ which holds on $\mathbb{Z}$ the sharp inequality $|\partial\Omega|\geqslant 2.n|\Omega|^{(n-1)/n}$ on $\mathbb{Z}^n$.

Proof of Proposition 8. We use again the equivalence between analytic inequalities and isoperimetric inequalities of Section 2 and write the equivalent analytic form of the isoperimetric inequality

$$|\partial\Omega|\geqslant c_{n_i}(G_i)|\Omega|^{(n_i-1)/n_i},$$

which holds for every subset Ω of V_i which is the vertex set of G_i. Here $i\in\{1,2\}$. This equivalent form states that for every function f defined on V_i we have

$$\int_{E_i}|\nabla f|\geqslant c_{n_i}(G_i)\left(\sum_{x_i\in V_i}|f(x_i)|^{n_i'}\right)^{1/n_i'},$$

where n_i' is the dual exponent of n_i, i.e. $1=1/n_i+1/n_i'$.

To get the desired isoperimetric inequality on $G_1\times G_2$ with vertex set V and edge set E, we first obtain an analytic equivalent form and proceed as follows (here $f(x_1,x_2)$ is a function defined on $V=V_1\times V_2$):

$$\begin{aligned}
\int_E|\nabla f| &= \sum_{x_2\in V_2}\int_{E_1}|\nabla f(\cdot,x_2)|+\sum_{x_1\in V_1}\int_{E_2}|\nabla f(x_1,\cdot)|\\
&\geqslant \sum_{x_2\in V_2} c_{n_1}(G_1)\left(\sum_{x_1\in V_1}|f(x_1,x_2)|^{n_1'}\right)^{1/n_1'}\\
&\quad+\sum_{x_1\in V_1} c_{n_2}(G_2)\left(\sum_{x_2\in V_2}|f(x_1,x_2)|^{n_2'}\right)^{1/n_2'}\\
&\geqslant c_{n_1}(G_1)P+c_{n_2}(G_2)Q\\
&\geqslant (n_1+n_2)\left(\frac{n_1}{n_1+n_2}\frac{c_{n_1}(G_1)}{n_1}P+\frac{n_2}{n_1+n_2}\frac{c_{n_2}(G_2)}{n_2}Q\right),
\end{aligned}$$

where $P=\sum_{x_2\in V_2}(\sum_{x_1\in V_1}|f(x_1,x_2)|^{n_1'})^{1/n_1'}$ and $Q=\sum_{x_1\in V_1}(\sum_{x_2\in V_2}|f(x_1,x_2)|^{n_2'})^{1/n_2'}$. By using $\alpha x+\beta y\geqslant x^\alpha y^\beta$ which holds for $\alpha,\beta,x,y\geqslant 0$ and $\alpha+\beta=1$ we obtain

$$\int_E|\nabla f|\geqslant(n_1+n_2)\left(\frac{c_{n_1}(G_1)}{n_1}\right)^{n_1/(n_1+n_2)}P^{n_1/(n_1+n_2)}\left(\frac{c_{n_2}(G_2)}{n_2}\right)^{n_2/(n_1+n_2)}Q^{n_2/(n_1+n_2)}.$$

We now claim that

$$P^{n_1/(n_1+n_2)}Q^{n_2/(n_1+n_2)} \geqslant \left\{ \sum_{x_1\in V_1, x_2\in V_2} |f(x_1,x_2)|^{n'} \right\}^{1/n'},$$

where n' is the dual exponent of $n = n_1 + n_2$. This is a consequence of

$$\left\{ \sum_{x_1\in V_1, x_2\in V_2} |f(x_1,x_2)|^{n'} \right\}^{1/n'} \leqslant \left\{ \sum_{x_2\in V_2} \left(\sum_{x_1\in V_1} |f(x_1,x_2)|^{n_1'} \right)^{1/n_1'} \right\}^{n_1/(n_1+n_2)}$$
$$\left\{ \sum_{x_2\in V_2} \left(\sum_{x_1\in V_1} |f(x_1,x_2)| \right)^{n_2'} \right\}^{(1/n_2')n_2/(n_1+n_2)}$$

which will be proved in the appendix (see Lemma B.1 in Appendix B and apply it with $R=(n',n')$, $P=(n_1',1)$, $Q=(1,n_2')$ and $t=n_1/(n_1+n_2)$), and

$$\left\{ \sum_{x_2\in V_2} \left(\sum_{x_1\in V_1} |f(x_1,x_2)| \right)^{n_2'} \right\}^{1/n_2'} \leqslant \sum_{x_1\in V_1} \left(\sum_{x_2\in V_2} |f(x_1,x_2)|^{n_2'} \right)^{1/n_2'}$$

The last inequality is only a special case of Minkowski's inequality.

Therefore,

$$\int_E |\nabla f| \geqslant (n_1+n_2) \left(\frac{c_{n_1}(G_1)}{n_1} \right)^{n_1/(n_1+n_2)} \left(\frac{c_{n_2}(G_2)}{n_2} \right)^{n_2/(n_1+n_2)}$$
$$\left\{ \sum_{x_1\in V_1, x_2\in V_2} |f(x_1,x_2)|^{n'} \right\}^{1/n'}.$$

Using once more the equivalence between analytic inequalities and isoperimetric inequalities we obtain the first statement of the proposition. □

6. Additional remarks

6.1. Comparison with isoperimetric inequalities for Riemannian manifolds

We wish to point out here, that although most of the results obtained here have been inspired by results obtained for Riemannian manifolds (especially [33]), there are subtle differences between both settings. For instance for products of Riemannian manifolds the corresponding formula of Lemma 2 is only true for $p=2$, i.e.

$$\int_{\mathscr{M}_1\times\mathscr{M}_2} |\nabla f|^2 = \int_{\mathscr{M}_1}\int_{\mathscr{M}_2} |\nabla f(x_1,\cdot)|^2 + \int_{\mathscr{M}_2}\int_{\mathscr{M}_1} |\nabla f(\cdot,x_2)|^2.$$

A consequence of this is that there is not really a corresponding theorem to Theorem1 for Riemannian manifolds. An illustration of this is given by the following example. The results obtained by [33] in Section 1, Example 2, p. 303 show that for subsets of $\mathbb{R}^n$ we have the following isoperimetric inequality:

$$\sigma(\partial\Omega) \geqslant e\sqrt[n]{\omega_n}\mu(\Omega)\ln\left(\frac{1}{\mu(\Omega)}\right),$$

where μ is the Lebesgue measure, $\sigma(\partial\Omega)$ the surface measure of the boundary of Ω, and ω_n the volume of the unit ball, that is $\pi^{n/2}/\Gamma(1+n/2)$. $e\sqrt[n]{\omega_n}$ is the best-possible constant in this isoperimeric inequality and depends on n. In the graph-theoretic setting we have seen in Theorem 1 that a much stronger result holds, i.e. the biggest constant C in the isoperimetric inequality $|\partial\Omega| \geqslant C|\Omega|\ln(|V|/|\Omega|)$ over a product graph G^n does not depend on n.

6.2. Sharpness of the isoperimetric inequality of Theorem 1

When we have a product graph G^n where n is large, it should be noted that unlike the isoperimetric inequality of the kind $|\partial\Omega| \geqslant C|\Omega||\bar{\Omega}|/|V|$, the isoperimetric inequality $|\partial\Omega| \geqslant C|\Omega|\ln(|V|/|\Omega|)$ of Theorem 1 is sharp for a whole family of sets ranging from very small sets to sets of linear size.

There is also a trivial improvement of the isoperimetric inequality of Theorem 1. Since $|\partial\Omega| = |\partial\bar{\Omega}|$, and $x\ln(1/x) \geqslant (1-x)\ln(1/(1-x))$ for $x \in [0, 1/2]$, we deduce from the isoperimetric inequality $|\partial\Omega| \geqslant C|\Omega|\ln(|V|/|\Omega|)$ the stronger isoperimetric inequality

$$|\partial\Omega| \geqslant \begin{cases} C|\Omega|\ln(|V|/|\Omega|) & \text{for } |\Omega| \leqslant |V|/2, \\ C|\bar{\Omega}|\ln(|V|/|\bar{\Omega}|) & \text{for } |\Omega| \geqslant |V|/2. \end{cases}$$

6.3. Dealing with analytic inequalities rather than with isoperimetric inequalities

Although several results given here could have been obtained without using analytic inequalities (Proposition 1 can be proved very easily by using straightforward combinatorial arguments; this can be discovered by looking at the proof of Lemma 3 and of Proposition 1, by letting f to be a characteristic function of Ω and by understanding the combinatorial meaning of the expressions which appear during the proof) we have chosen to use analytic inequalities in all cases mainly to show that

- there is little specificity about sets and isoperimetric inequalities,
- isoperimetric inequalities seem to be hidden behind more general analytical inequalities involving functions defined on the vertex set of the graph.

Acknowledgements

I would like to thank S. Bezrukov for helpful discussions about isoperimetric problems on graphs.

Appendix A. Quasilinearization

We prove here that the functionals which appear in the examples of Section 2 admit a 'quasi-linearized' representation.

$\mathscr{F}(f)=\sum_{x\in V}|f(x)-E(f)|$: Here $U=\{u|E(u)=0,\ \sup u-\inf u\leqslant 2\}$. Let $u\in U$. We claim that $\mathscr{F}(f)\geqslant\sum_{x\in V}u(x)f(x)$. Indeed for any $a\in\mathbb{R}$:

$$\begin{aligned}\sum_{x\in V}u(x)f(x) &= \sum_{x\in V}u(x)(f(x)-E(f))\\ &= \sum_{x\in V}(u(x)-a)(f(x)-E(f))\\ &\leqslant \sup_{x\in V}|u(x)-a|\sum_{x\in V}|f(x)-E(f)|.\end{aligned}$$

Therefore,

$$\sum_{x\in V}u(x)f(x)\leqslant \inf_{a\in\mathbb{R}}\ \sup_{x\in V}|u(x)-a|\sum_{x\in V}|f(x)-E(f)|\leqslant\sum_{x\in V}|f(x)-E(f)|.$$

The last inequality is a consequence of $\sup u-\inf u\leqslant 2$.

To prove that the maximum of $\sum_{x\in V}u(x)f(x)$ is indeed $\mathscr{F}(f)$; let us note that when u is chosen to be $u(x)=\mathrm{sign}(f(x)-E(f))+a$, where a is such that $E(u)=0$, then $u\in U$ and

$$\begin{aligned}\sum_{x\in V}u(x)f(x) &= \sum_{x\in V}u(x)(f(x)-E(f))\\ &= \sum_{x\in V}(u(x)-a)(f(x)-E(f))\\ &= \sum_{x\in V}|f(x)-E(f)|\\ &= \mathscr{F}(f)\end{aligned}$$

$\mathscr{F}(f)=\{\sum_{x\in V}|f(x)|^p\}^{1/p}$: Here $U=\{u|\sum_{x\in V}|u(x)|^q\leqslant 1\}$, where q is the dual exponent of p $(1=1/p+1/q)$. Let $u\in U$. We claim that $\mathscr{F}(f)\geqslant\sum_{x\in V}u(x)f(x)$. This follows from Hölder's inequality

$$\begin{aligned}\sum_{x\in V}u(x)f(x) &\leqslant \left\{\sum_{x\in V}|u(x)|^q\right\}^{1/q}\left\{\sum_{x\in V}|f(x)|^p\right\}^{1/p}\\ &\leqslant \left\{\sum_{x\in V}|f(x)|^p\right\}^{1/p}.\end{aligned}$$

On the other hand if we put $u(x)=\mathrm{sign}\,f(x)|f(x)|^{p-1}/\{\sum_{x\in V}|f(x)|^p\}^{(p-1)/p}$, it is straightforward to check that $\sum_{x\in V}|u(x)|^q=\sum_{x\in V}|u(x)|^{p/(p-1)}=1$. Hence $u\in U$.

Moreover,

$$\sum_{x\in V} u(x)f(x) = \left\{\sum_{x\in V} |f(x)|^p\right\}^{1/p} = \mathscr{F}(f).$$

$\mathscr{F}(f) = \inf_a\{\sum_{x\in V} |f(x) - a|^p\}^{1/p}$: Here $U = \{u|E(u) = 0, \sum_{x\in V} |u(x)|^q \leqslant 1\}$, where q is the dual exponent of p $(1 = 1/p + 1/q)$. Let $u \in U$. We claim that $\mathscr{F}(f) \geqslant \sum_{x\in V} u(x)f(x)$. Indeed for any $a \in \mathbb{R}$:

$$\sum_{x\in V} u(x)f(x) = \sum_{x\in V} u(x)(f(x) - a)$$
$$\leqslant \left\{\sum_{x\in V} |u(x)|^q\right\}^{1/q} \left\{\sum_{x\in V} |f(x) - a|^p\right\}^{1/p}.$$

Therefore,

$$\sum_{x\in V} u(x)f(x) \leqslant \inf_a \left\{\sum_{x\in V} |f(x) - a|^p\right\}^{1/p} = \mathscr{F}(f).$$

We conclude by letting $u(x) = \operatorname{sign}(f(x) - a)|f(x) - a|^{p-1}/\{\sum_{x\in V} |f(x) - a|^p\}^{(p-1)/p}$, where a is now a number which attains the infimum in $\inf_a\{\sum_{x\in V} |f(x) - a|^p\}^{1/p}$. It is easy to show that a number a for which this infimum is attained satisfies $\sum_{x\in V}$ sign $(f(x) - a)|f(x) - a|^{p-1} = 0$; this implies that $E(u) = 0$ and it is straightforward to check that $\sum_{x\in V} |u(x)|^q = \sum_{x\in V} |u(x)|^{p/(p-1)} = 1$. Hence $u \in U$. Moreover,

$$\sum_{x\in V} u(x)f(x) = \left\{\sum_{x\in V} |f(x) - a|^p\right\}^{1/p} = \mathscr{F}(f).$$

$\mathscr{F}(f) = \sum_{x\in V} |f(x)| \ln|f(x)|/E(|f|)$: Here $U = \{u| \sum_{x\in V} \mathrm{e}^{u(x)} \leqslant |V|\}$. Let $u \in U$. We claim that $\mathscr{F}(f) \geqslant \sum_{x\in V} u(x)|f(x)|$. This is a consequence of Young's inequality $st \leqslant \mathrm{e}^s + t\ln t - t$ (which holds for any s and any $t > 0$):

$$\sum_{x\in V} u(x)|f(x)| = E(|f|)\sum_{x\in V} u(x)\frac{|f(x)|}{E(|f|)}$$
$$\leqslant E(|f|)\sum_{x\in V} \left\{\mathrm{e}^{u(x)} + \frac{|f(x)|}{E(|f|)}\ln\left(\frac{|f(x)|}{E(|f|)}\right) - \frac{|f(x)|}{E(|f|)}\right\}$$
$$\leqslant \sum_{x\in V} |f(x)| \ln\frac{|f(x)|}{E(|f|)}.$$

We conclude by noting that the choice $u(x) = \ln(|f(x)|/E(|f|))$ leads to $\sum_{x\in V} \mathrm{e}^{u(x)} \leqslant |V|$ and $\sum_{x\in V} u(x)|f(x)| = \mathscr{F}(f)$.

Appendix B. A generalization of Hölder's inequality

We will prove here a standard result on mixed norms whose statement can be found in [9]. Nevertheless, since the proof of the inequality, we need is only roughly sketched in [9] we have decided to give a complete proof here.

Before proving the inequality that we have used in Section 5, we need a few notations. For a couple of numbers $P=(p_1,p_2)$ which are $\geqslant 1$ and a function $f(x_1,x_2)$ defined on $V_1 \times V_2$ we denote by

$$\|f\|_P = \left\{ \sum_{x_2 \in V_2} \left(\sum_{x_1 \in V_1} |f(x_1,x_2)|^{p_1} \right)^{p_2/p_1} \right\}^{1/p_2}.$$

Note that if further each p_i is equal to p:

$$\|f\|_{(p,p)} = \left\{ \sum_{x_1 \in V_1, x_2 \in V_2} |f(x_1,x_2)|^p \right\}^{1/p} = \|f\|_p.$$

We also recall that a slight generalization of Hölder's inequality can be written as

$$\|f\|_r \leqslant \|f\|_p^t \|f\|_q^{1-t} \tag{B.1}$$

for $p,\ q \geqslant 1$, $0 \leqslant t \leqslant 1$, and $1/r = t/p + (1-t)/q$. Here $\|f\|_p$ denotes $(\sum_{x\in V} |f(x)|^p)^{1/p}$, where f is a function defined on V.

Now we can state the result we have used in Section 5; this result can be considered as a further generalization of the previous 'Hölder's inequality

Lemma B.1. *For a function f defined on $V_1 \times V_2$ and couples $P=(p_1,p_2)$, $Q=(q_1,q_2)$, $R=(r_1,r_2)$ of numbers $\geqslant 1$ such that there exists $0 \leqslant t \leqslant 1$ for which $1/r_i = t/p_i + (1-t)/q_i$*

$$\|f\|_R \leqslant \|f\|_P^t \|f\|_Q^{1-t}.$$

Proof. A first application of (4) with $r=r_1,\ p=p_1,\ q=q_1$ and $f=f(\cdot,x_2)$ gives

$$\|f\|_R = \left\{ \sum_{x_2 \in V_2} \left(\sum_{x_1 \in V_1} |f(x_1,x_2)|^{r_1} \right)^{r_2/r_1} \right\}^{1/r_2}$$

$$\leqslant \left\{ \sum_{x_2 \in V_2} \|f(\cdot,x_2)\|_{p_1}^{r_2 t} \|f(\cdot,x_2)\|_{q_1}^{r_2(1-t)} \right\}^{1/r_2}.$$

Now we use Hölder's inequality for functions defined on V_2 in its more usual form

$$\sum_{x_2 \in V_2} f(x)g(x) \leqslant \|f\|_\alpha \|g\|_{\alpha'},$$

where α' is the dual exponent of α, i.e $1 = 1/\alpha + 1/\alpha'$. We choose here $\alpha = p_2/r_2 t$ which is clearly $\geqslant 1$, and note that $\alpha' = \alpha/\alpha - 1 = q_2/r_2(1-t)$ to get

$$\|f\|_R \leqslant \left\{ \sum_{x_2 \in V_2} \|f(\cdot, x_2)\|_{p_1}^{p_2} \right\}^{1/\alpha r_2} \left\{ \sum_{x_2 \in V_2} \|f(\cdot, x_2)\|_{q_1}^{q_2} \right\}^{1/\alpha' r_2}$$

$$\leqslant \|f\|_P^t \|f\|_Q^{1-t}$$

This concludes the proof. $\square$

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