Motion estimation using Data Assimilation in a reduced order model

Karim Drifi

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A few words about **CLIME**

Two main topics of the team

- Image assimilation,
- Air quality modelling.

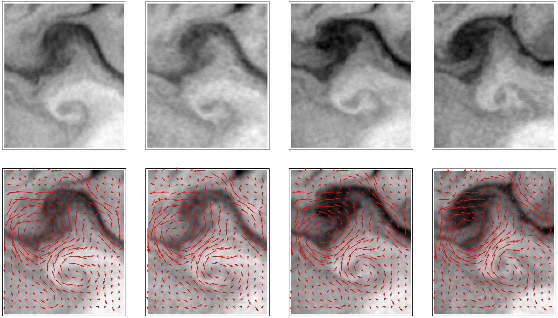
The common part for both topics is **DATA ASSIMILATION**.

A possible definition

Data assimilation is about making a **Compromise** between a **Model** and **Observations**.

Objective of Image Assimilation for Oceanography

Estimation of motion fields (surface flow) using satellite temperature image sequence.



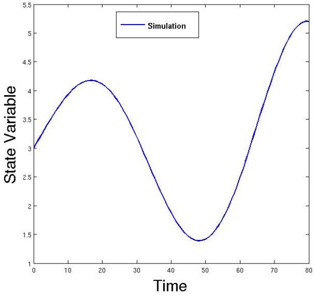
Satellite data acquired over the Black Sea and motion result

Simulation

- $X(t)$, a state vector, defined on $t \in [0, \mathbf{T}]$
- Find $X(t)$:

$$\frac{dX}{dt}(t) + \mathbb{M}(X)(t) = 0 \quad (1)$$

$$X(0) = X_0 \quad (2)$$

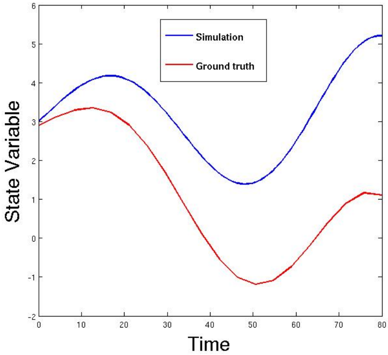


Butterfly effect

But the initial condition X_0 is not perfectly known :

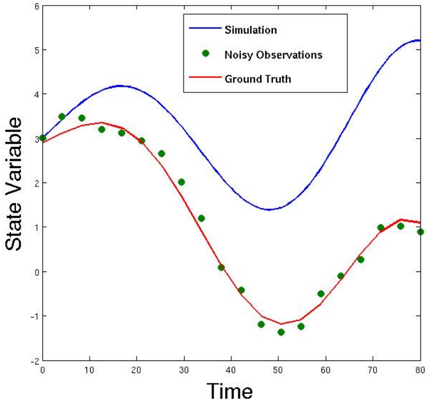
$$X(0) \quad = \quad X_0 + \mathcal{E}_b, \qquad (3)$$

$\mathcal{E}_b$ being the background error.



Need to **Assimilate** (Noisy) Observations

$$Y(t) \quad = \quad \mathbb{H}(X(t)) + \mathcal{E}_o(t), \quad \mathcal{E}_o(t) \text{ being the observation error}$$



Minimizing the errors

- System to be solved, find X:

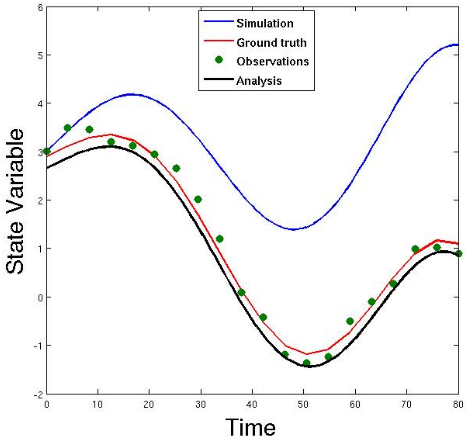
$$\frac{dX}{dt}(t) + \mathbb{M}(X)(t) = 0 \quad (4)$$

$$X(0) = X_0 + \mathcal{E}_b \quad (5)$$

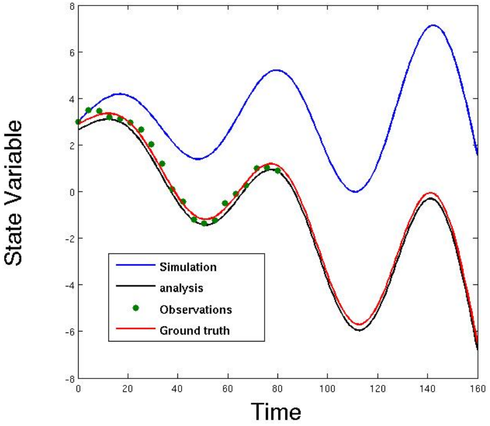
$$Y(t) = IH(X(t)) + \mathcal{E}_o(t) \quad (6)$$

- Minimizing the errors $\|\mathcal{E}_b\| + \|\mathcal{E}_o\|$,
- Variational formulation $\rightarrow$ 4D-Var algorithm.

Analysis (result) fits observations

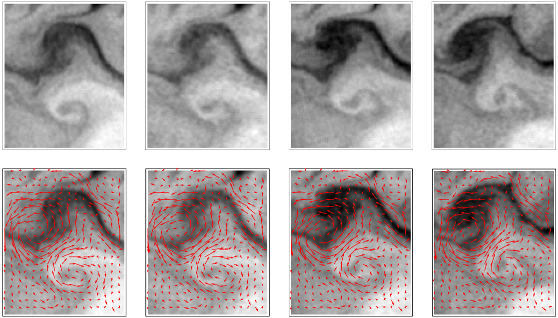


Better Forecast



Back to Image assimilation

Estimation of motion $\mathbf{w}(\mathbf{x}, t)$ on an image sequence $T(\mathbf{x}, t_k)$.
Domain : $(\mathbf{x}, t) \in A = \Omega \times [0, \mathbf{T}]$.



Satellite data acquired over the Black Sea and motion result

Image assimilation

- State Vector defined on $(\mathbf{x}, t) \in A = \Omega \times [0, \mathbf{T}]$:

$$\mathbf{X}(\mathbf{x}, t) = \begin{pmatrix} \mathbf{w}(\mathbf{x}, t) \\ T_s(\mathbf{x}, t) \end{pmatrix}$$

- Model equation

$$\frac{\partial \mathbf{X}}{\partial t}(\mathbf{x}, t) + \mathbb{M}(\mathbf{X})(\mathbf{x}, t) = 0 \tag{7}$$

- Background Equation:

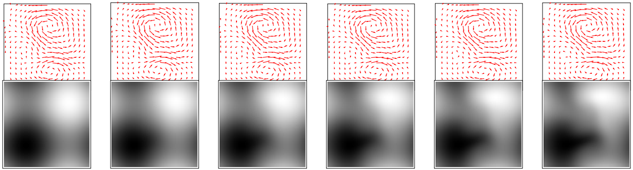
$$\mathbf{X}(\mathbf{x}, 0) = \mathbf{X}_b(\mathbf{x}) + \mathcal{E}_b(\mathbf{x}) \qquad \mathbf{X}_b(\mathbf{x}) = \begin{pmatrix} 0 \\ T(\mathbf{x}, t_1) \end{pmatrix}$$

- Observation Equation:

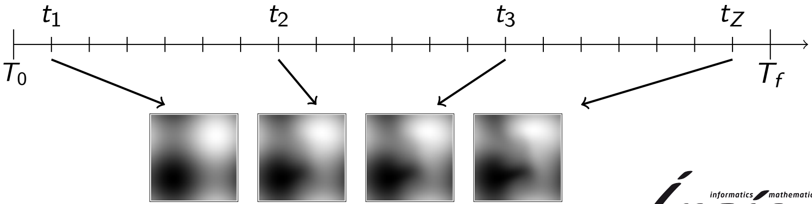
$$T^{obs}(\mathbf{x}, t) = T_s(\mathbf{x}, t) + \mathcal{E}_o(\mathbf{x}, t)$$

Twin experiment

- **Synthetic data** : Simulation of $\mathbb{M}$ from initial conditions $(\mathbf{w}_0(\mathbf{x}), T_0(\mathbf{x})) \rightarrow (\mathbf{w}(\mathbf{x}, t), T_s(\mathbf{x}, t))$:

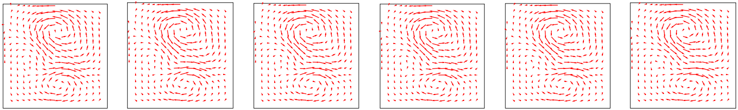


- Choice of observation dates t_z :

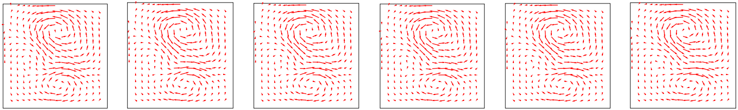


Results

Ground-truth:



Motion estimation:



Drawbacks

- Data size : 8192 variables in our experiment (about 10^8 variables for real data).
- Computation time : Huge !!!

Reduction Twin experiment

Tool

Estimate a small subspace, the solution should be close to the subspace. Define a **reduced model** $\mathbb{M}_R$ by the Galerkin projection of **the full model** $\mathbb{M}$ on the subspace.

- Base $\Psi = \{\psi_j(\mathbf{x})\}_{j=1...L}$ is obtained by applying POD to the image sequence $T(\mathbf{x}, t_k)$.
- Base $\Phi = \{\phi_i(\mathbf{x})\}_{i=1...K}$ is obtained by applying POD to the motion sequence.
- It comes:

$$\mathbf{w}(\mathbf{x}, t) \approx \sum_{i=1}^{K} a_i(t)\phi_i(\mathbf{x})$$

$$T_s(\mathbf{x}, t) \approx \sum_{j=1}^{L} b_j(t)\psi_j(\mathbf{x})$$

Reduced Model

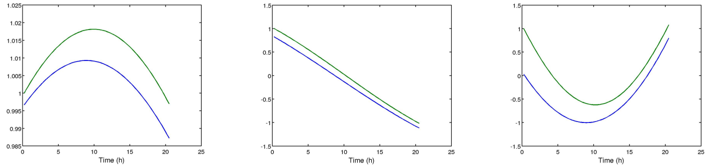
- $a(t) = (a_1(t), \ldots, a_K(t))^T$, $b(t) = (b_1(t), \ldots, b_L(t))^T$
- A reduced state vector: $\mathbf{X}_R(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ $K+L$ components, **less than** 10 in our experiment.
- A reduced model $\mathbb{M}_R$ derived from $\mathbb{M}$:

$$\frac{d\mathbf{X}_R}{dt}(t) + \mathbb{M}_R(\mathbf{X}_R)(t) = 0. \tag{8}$$

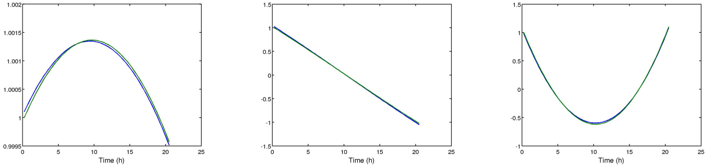
Assimilation in the reduced model

The image data $T(\mathbf{x}, t_k)$ are projected on Ψ with coefficients $b_j^{obs}(t_k)$. The observations $b_j^{obs}(t_k)$ are assimilated in the reduced model $\mathbb{M}_R$ to estimate the coefficients $a_i(t)$.

Results



Motion coefficients – Ground-truth (green) – assimilation results (blue).



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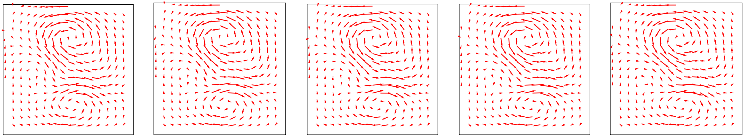
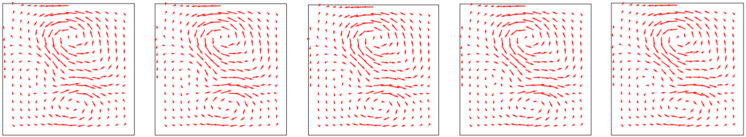
Results

Ground-truth:



Estimation of motion by the reduced model:



Problem – Solution

Ψ et Φ

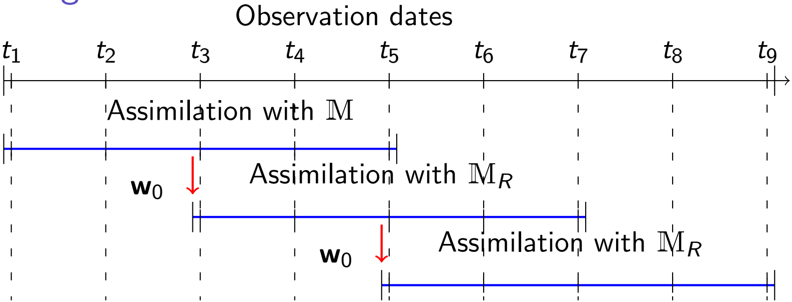
In the twin experiment:
$(\mathbf{w}_0(\mathbf{x}), T_0(\mathbf{x})) \rightarrow$ Simulation $(\mathbf{w}(\mathbf{x}, t), T_s(\mathbf{x}, t)) \quad \rightarrow$ POD $\rightarrow \quad \Phi, \Psi$
which defines the subspaces
Real case:

- Ψ is obtained by applying POD to the image sequence.
- Computation of Φ **requires an initial motion field $\mathbf{w}_0(\mathbf{x})$**. This defines the admissible subspace.

How to get $\mathbf{w}_0$?

Sliding temporal windows : **Coupling** full and reduced models.

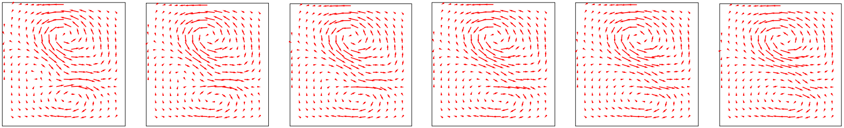
Sliding windows



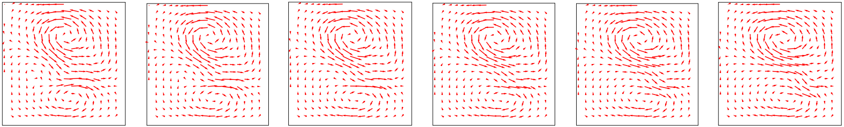
- In the first temporal window: assimilation in the full model.
- $\mathbf{w}_0$ is used to define the reduced model in the second window.
- Assimilation in the reduced model (second window)
- Iteration of the process: 6 consecutive windows for the reduced model.

Results

Ground-truth:



Results of the reduced model on the first frame of windows 1 to 6:



Conclusions and perspectives on model reduction

- The main objective was to reduce computation time and memory size:
  - Computation time:
    Full model: 4 h for 1 temporal window of 20 h.
    Reduced model: < 1 min for 6 temporal windows corresponding to 60 h.
  - State vector:
    Full model: 8192 components.
    Reduced model: 6 components.
- Next step: robust POD?