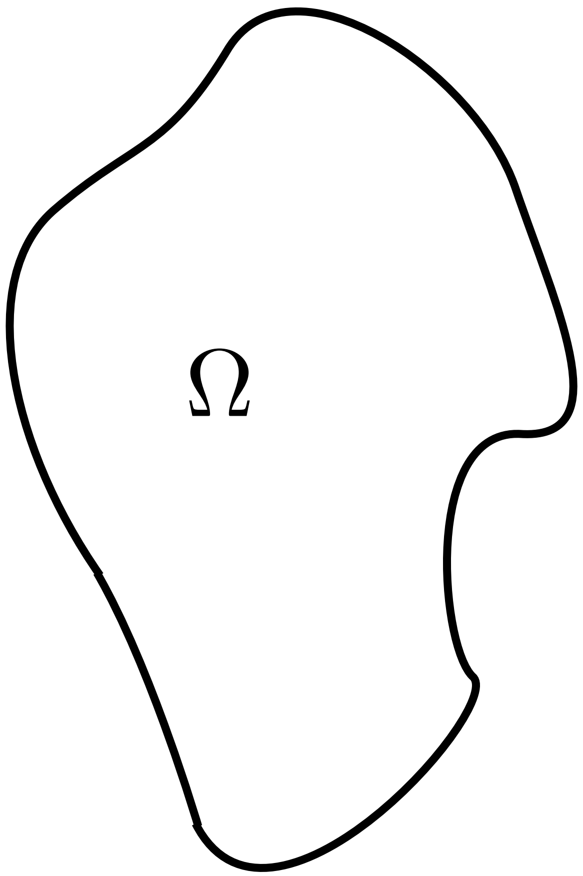
Elastic Wave Propagation into Soft Elastic Materials with Thin Layers

Aliénor Burel

(Project POems INRIA + Paris-Sud XI)

Joint work with Patrick Joly (Project POems INRIA/ENSTA/CNRS)
Sébastien Imperiale (Columbia University, NY)
Marc Duruflé (Project Bacchus INRIA Bordeaux)

What is Numerical Analysis?



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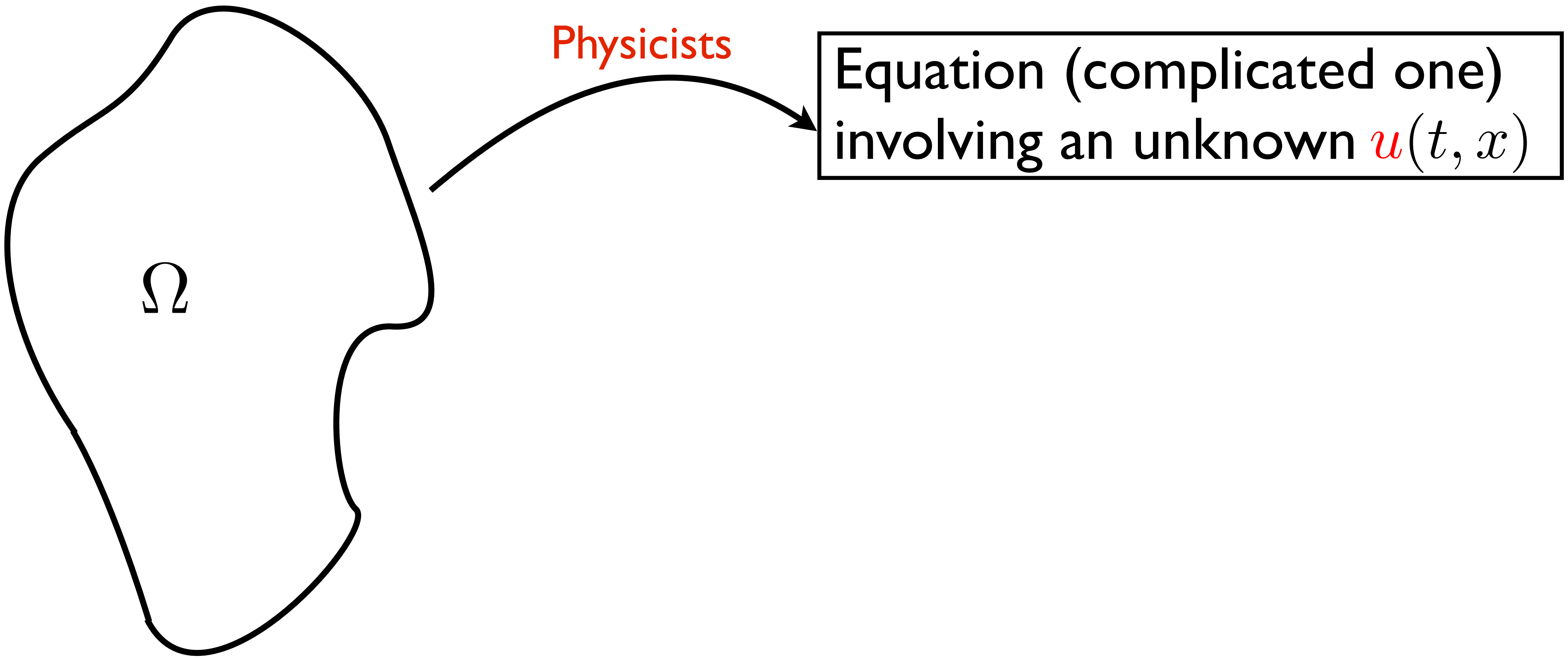

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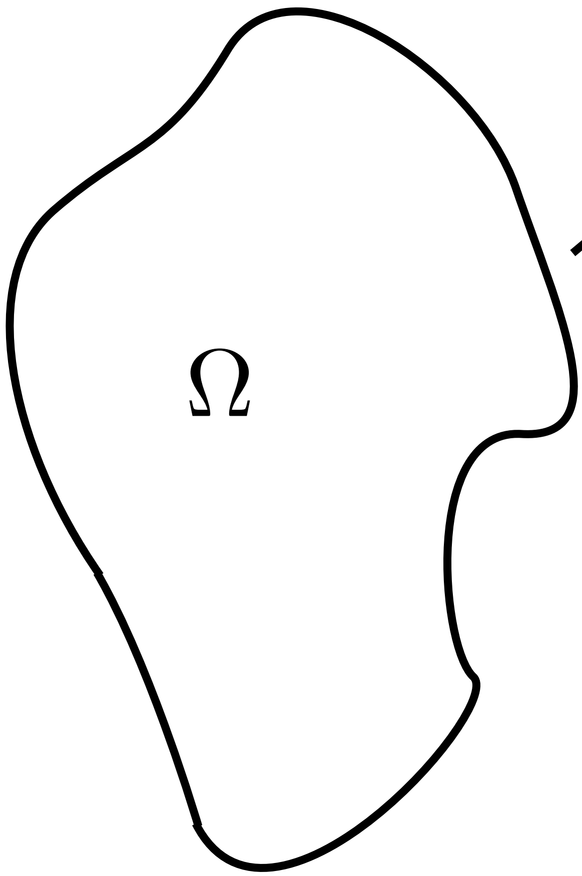

Physicists

Equation (complicated one) involving an unknown $u(t,x)$

Theorical Mathematicians

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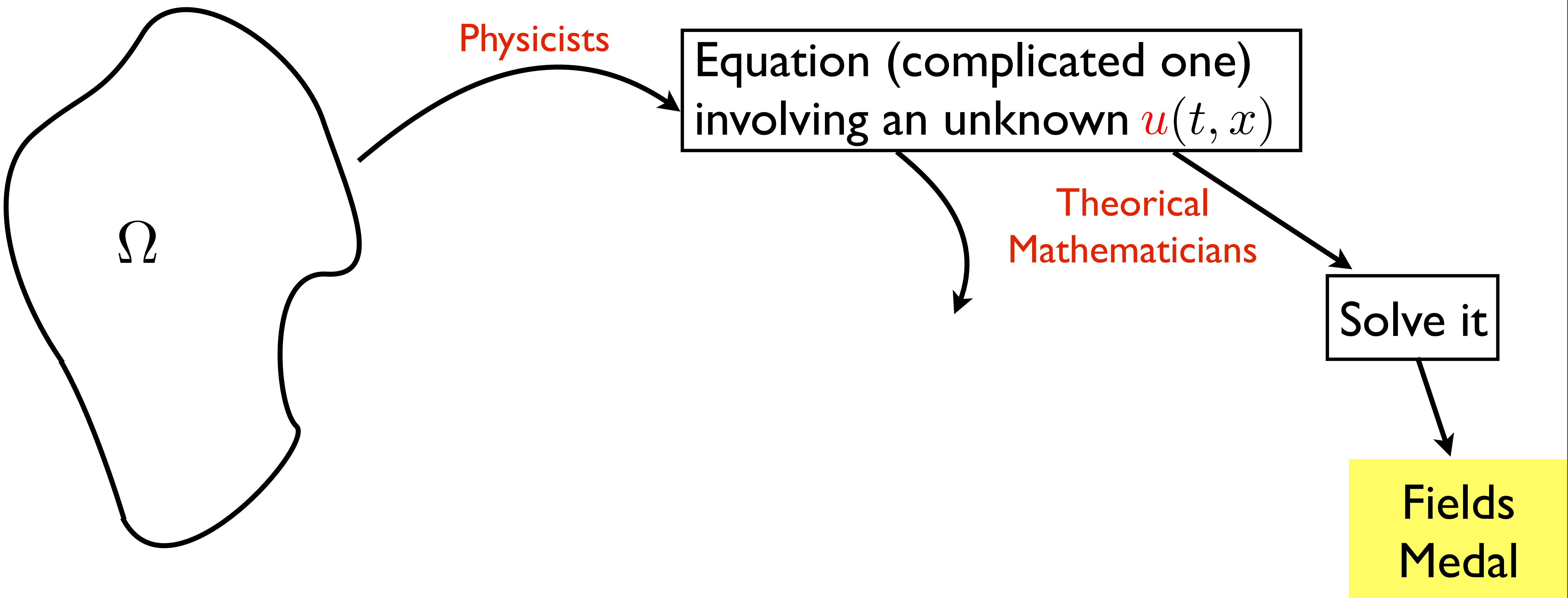

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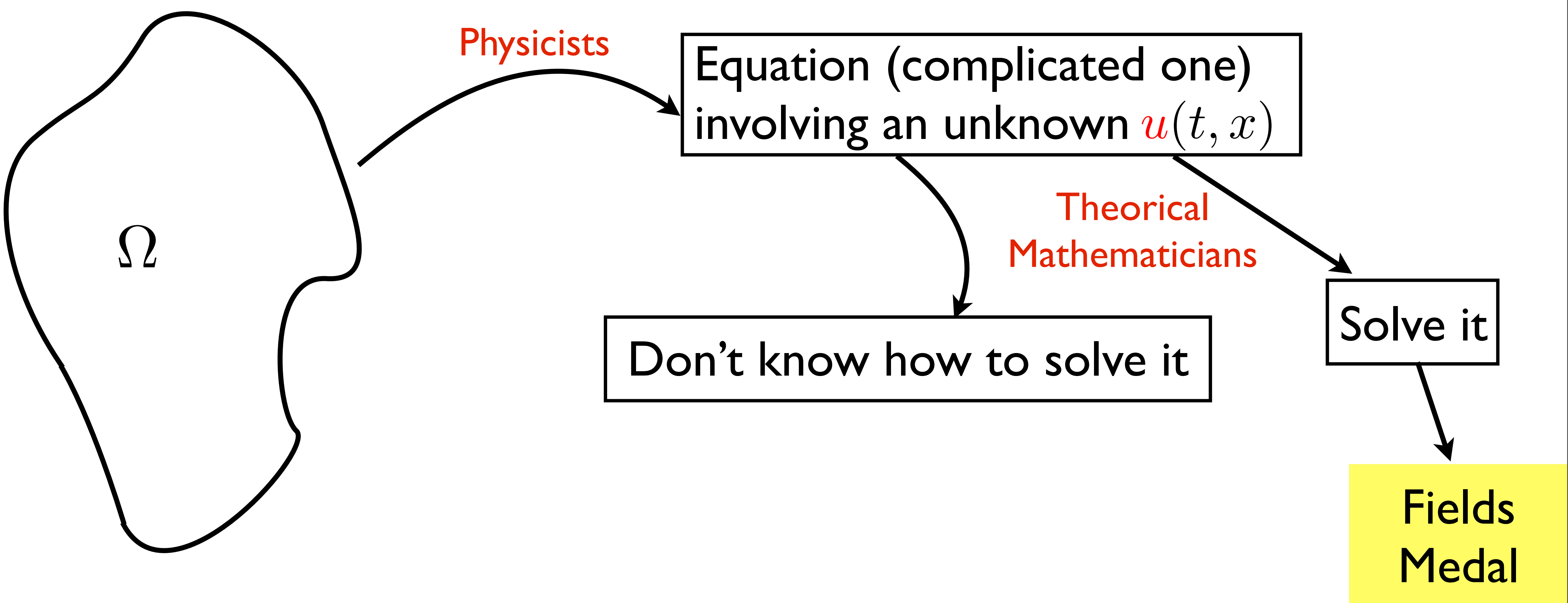

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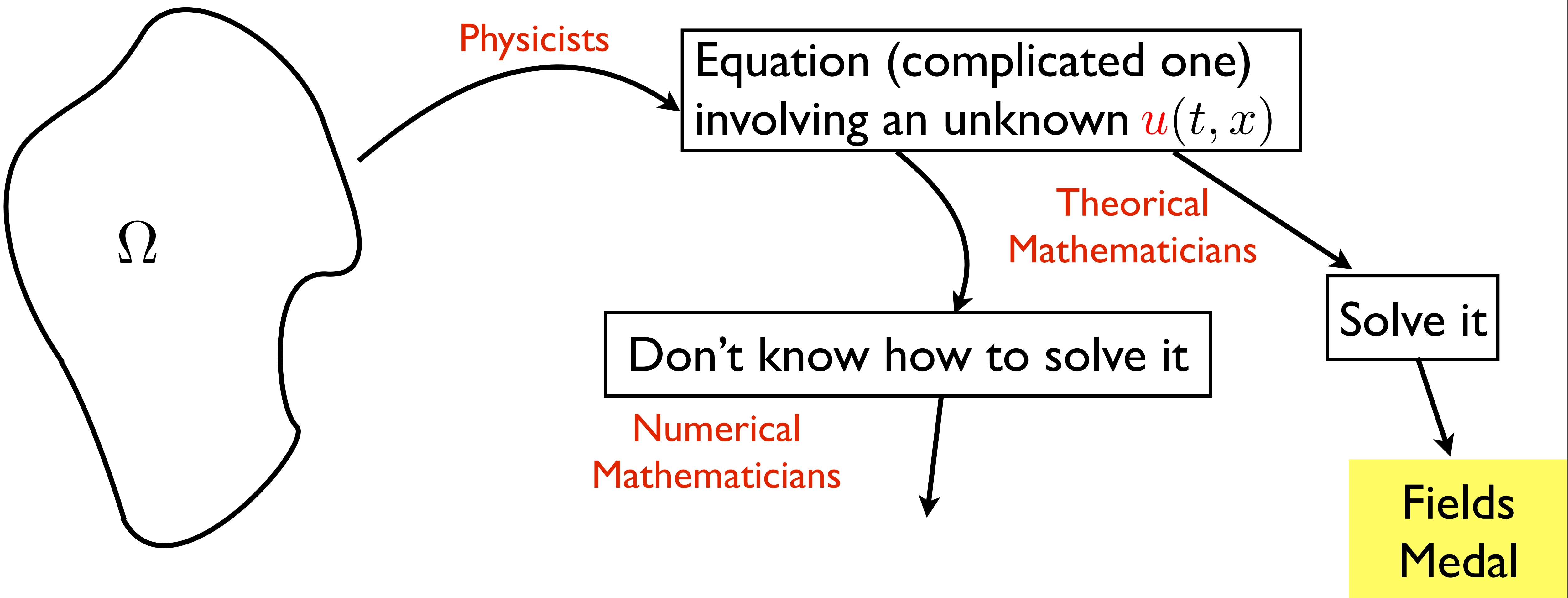

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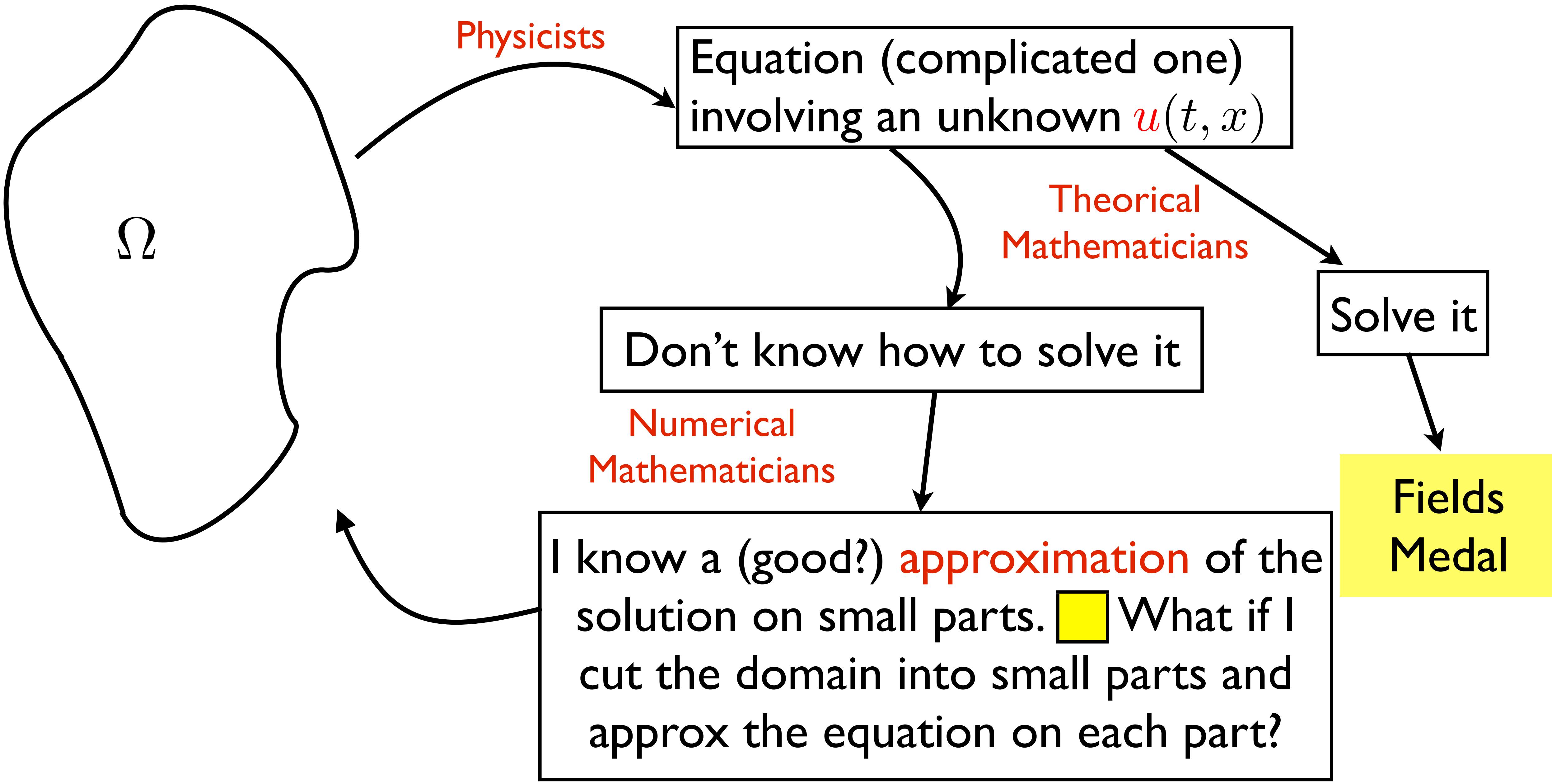

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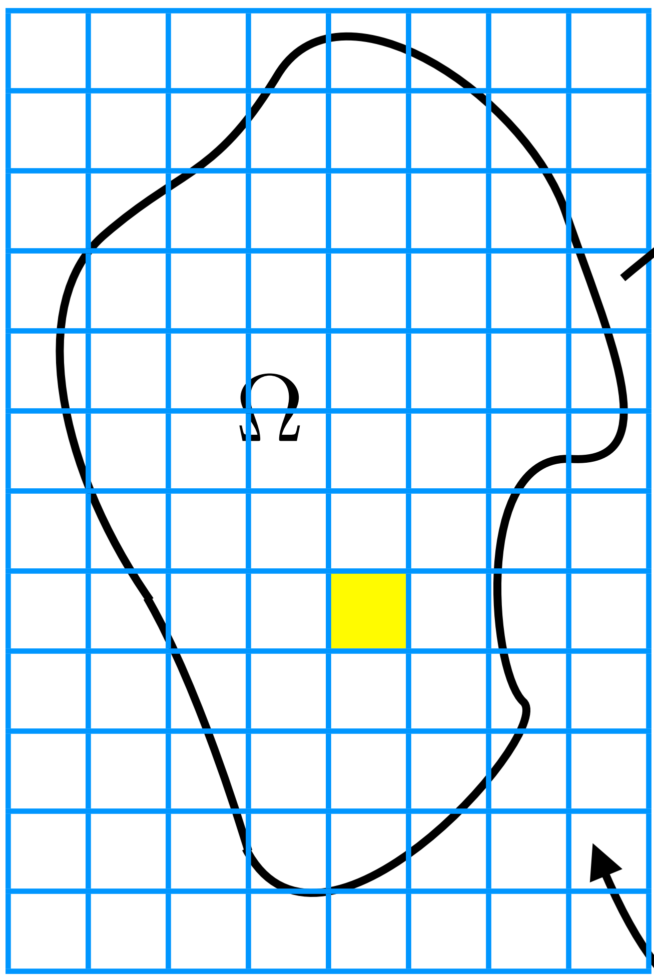

Mesh

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Equation (complicated one) involving an unknown $u(t, x)$

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Solve it

Fields Medal

Don't know how to solve it

Numerical Mathematicians

I know a (good?) approximation of the solution on small parts. What if I cut the domain into small parts and approx the equation on each part?

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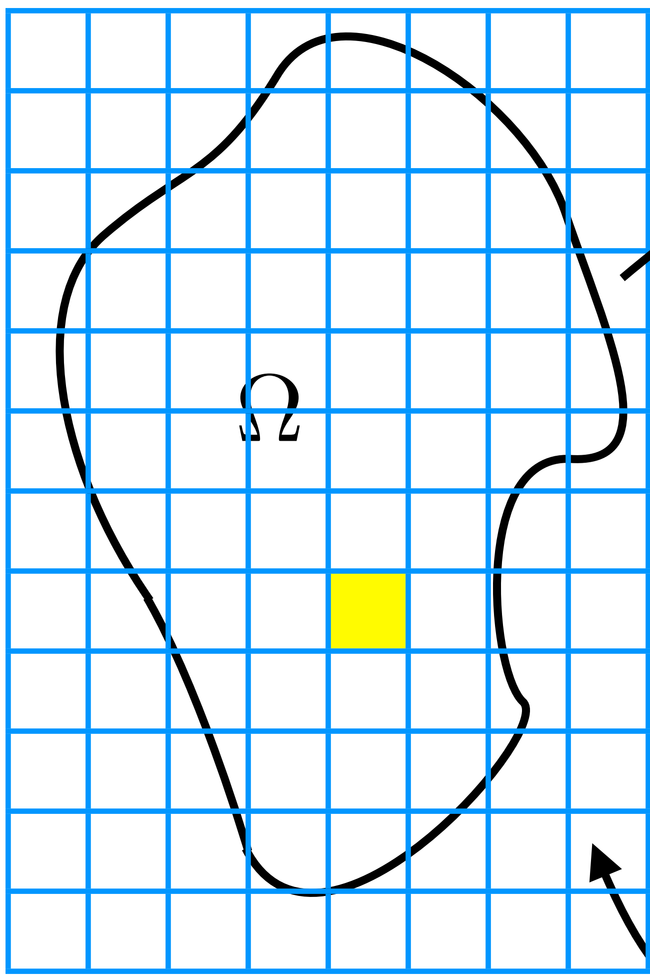

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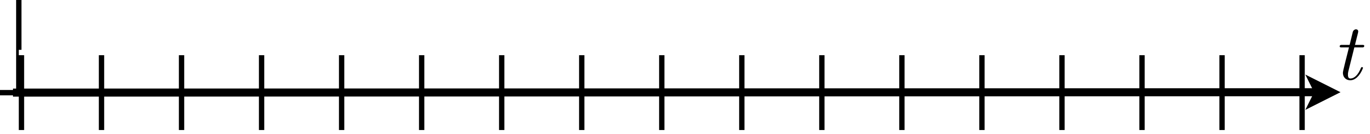
If my equation is dependent on time, I also choose to cut the time into small time steps



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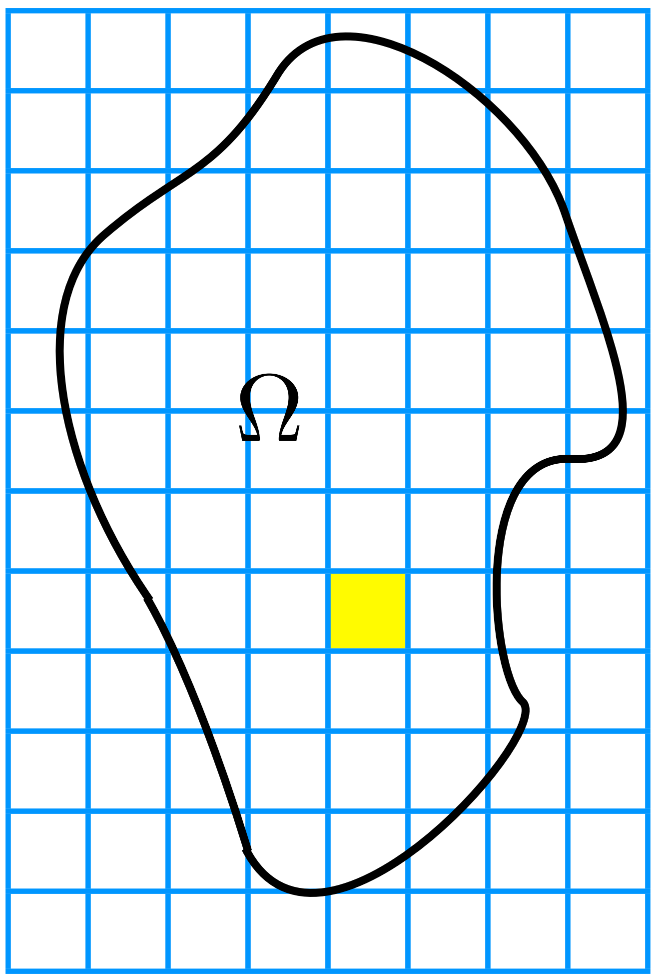

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From now, what are the problems ?

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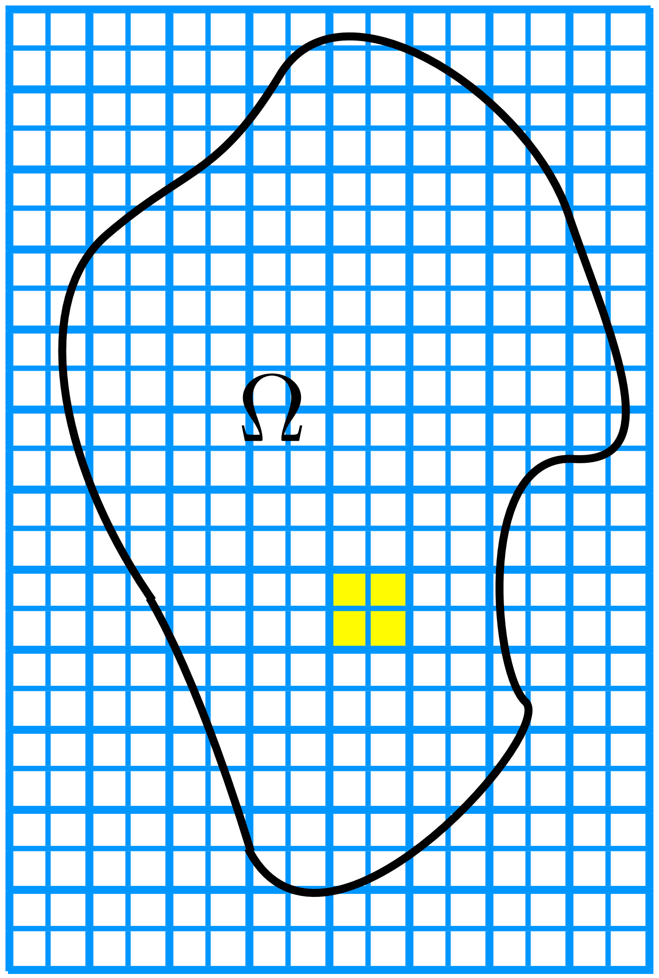

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Size of the mesh

The more squares, the better the approximation

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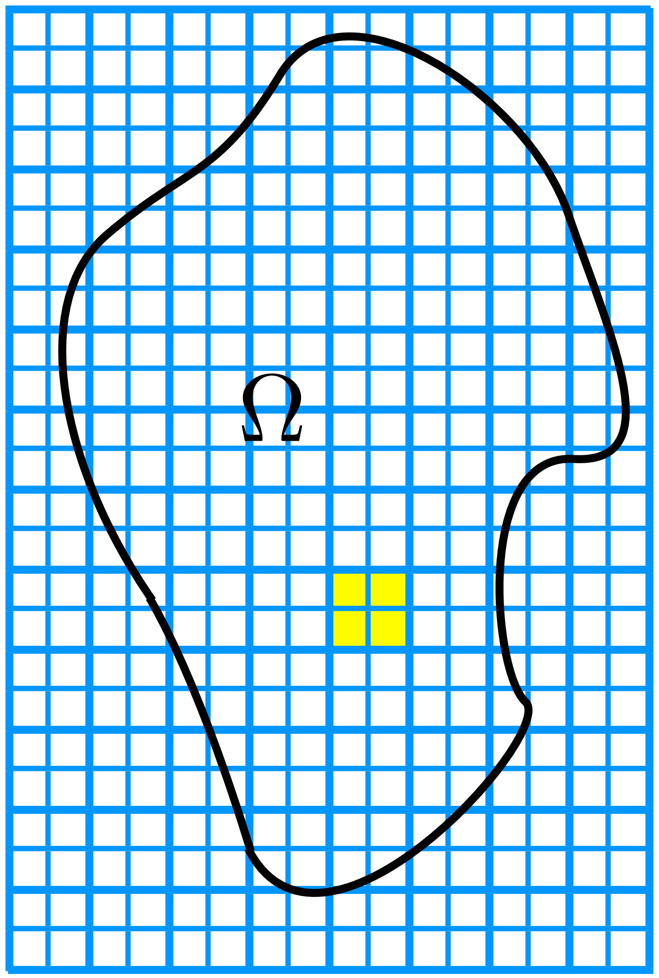

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From now, what are the problems ?

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- Quality of the approximation of the equation
  - Choice of a scheme

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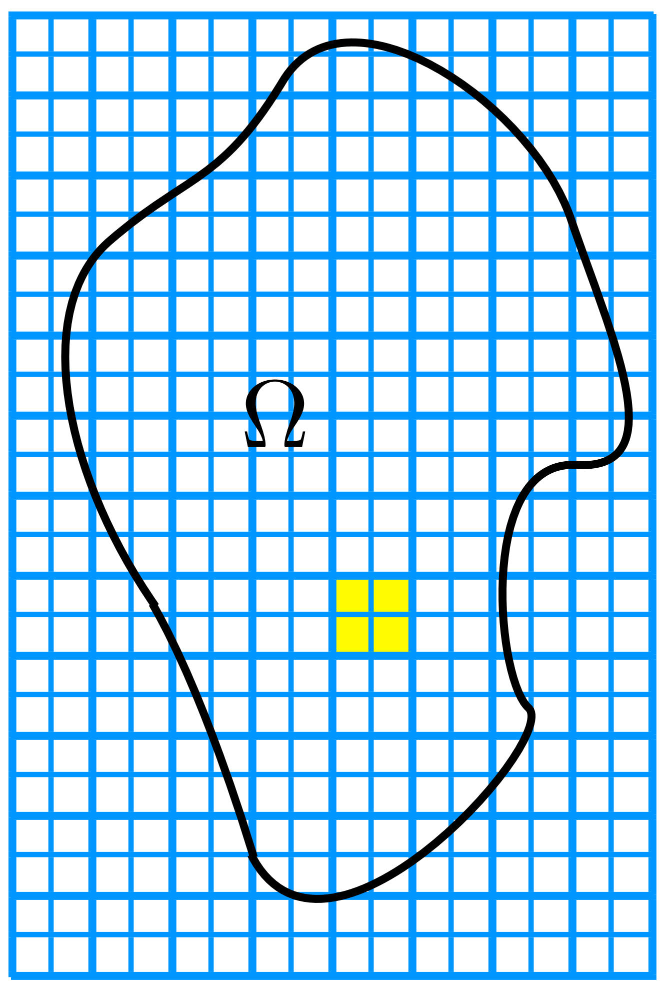

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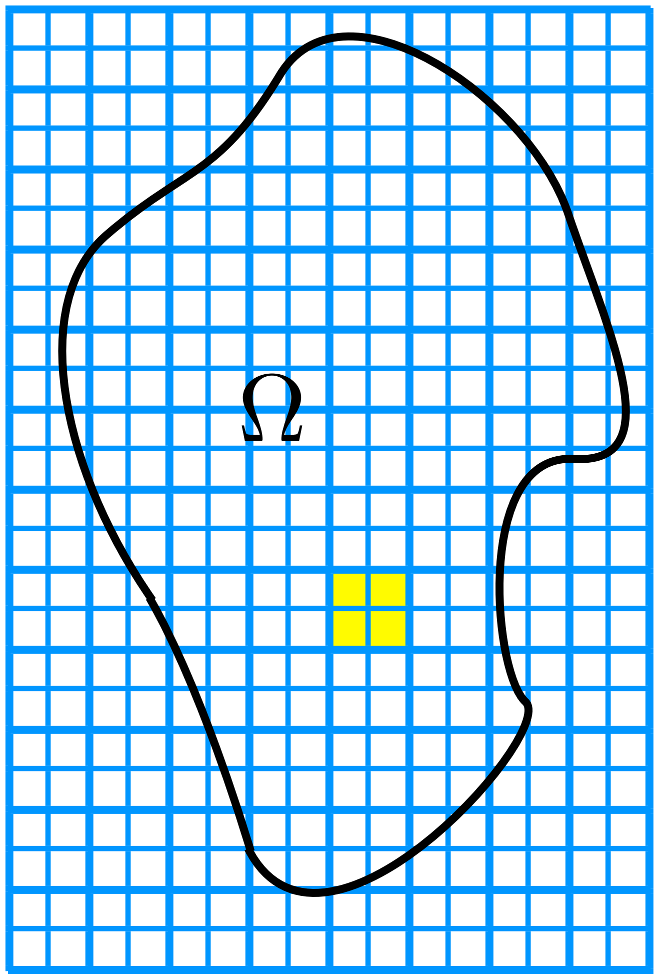

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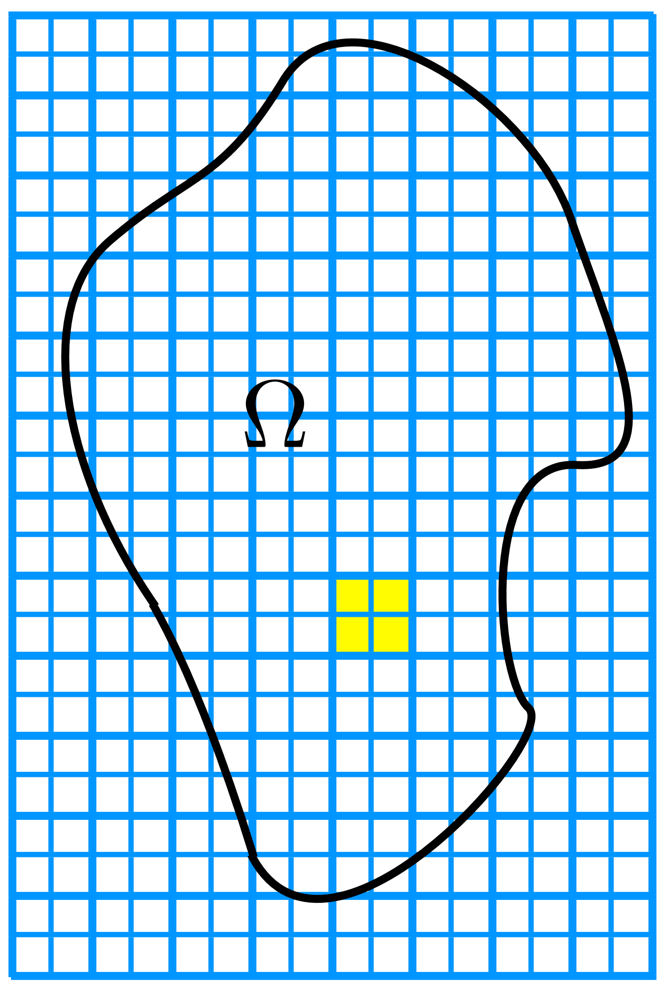

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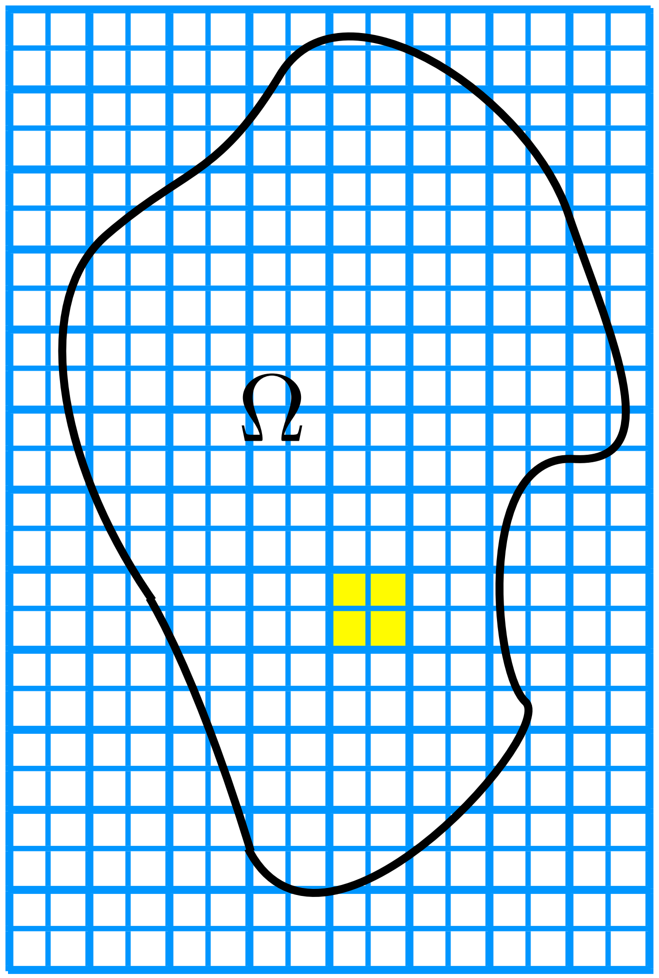

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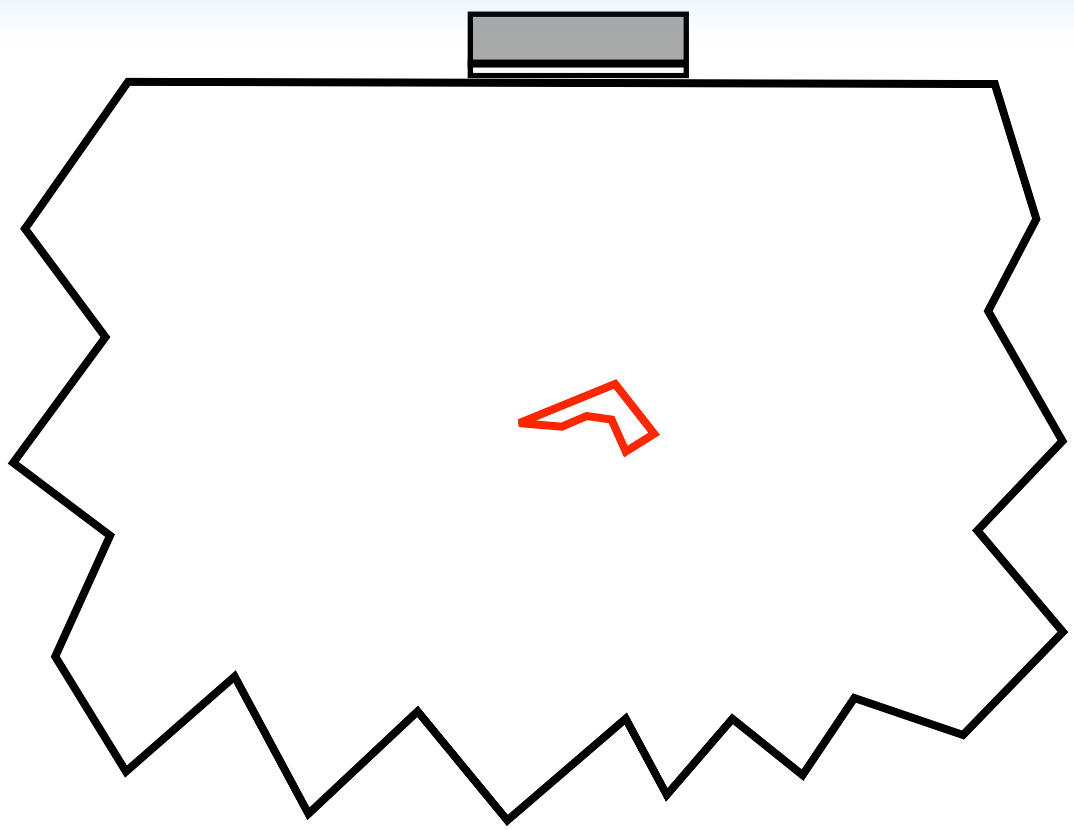
The equation given by the scheme is supposed to be consistent with the initial problem and the solution to be stable (doesn't explode) if the problem is well-posed

Frame

We study ultrasonic wave propagation in order to work on Non-Destructive Testing.



Aim: test the shape or the structure of a piece or an organ without destroying it...

Examples: ultrasonic testing of industrial pieces such as rubber, detection of stones in the kidney, ultrasound of organs...

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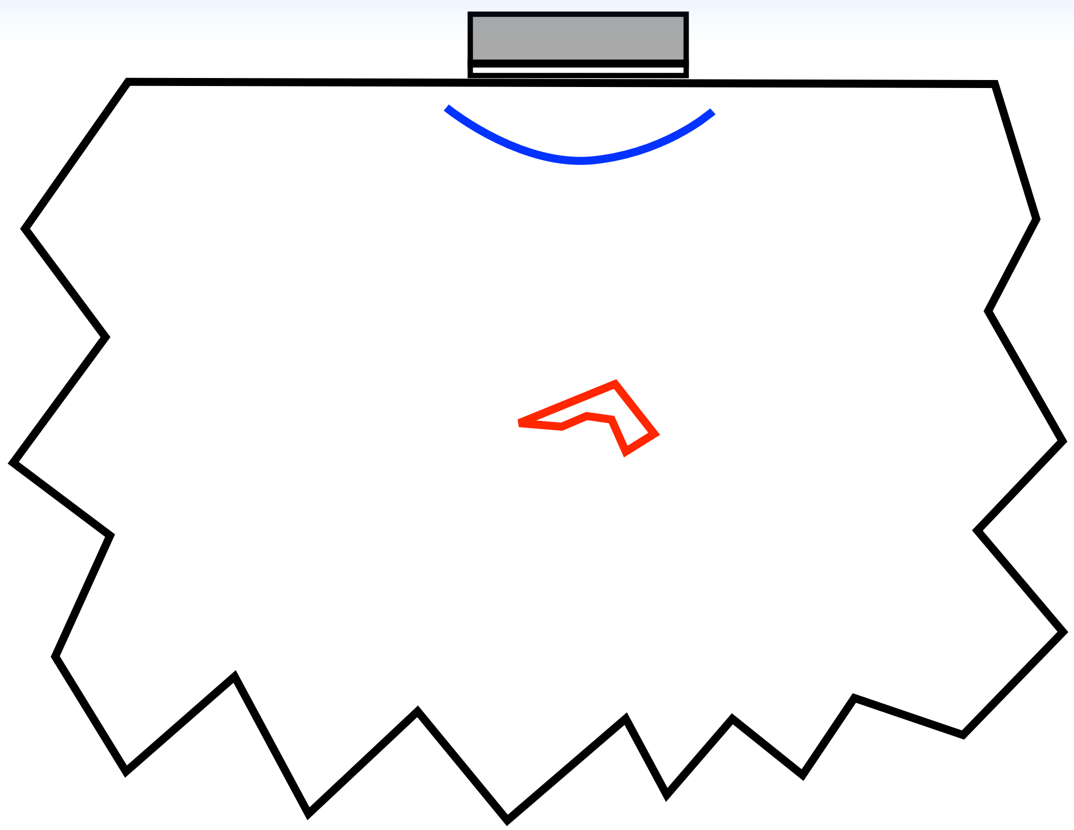

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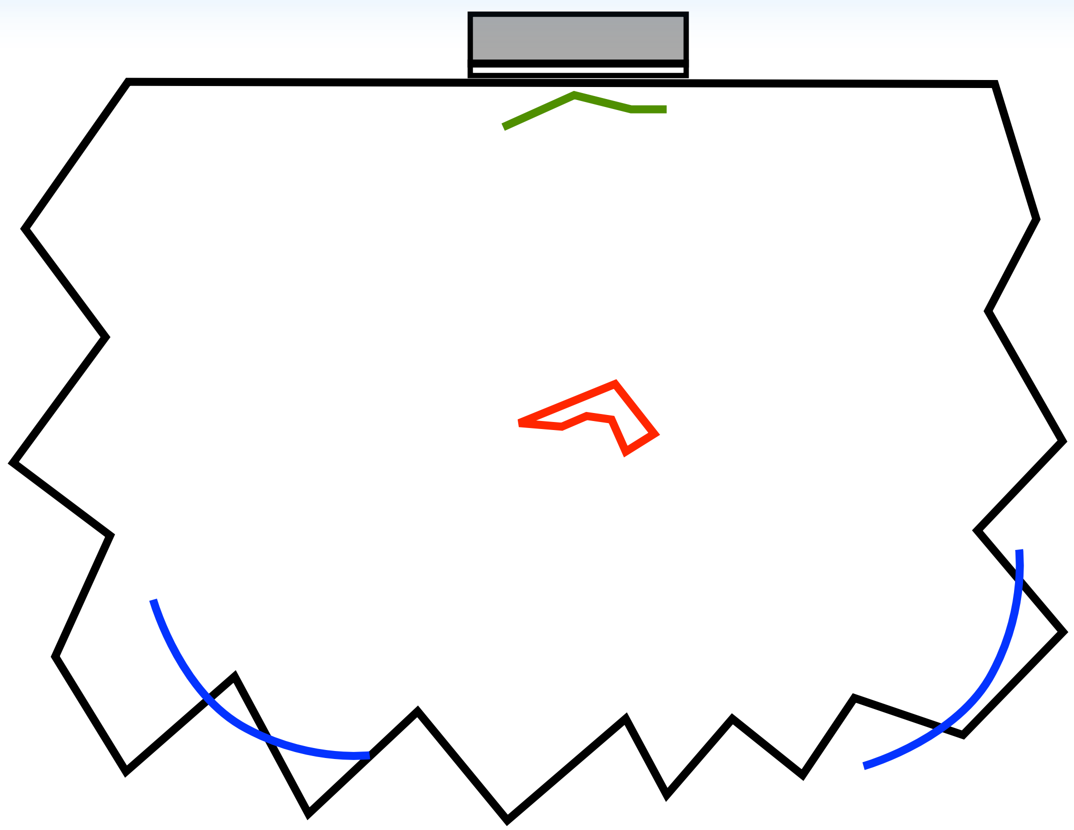

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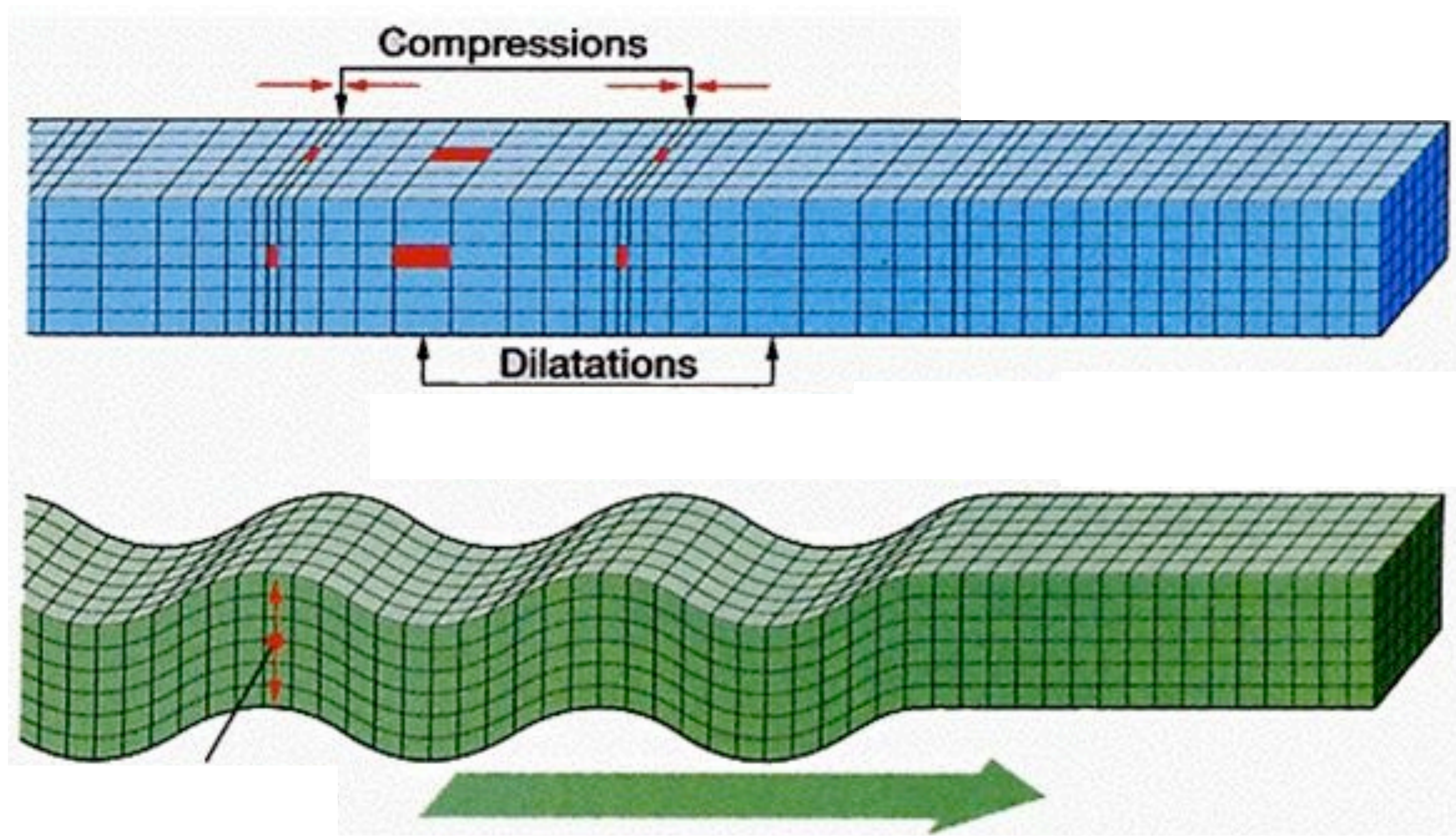
Motivations

Two wave types (P and S) in homogeneous isotropic materials with different speeds:

Pressure wave

Shear wave



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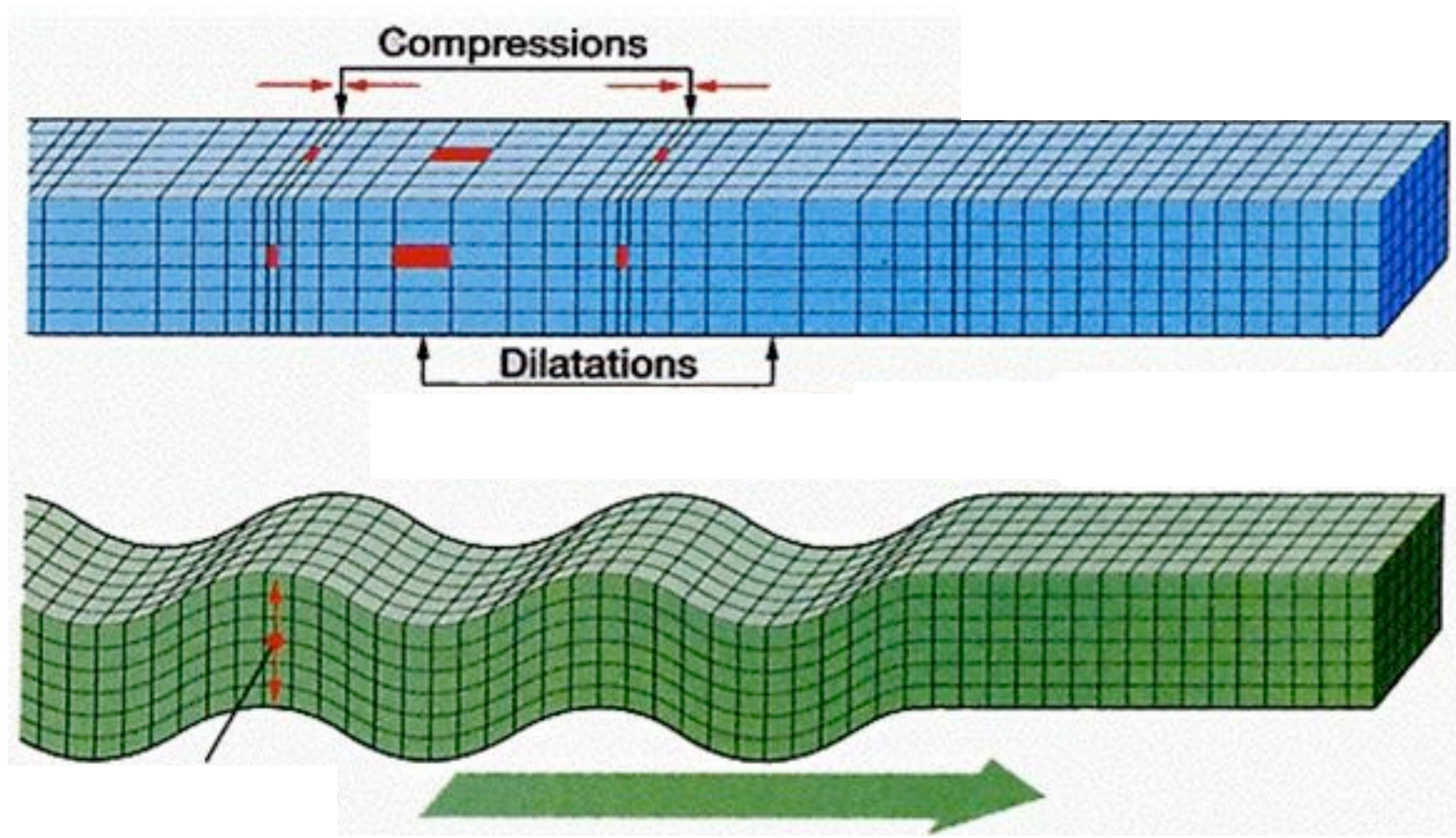

P-wave speed > S-wave speed

$$V_P > V_S$$

Thesis Motivations

Part 1

In soft materials (rubber, human organs...) the P-wavespeed can be very high compared to the S-wavespeed: a hundred times larger...

In the classical numerical studies, this knowledge is not taken into account.

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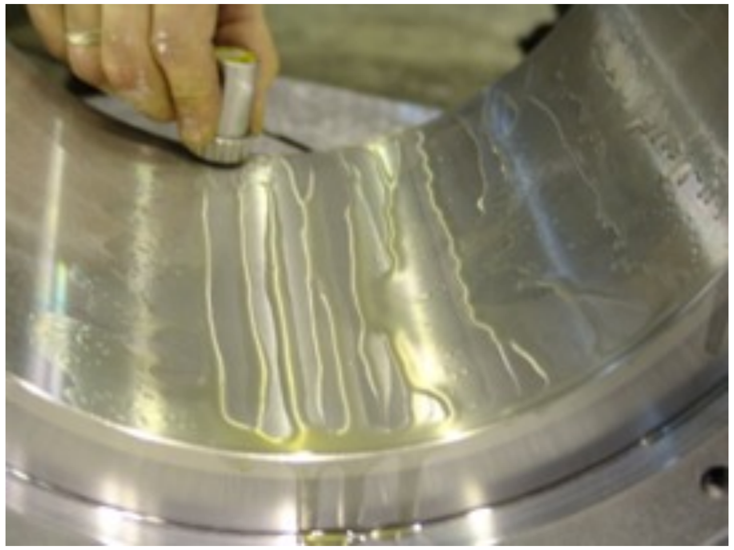
Compute separately the simulations of the P-waves and S-waves?

Thesis Motivations

Part 2

Ultrasonic waves amplitude decreases very fast in the air (doesn't exist in the vacuum). We need a gel which allows to keep the solid/solid contact.



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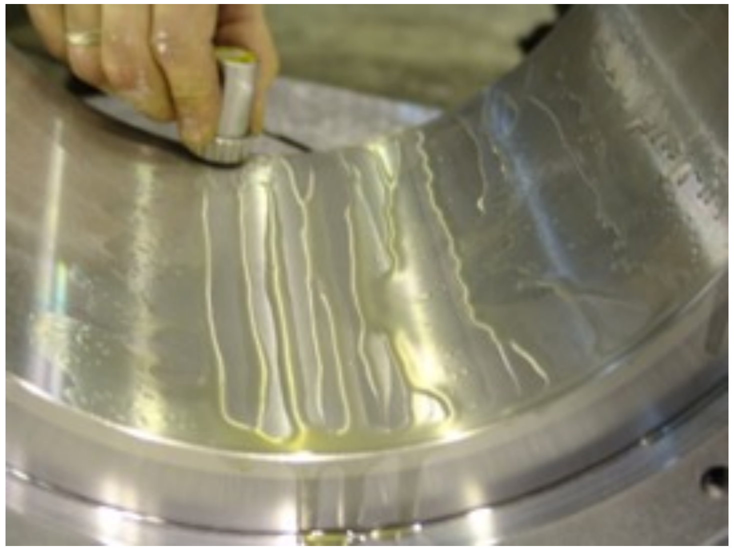

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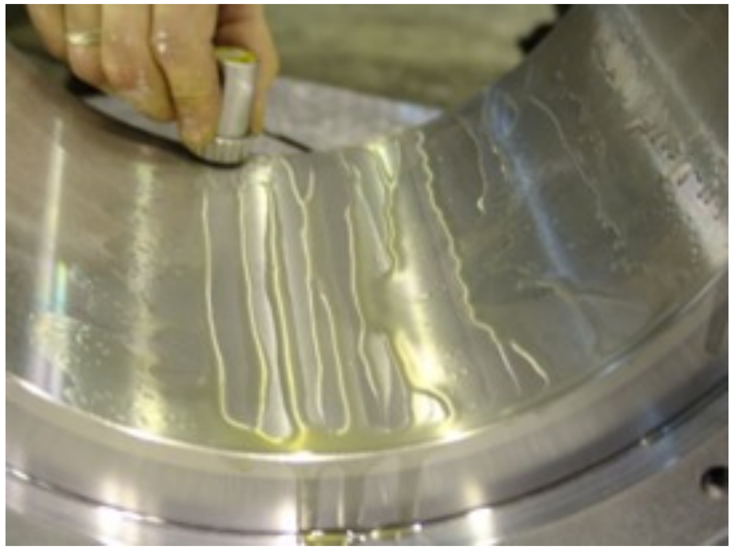

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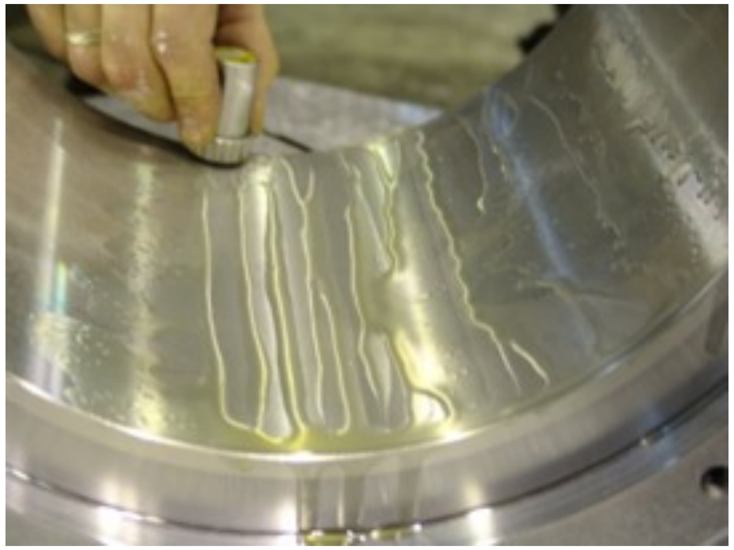

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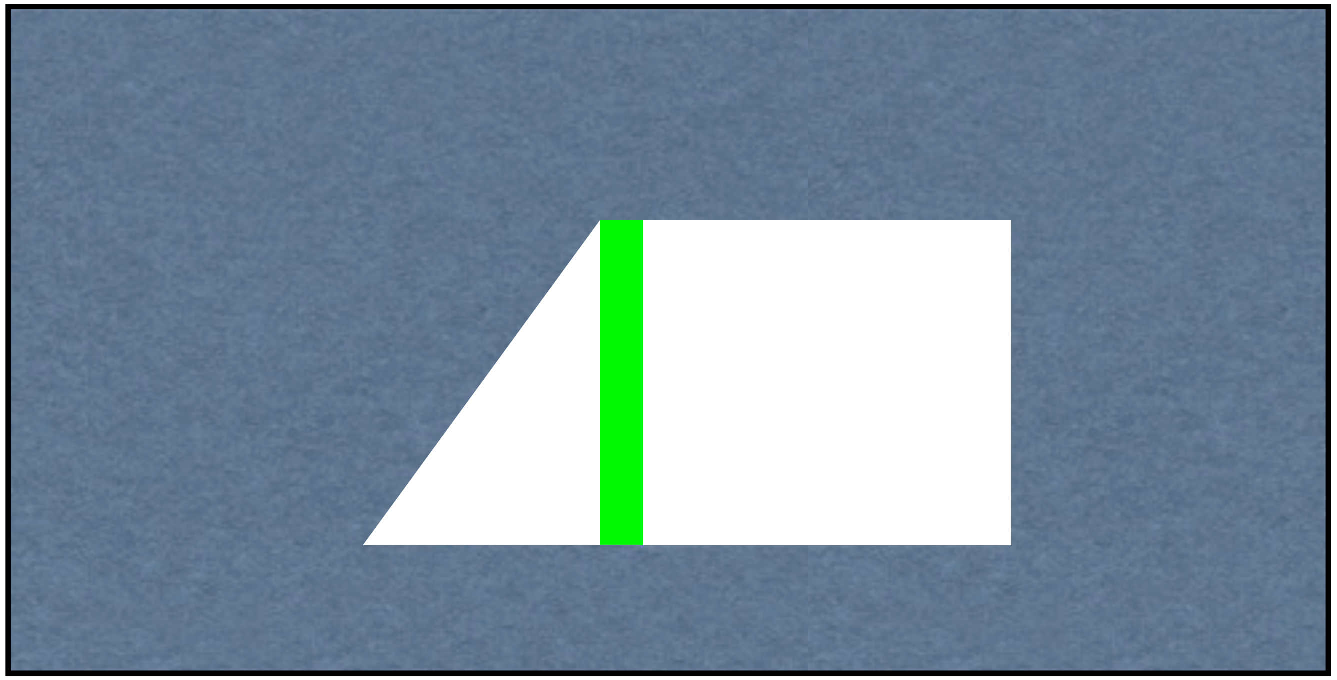

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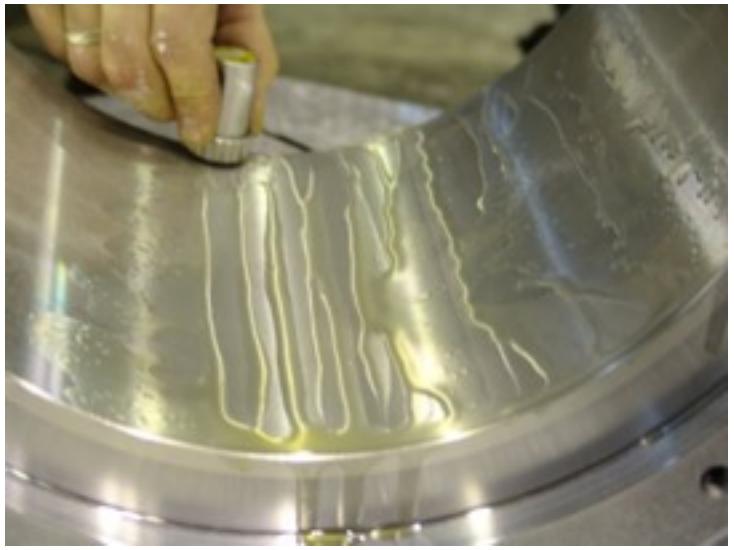

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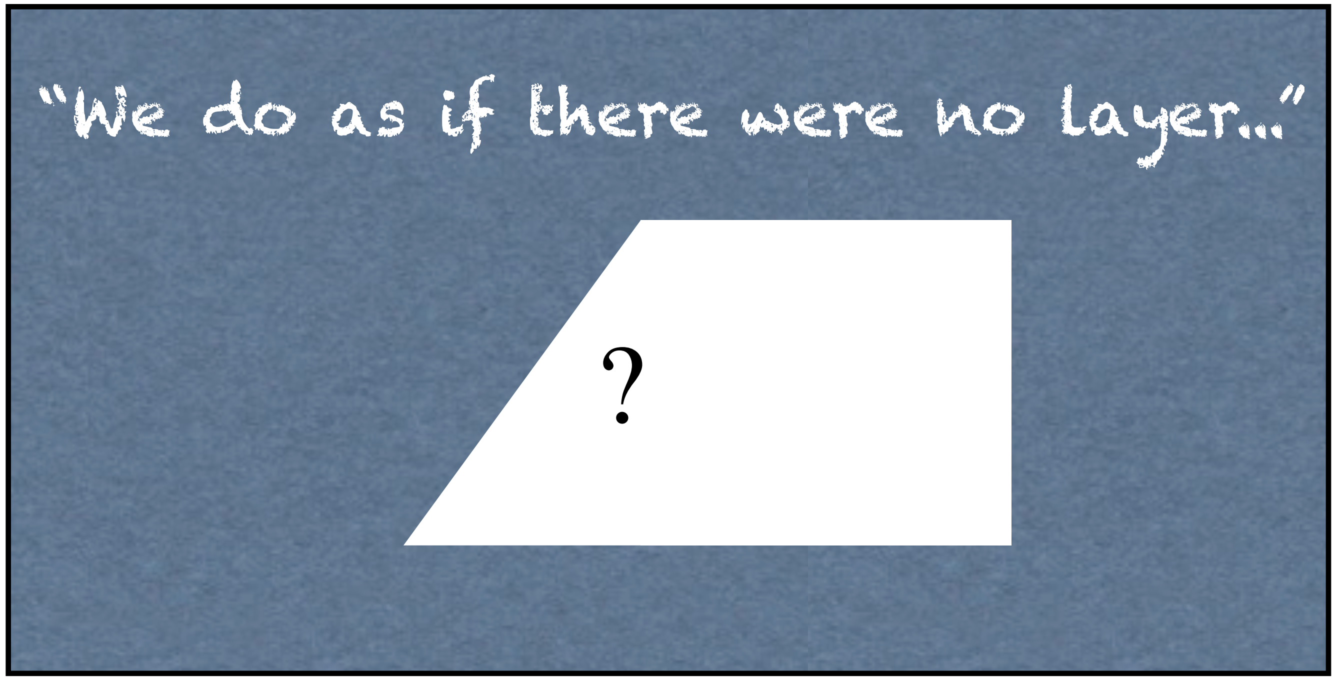

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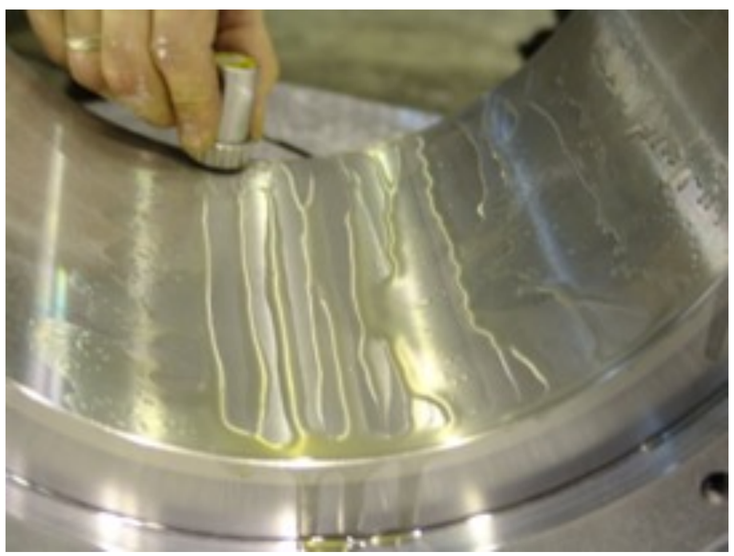

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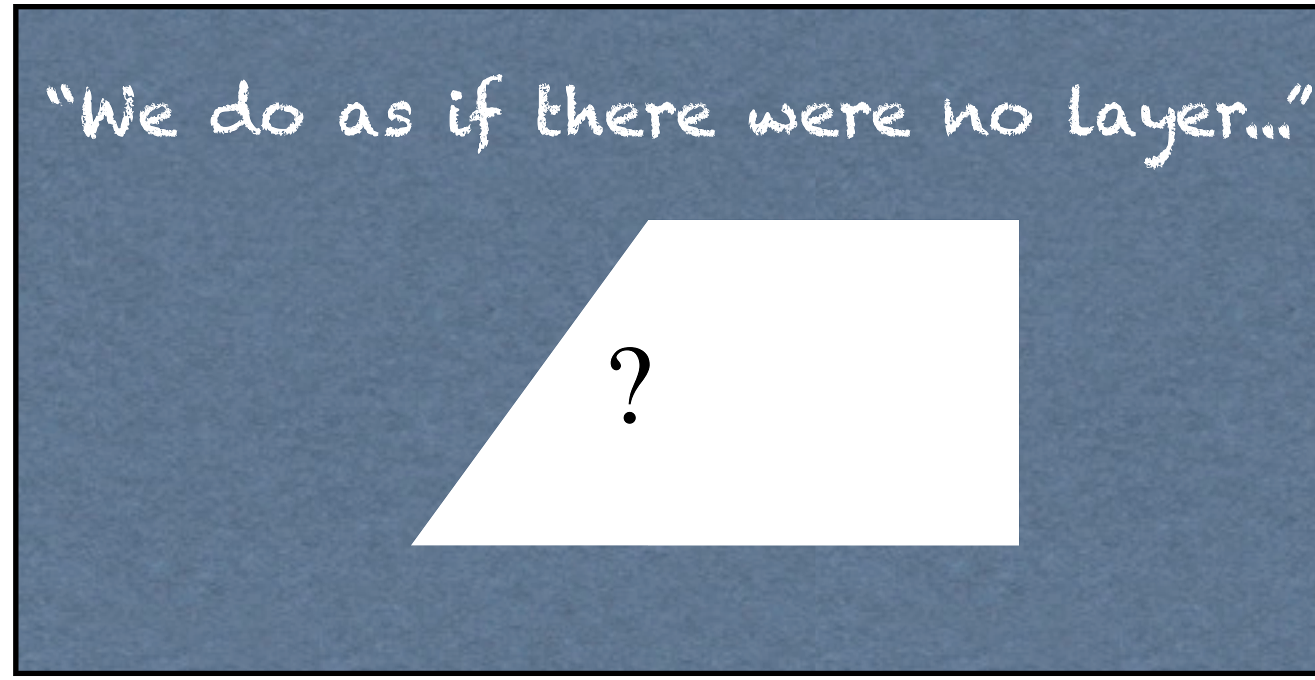

Collages between materials are very often forgotten in the simulations.

Necessity of a precise study of these thin layers and the conditions of its cancellation

Goals

Is this wing safe?

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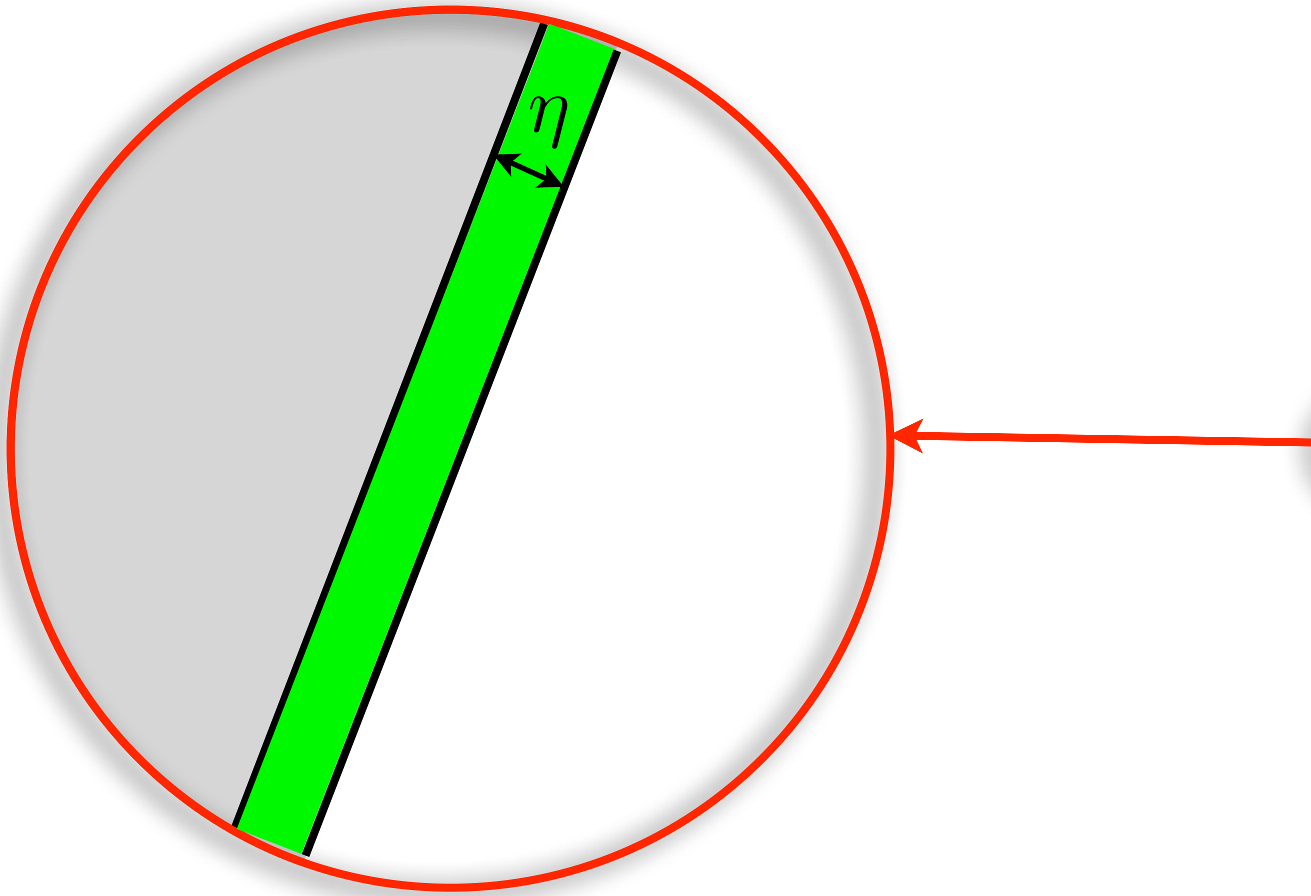

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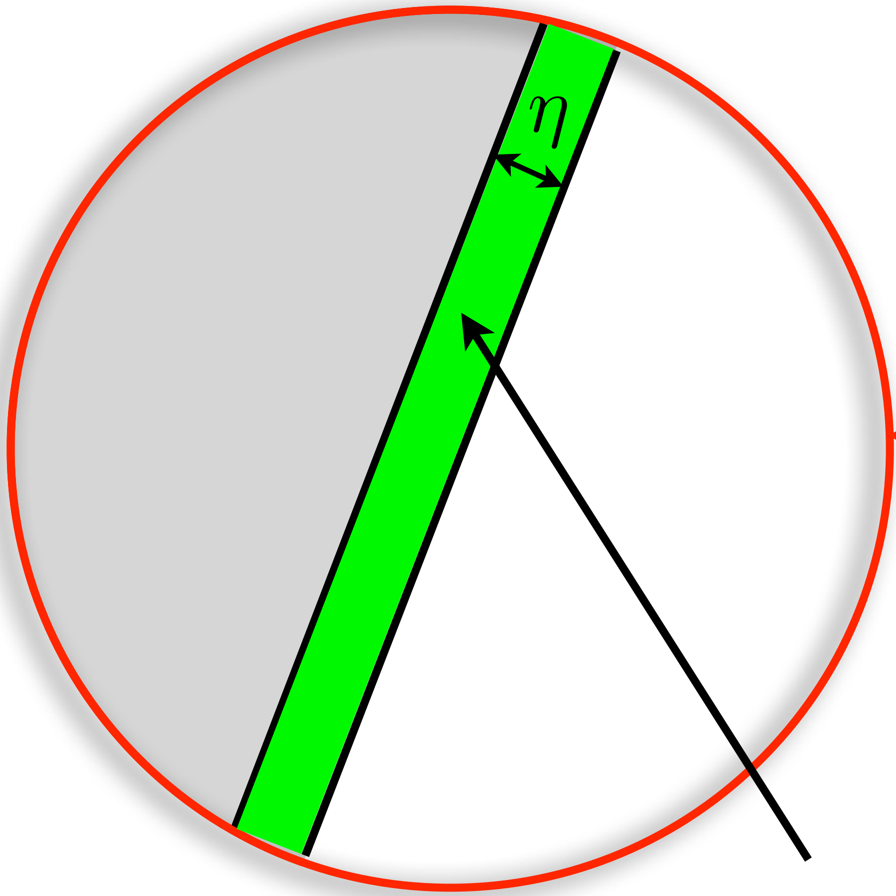

Glue: soft material
different wavespeeds

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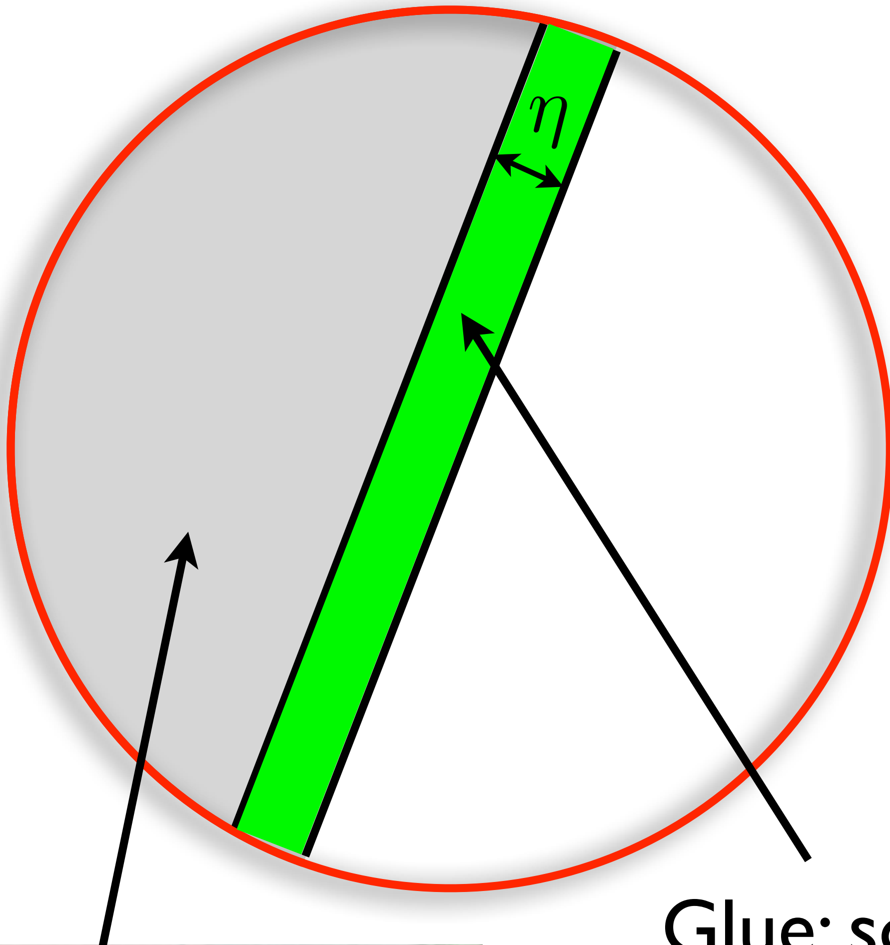

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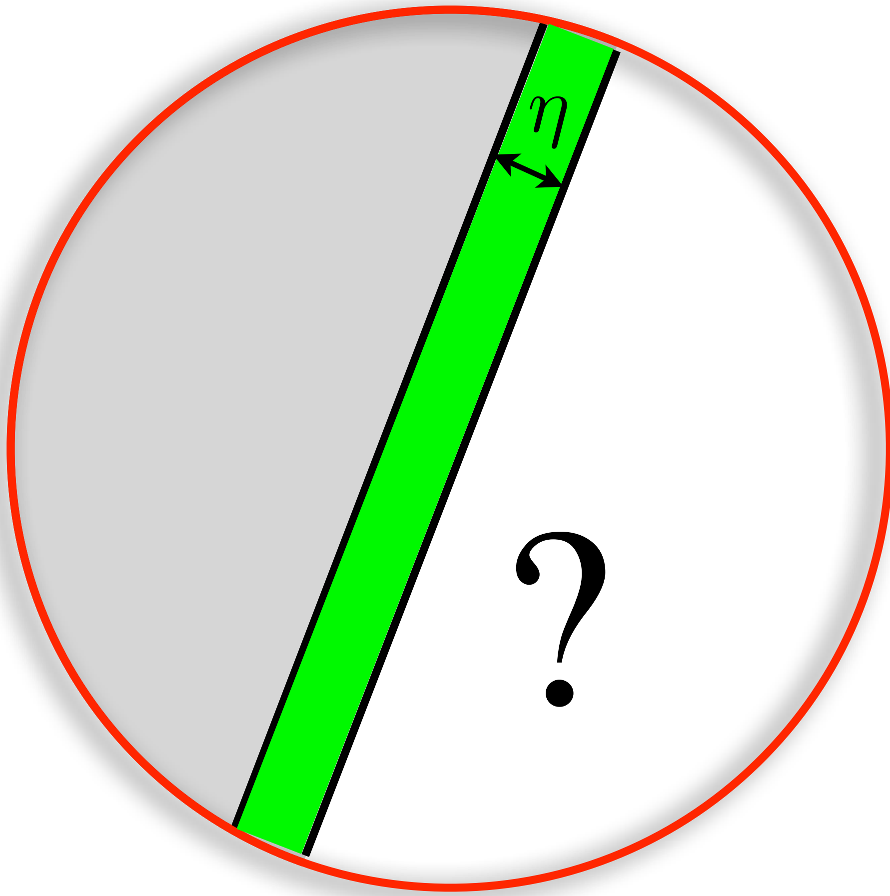

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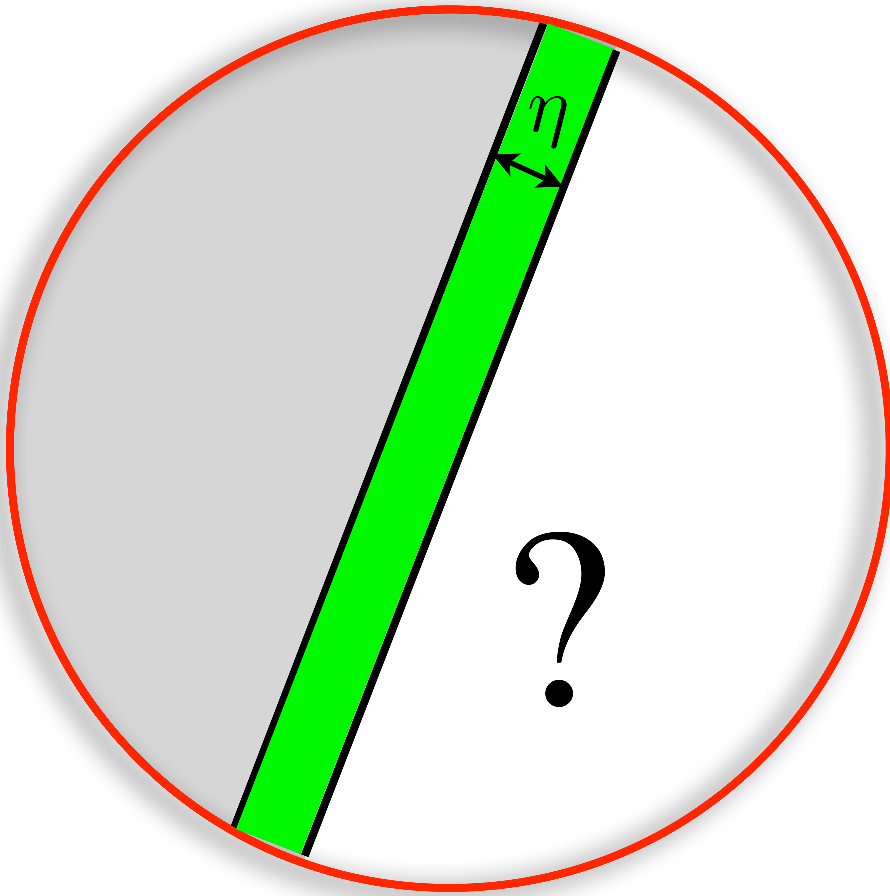

Is this wing safe?

Can we study more precisely the propagation of each wave type into the soft medium? → **Part I**

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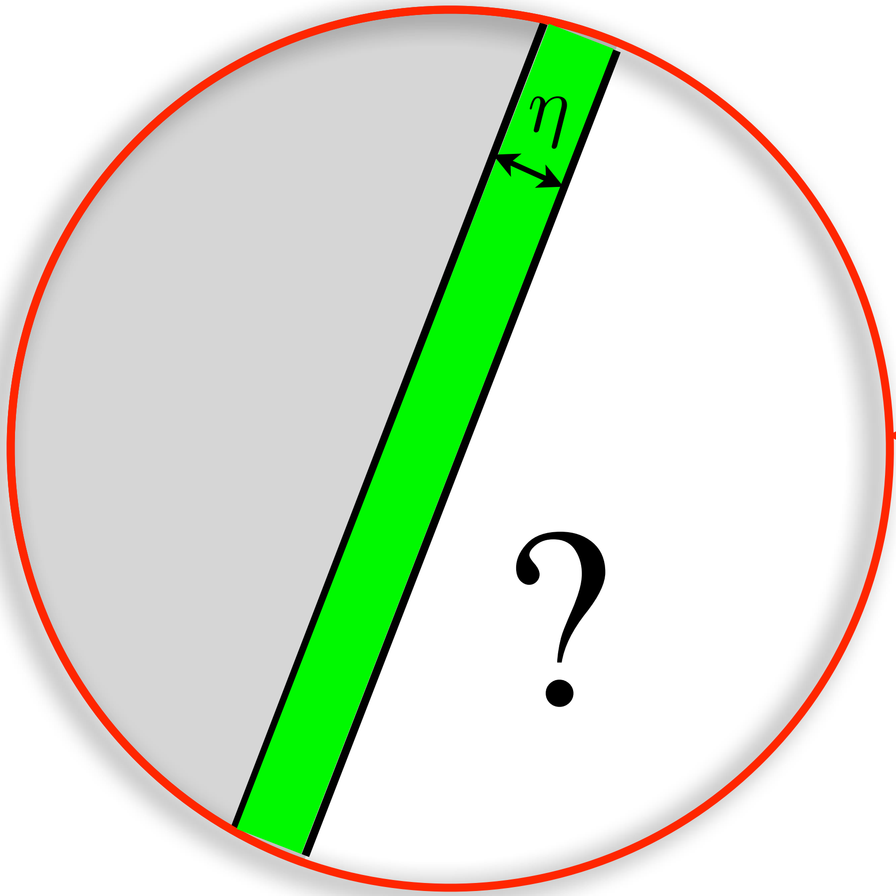

Is this wing safe?

Can we study more precisely the propagation of each wave type into the soft medium? → **Part 1**

If the width is small enough, can we compute "as if" there were no glue? → **Part 2**

PART I

Using potentials in elastodynamics: a challenge for finite elements methods

Motivations

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Yes, if we know how to...

Separate mathematically the P-waves and S-waves

Formulas, properties, tricks

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And if we know how to...

Separate numerically the P-waves and S-waves

Numerical scheme

Verification that our numerical methods are stable

Motivations

Can we study more precisely the propagation of each wave type into the soft medium?

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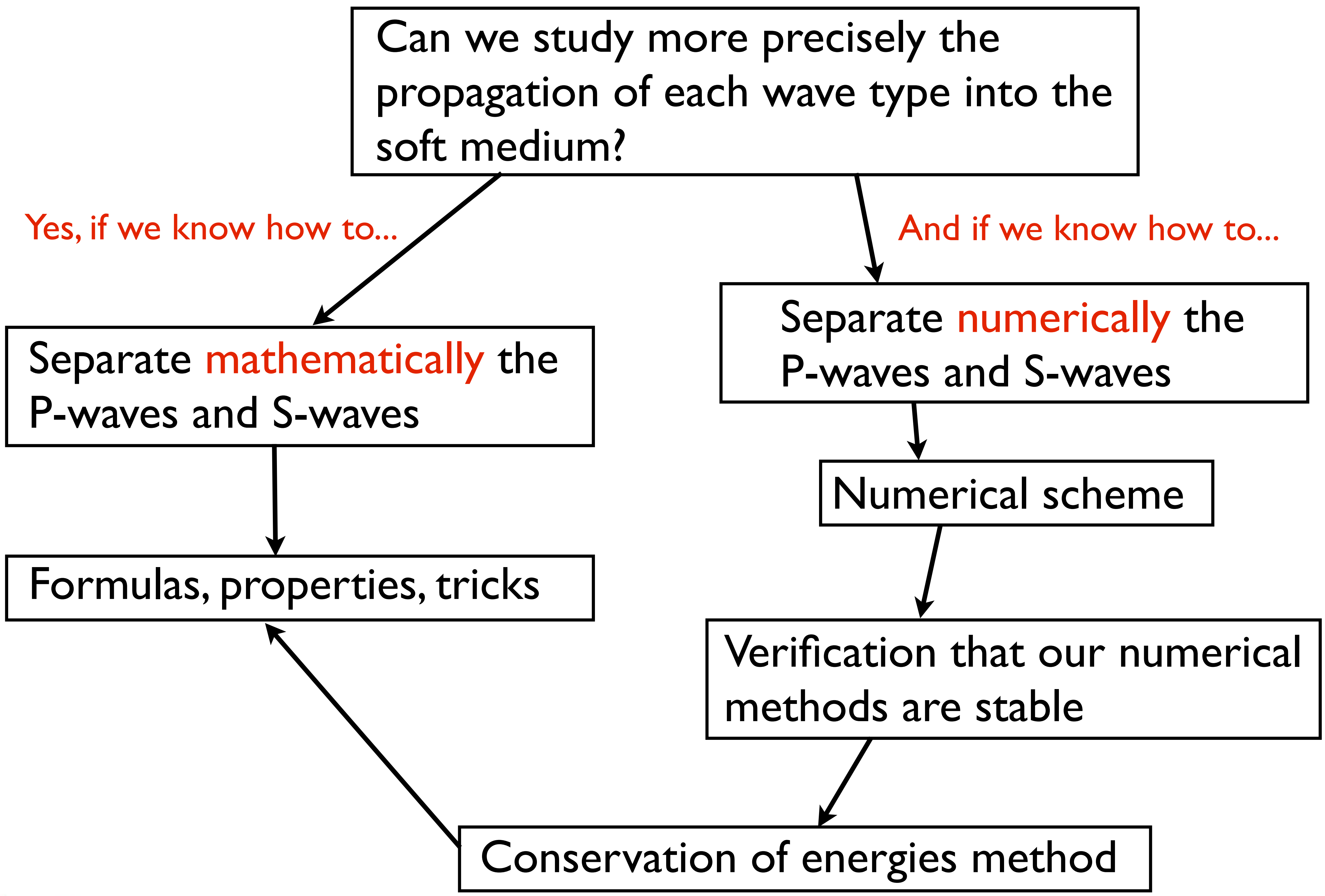
- Separate mathematically the P-waves and S-waves
- Formulas, properties, tricks

And if we know how to...

- Separate numerically the P-waves and S-waves
- Numerical scheme
- Verification that our numerical methods are stable
- Conservation of energies method

Motivations



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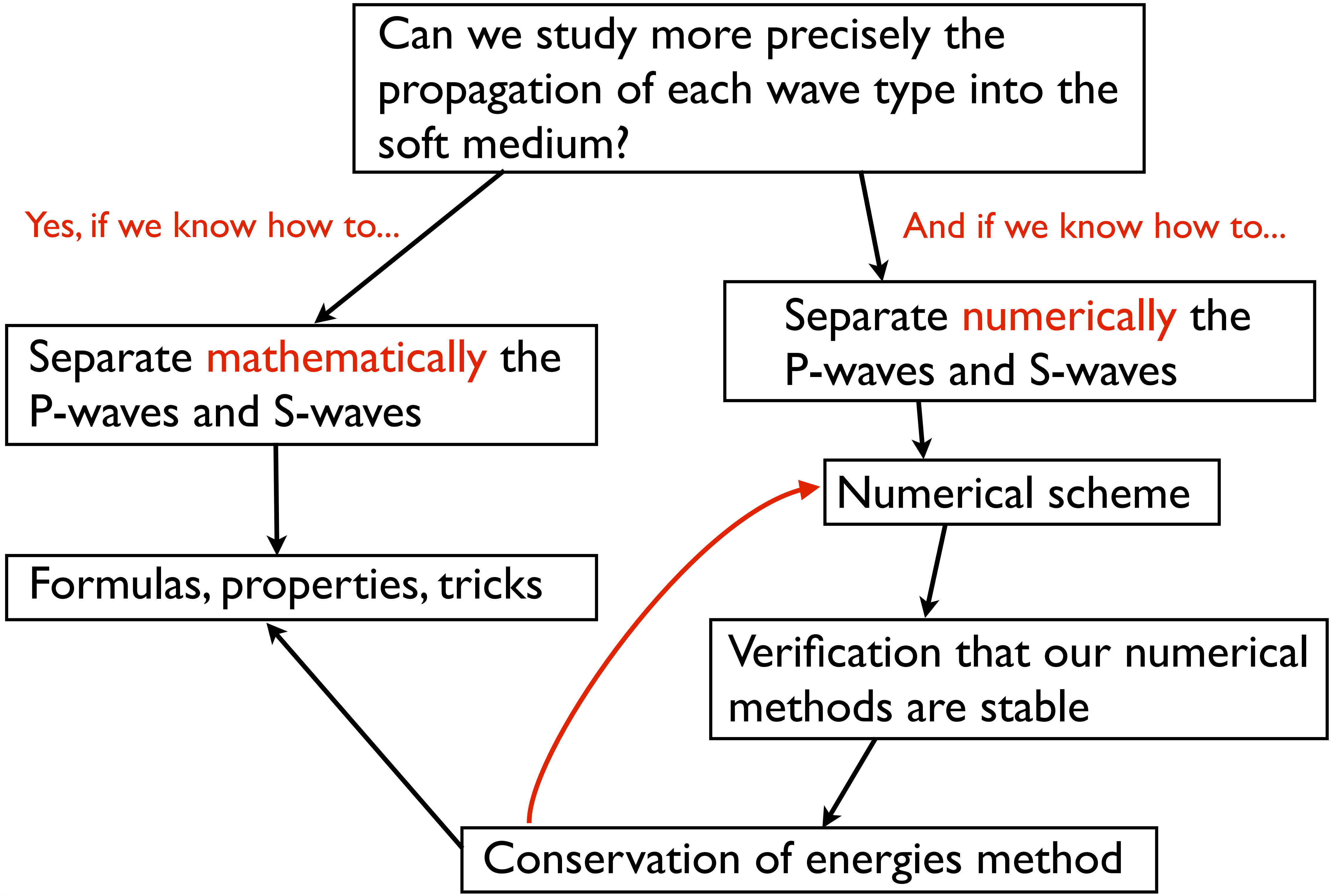
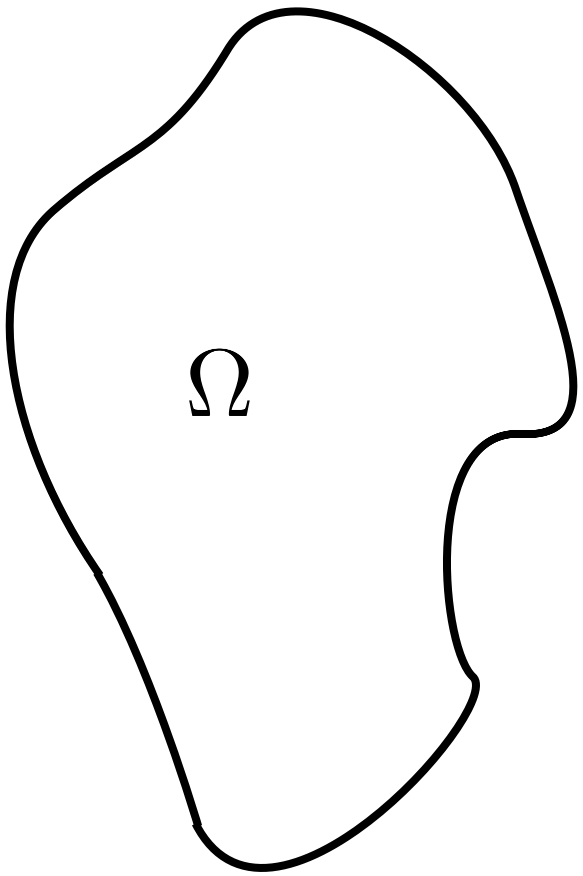


Wave propagation in elastodynamics



Homogeneous elastodynamic equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \operatorname{div} \sigma(u) = 0, \quad x \in \Omega, \quad t > 0.$$

In the isotropic 2D case, Hooke's law gives:

$$\sigma(u) = \lambda \operatorname{div} u + 2\,\mu\,\varepsilon(u)$$

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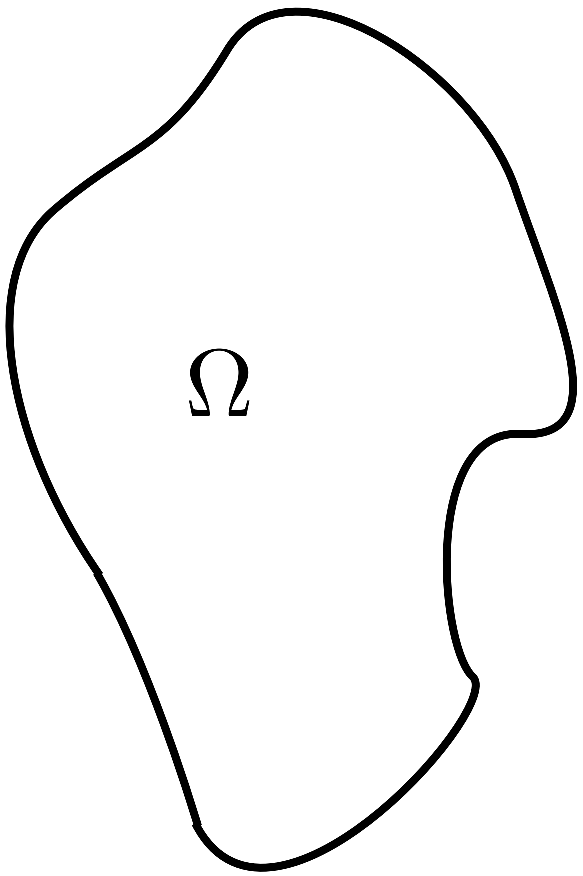

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$$\boxed{\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2\mu)\nabla(\operatorname{div} u) + \mu\, \overrightarrow{\operatorname{curl}}(\operatorname{curl} u) = 0}$$

Classical equation

$\lambda \quad \mu$ constants

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\overrightarrow{\operatorname{curl}}\, \varphi = \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}\right)^t \qquad \operatorname{curl} u = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}$$

Decomposition of the displacement field into P wave and S wave

$$\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2\mu)\nabla(\text{ div } u) + \mu \overrightarrow{\text{curl}}(\text{ curl } u) = 0$$

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$$\rho \frac{\partial^2}{\partial t^2}(\nabla\varphi_P + \overrightarrow{\text{curl}}\,\varphi_S) - (\lambda + 2\mu)\nabla(\Delta\varphi_P) - \mu\,\overrightarrow{\text{curl}}(\Delta\varphi_S) = 0$$

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$$\operatorname{curl}(\nabla) = 0$$

$$\operatorname{curl}(\overrightarrow{\operatorname{curl}}) = -\Delta$$

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$$\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2\mu)\nabla(\operatorname{div} u) + \mu \overrightarrow{\operatorname{curl}}(\operatorname{curl} u) = 0$$

$$\boxed{u = \nabla \varphi_P + \overrightarrow{\operatorname{curl}}\, \varphi_S}$$

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Pressure wave

$$\rho \frac{\partial^2 \varphi_S}{\partial t^2} - \mu \Delta \varphi_S = 0$$

Shear wave

Decomposition of the displacement field into potentials

$$\rho \frac{\partial^2 \varphi_P}{\partial t^2} - (\lambda + 2\mu)\Delta\varphi_P = 0$$

$$\rho \frac{\partial^2 \varphi_S}{\partial t^2} - \mu\Delta\varphi_S = 0$$

$$V_P^2 = \frac{\lambda + 2\mu}{\rho}$$

$$V_S^2 = \frac{\mu}{\rho}$$

V_P: pressure waves velocity

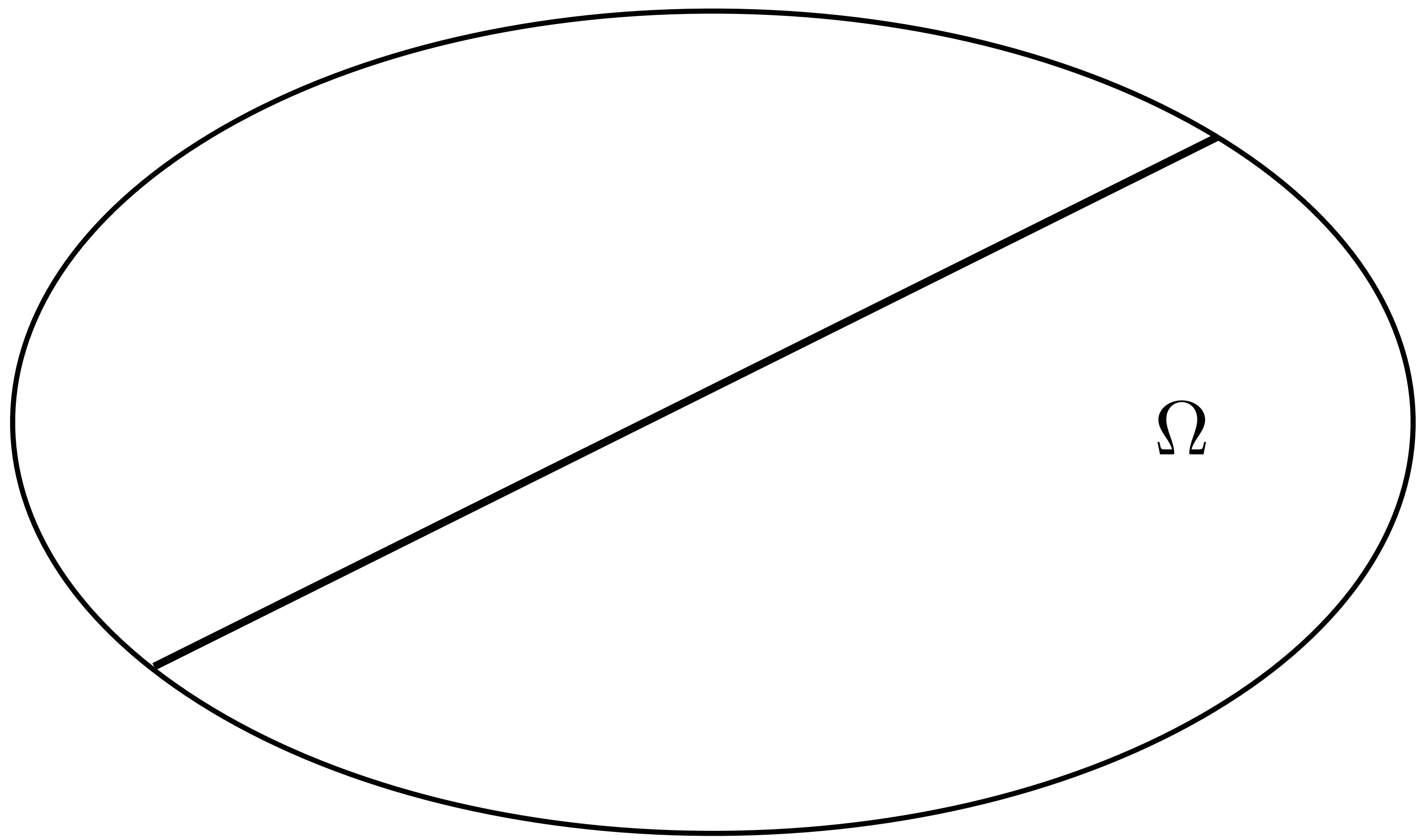
V_S : shear waves velocity

$$\frac{1}{V_P^2}\frac{\partial^2 \varphi_P}{\partial t^2} - \Delta\varphi_P = 0$$

$$\frac{1}{V_S^2}\frac{\partial^2 \varphi_S}{\partial t^2} - \Delta\varphi_S = 0$$

Coupling of the P-waves and S-waves

In a piecewise homogeneous medium, the coupling appear on the boundaries and on the interfaces.



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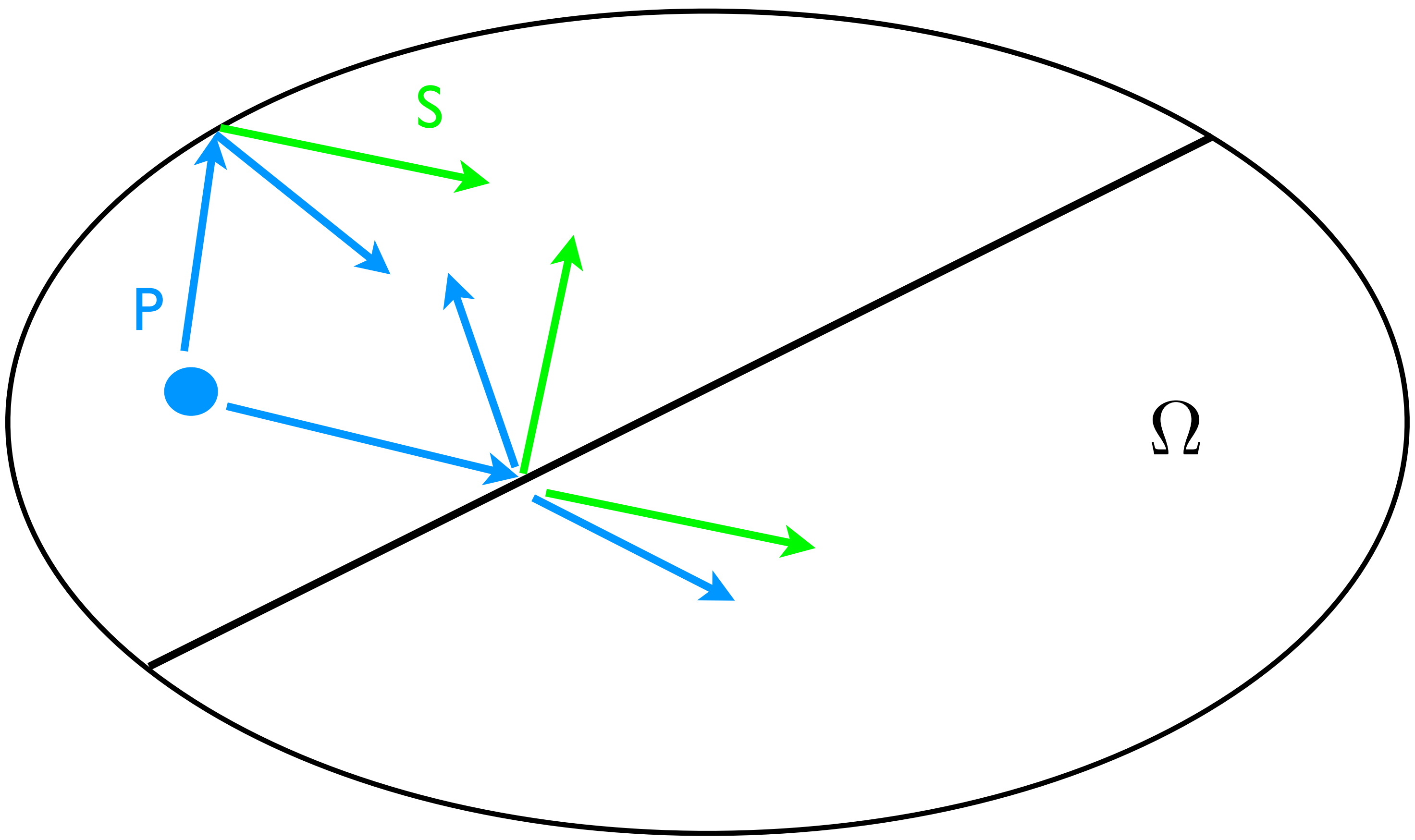
Boundary Conditions

or $u = 0$

or $\sigma(u)\,n = 0$

Perfect Bounding Interface Conditions

$$\begin{cases} [u] = 0 \\ [\sigma(u)\,n] = 0 \end{cases}$$



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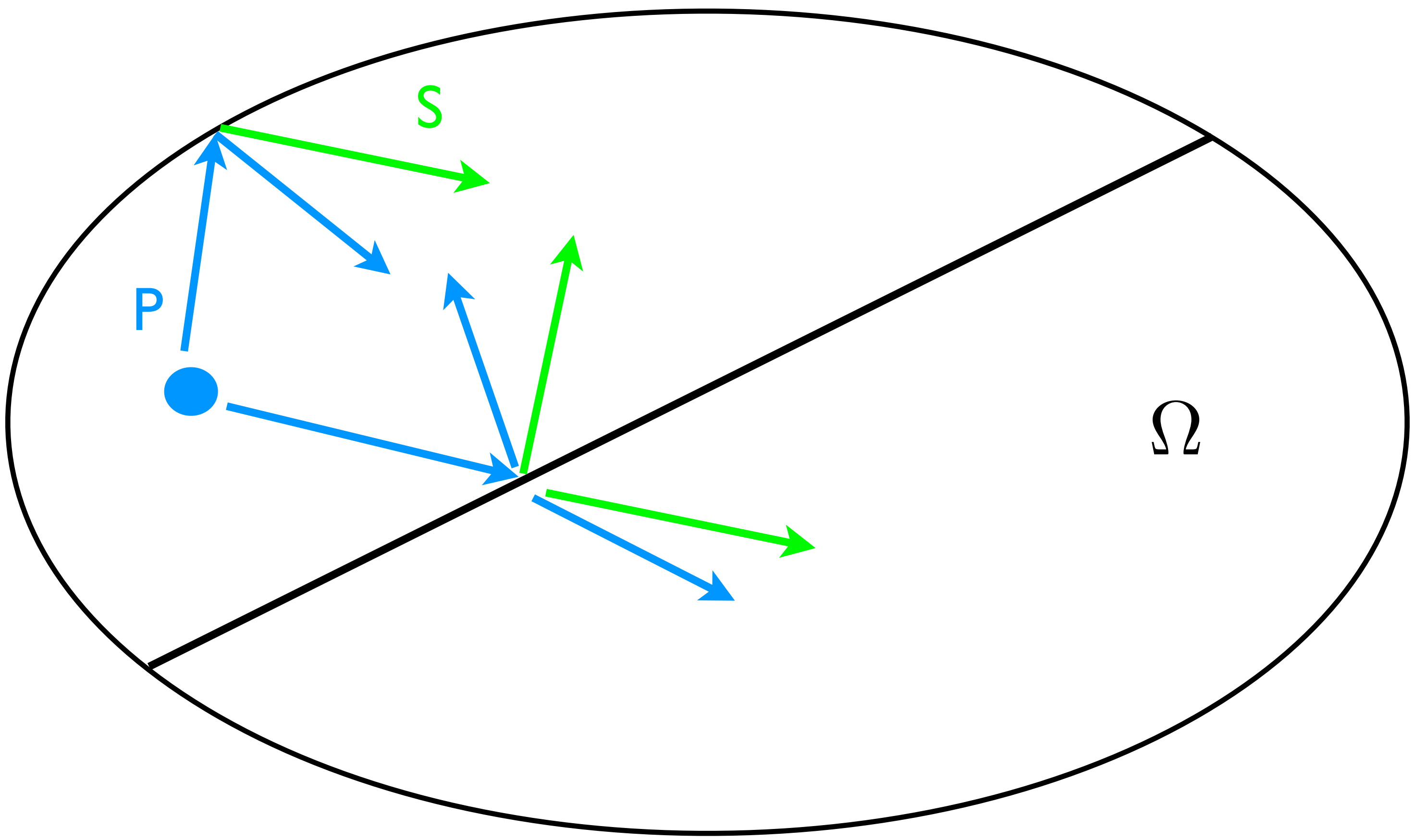

Main issue: treat these couplings by potentials in a stable way.

Dirichlet boundary conditions

Example: case of a rigid boundary $u = 0$ on $\Gamma = \partial\Omega$

$$\nabla \varphi_P + \overrightarrow{\text{curl}}\, \varphi_S = 0 \quad \text{on} \quad \Gamma = \partial\Omega$$

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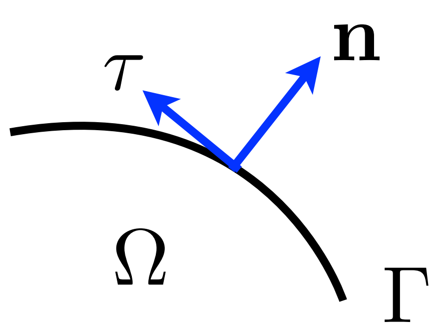

$$\overrightarrow{\mathrm{curl}}\, \varphi \cdot n = -\frac{\partial \varphi}{\partial \tau} \qquad \overrightarrow{\mathrm{curl}}\, \varphi \times n = \frac{\partial \varphi}{\partial n}$$

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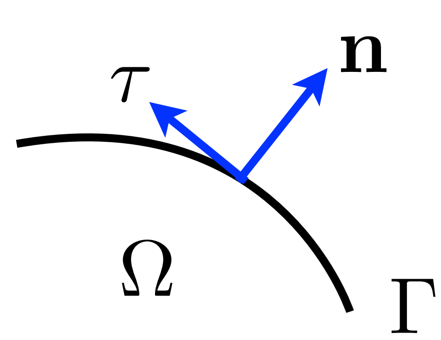

$$\overrightarrow{\text{curl}}\,\varphi \cdot n = -\frac{\partial\varphi}{\partial\tau} \qquad \overrightarrow{\text{curl}}\,\varphi \times n = \frac{\partial\varphi}{\partial n}$$

After projection along the normal $(\cdot\mathbf{n})$ and the tangent $(\times\mathbf{n})$ to Γ, we get

$$\frac{\partial\varphi_S}{\partial n} + \frac{\partial\varphi_P}{\partial\tau} = 0 \qquad \Gamma = \partial\Omega \qquad (\Longleftrightarrow \; u\cdot\tau = 0)$$

$$\frac{\partial\varphi_P}{\partial n} - \frac{\partial\varphi_S}{\partial\tau} = 0 \qquad \Gamma = \partial\Omega \qquad (\Longleftrightarrow \; u\cdot\mathbf{n} = 0)$$

The energy identity

$$\begin{cases} \dfrac{1}{V_P^2} \dfrac{\partial^2 \varphi_P}{\partial t^2} - \Delta \varphi_P = 0 & x \in \Omega, \quad t > 0 \\ \dfrac{1}{V_S^2} \dfrac{\partial^2 \varphi_S}{\partial t^2} - \Delta \varphi_S = 0 & x \in \Omega, \quad t > 0 \end{cases}$$

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Discretization of these boundary conditions?

The energy identity

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We demonstrated the following energy equality:

Proposition:

Let $E(t) := \frac{1}{2}\Big[\int_\Omega \Big(\frac{1}{V_P^2}\Big|\frac{\partial \varphi_P}{\partial t}\Big|^2 + |\nabla \varphi_P|^2\Big) + \int_\Omega \Big(\frac{1}{V_S^2}\Big|\frac{\partial \varphi_S}{\partial t}\Big|^2 + |\nabla \varphi_S|^2\Big)$

E is positive and conservative: $\dfrac{\partial}{\partial t}E(t) = 0.$

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The key to demonstrate the previous proposition is:

Lemma: $\forall\ \varphi_P, \varphi_S \ \in\ H^1(\Omega)^2$

$$\int_\Omega \left(|\nabla \varphi_P|^2 + |\nabla \varphi_S|^2 \right) - \int_\Gamma \left(\frac{\partial \varphi_S}{\partial \tau} \varphi_P - \frac{\partial \varphi_P}{\partial \tau} \varphi_S \right) = \int_\Omega \left| \nabla \varphi_P + \overrightarrow{\text{curl}}\ \varphi_S \right|^2$$

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And why are you happy with that?

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We found an energy identity so we are confident in the fact that we can find the same kind of equality for the discrete scheme.

And why are you happy with that?

Because if a quantity involving my discrete potentials is conserved in time, it means I can find a scheme that won't explode.

An energy-preserving scheme and a code

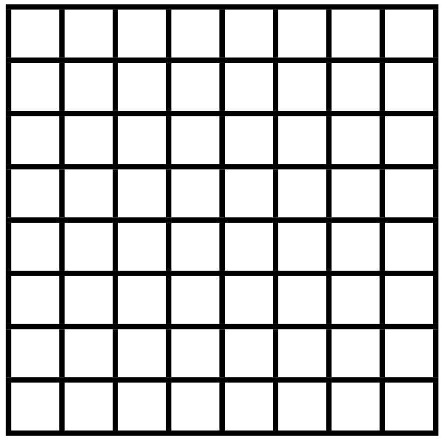
Remark : Since we don't want any of the two waves to be more penalizing than the other, it will be natural to choose our approximation spaces so that we can play with either the space step or the order of the scheme.

Finite Elements discretization

We choose two space steps h_P and h_S or two different orders

We take $V_h = V_{h_P} \times V_{h_S}$ discretization spaces

$\Rightarrow$ two meshes



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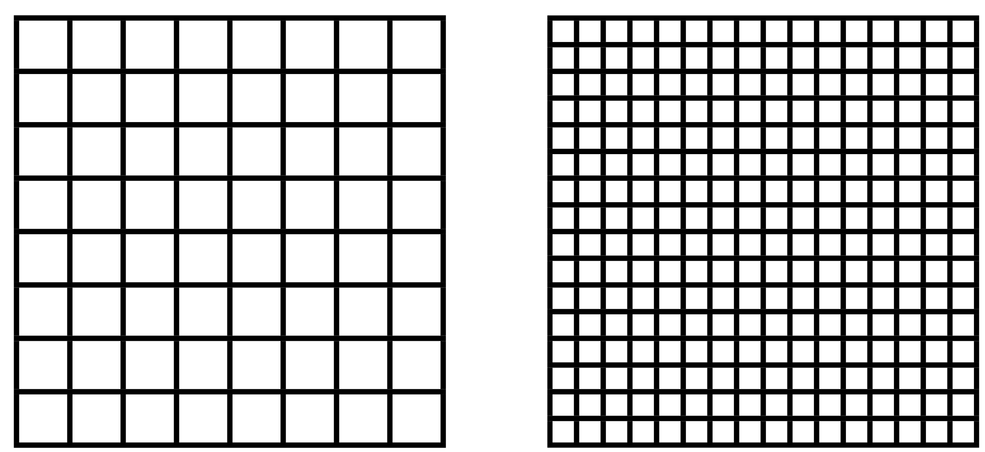

A numerical illustration

The modulus of the displacement field is represented (in color levels) as a function of time

$V_P \simeq 3\ V_S$

P-wave

S-wave

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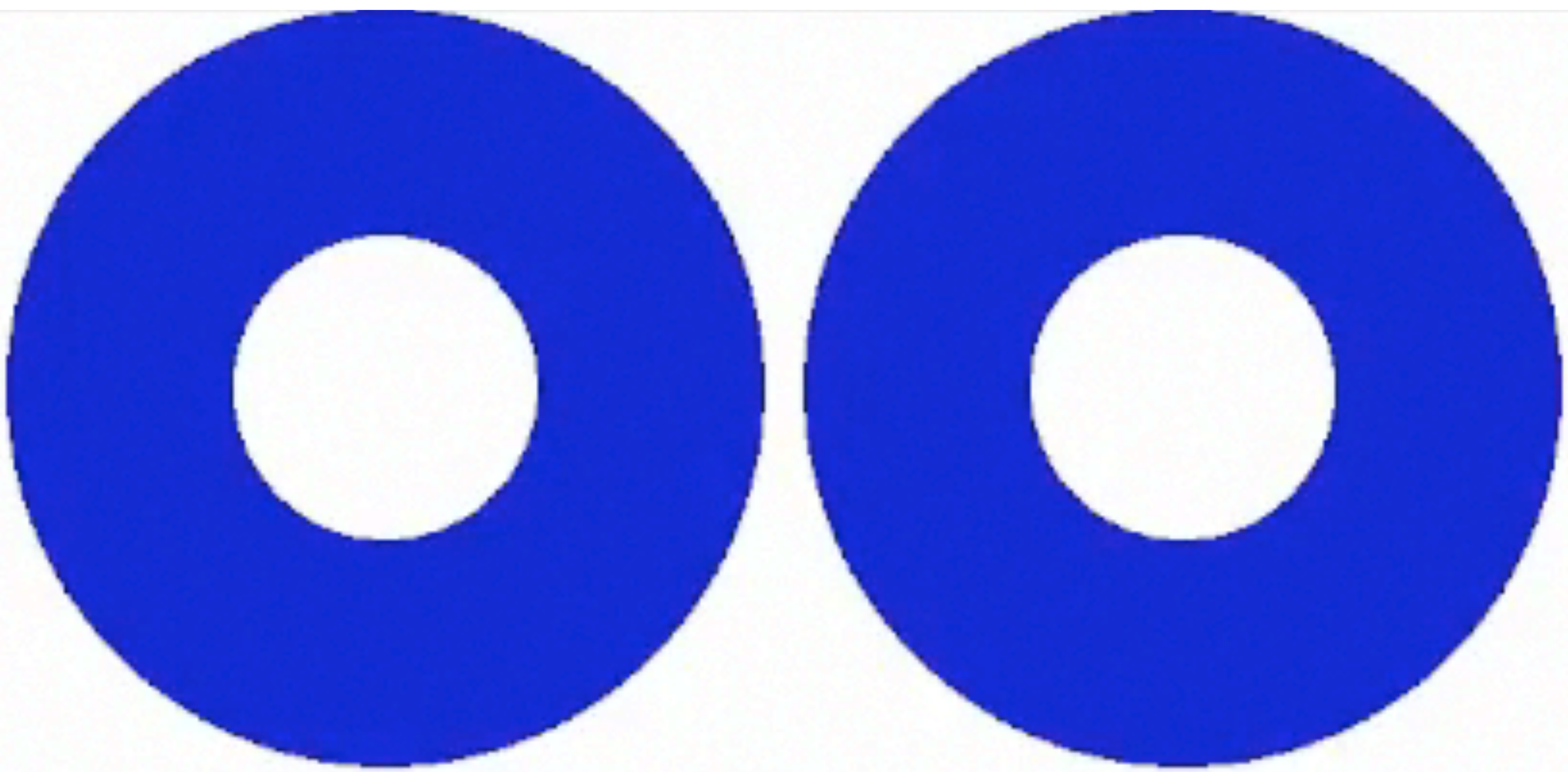

A numerical illustration

The modulus of the displacement field is represented (in color levels) as a function of time

Our method

Classical elastodynamics method

A numerical illustration

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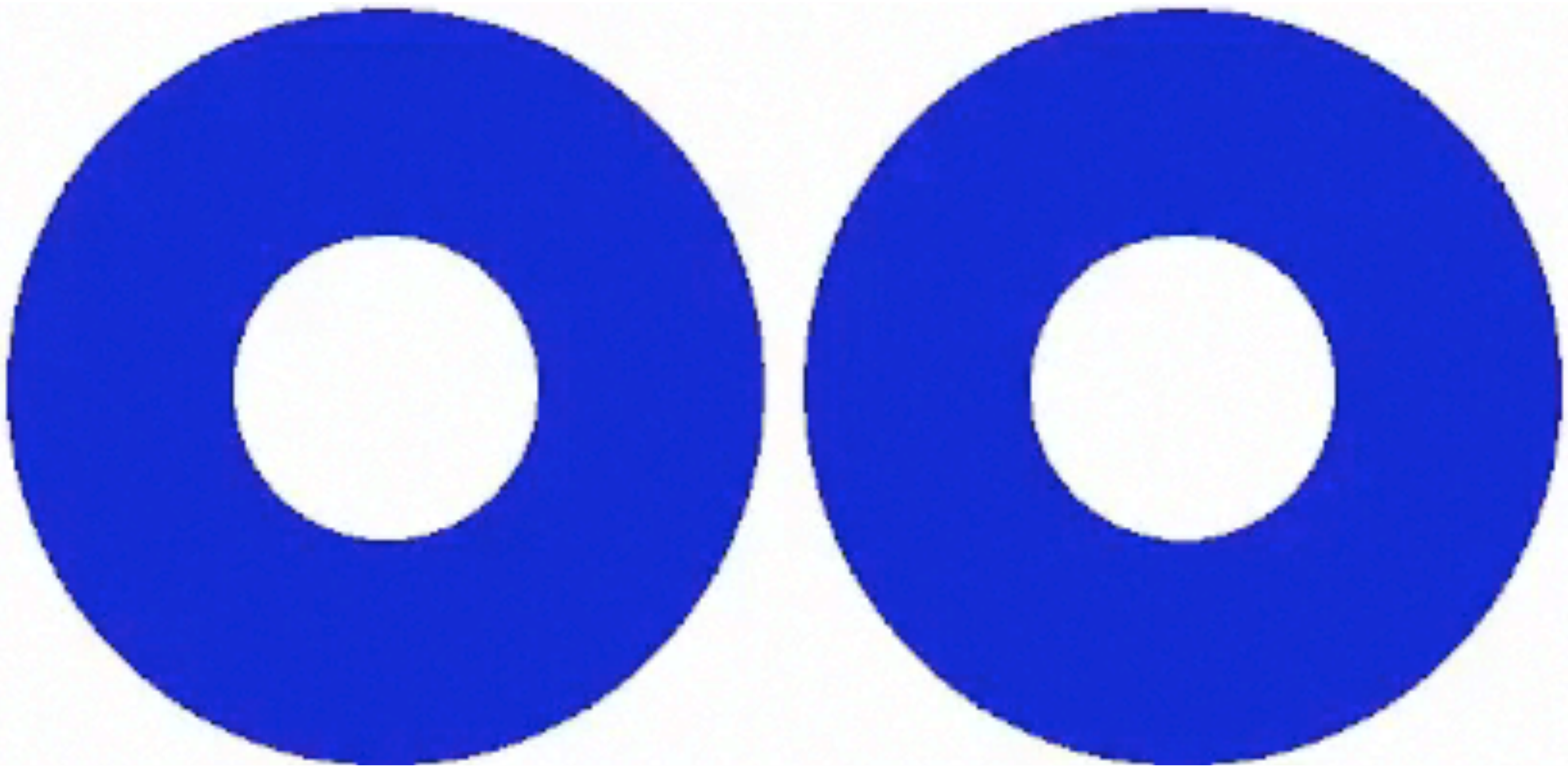


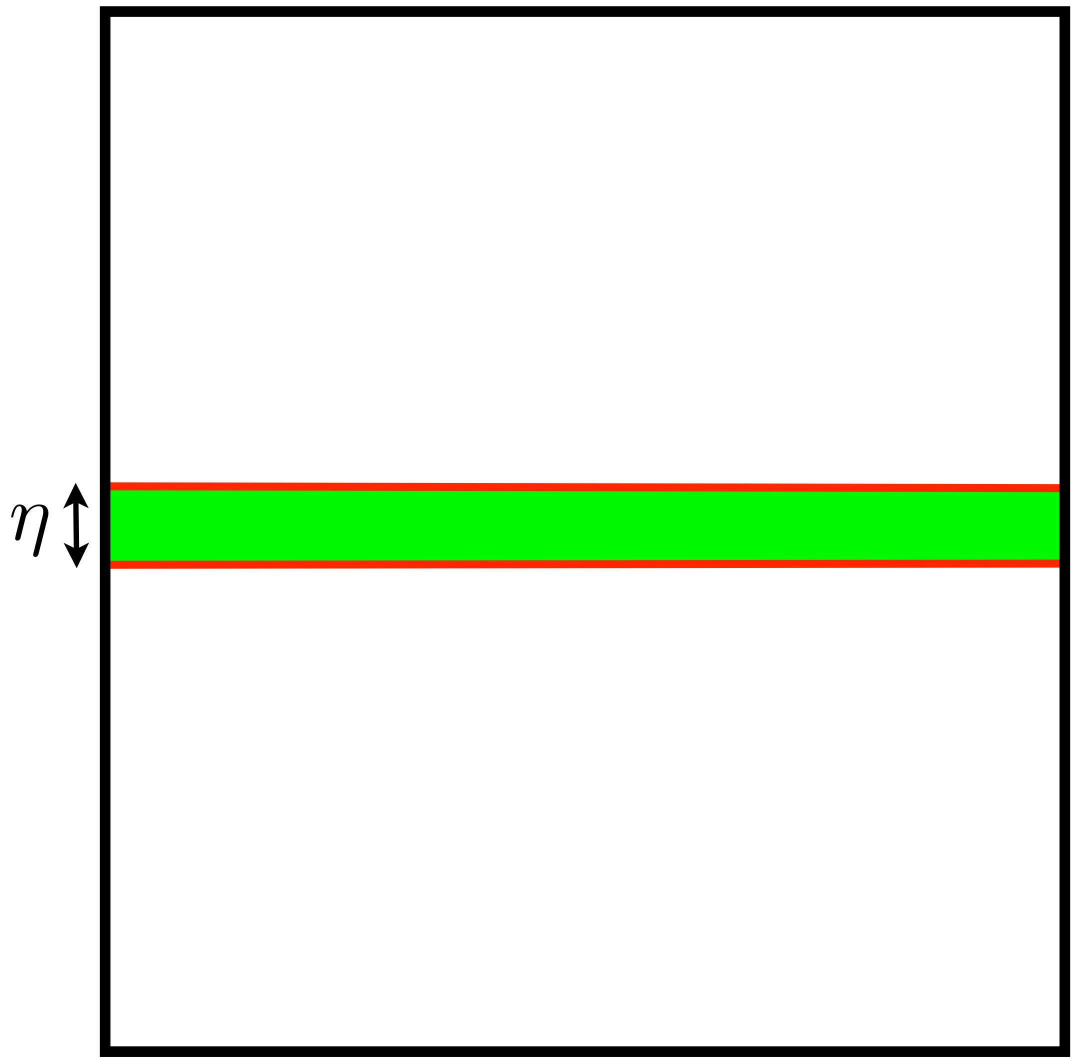
PART 2

Thin layers approximation for time-domain elastodynamics

Goal

If the width of the layer is small enough, can we compute "as if" there were no glue?

What does that mean?



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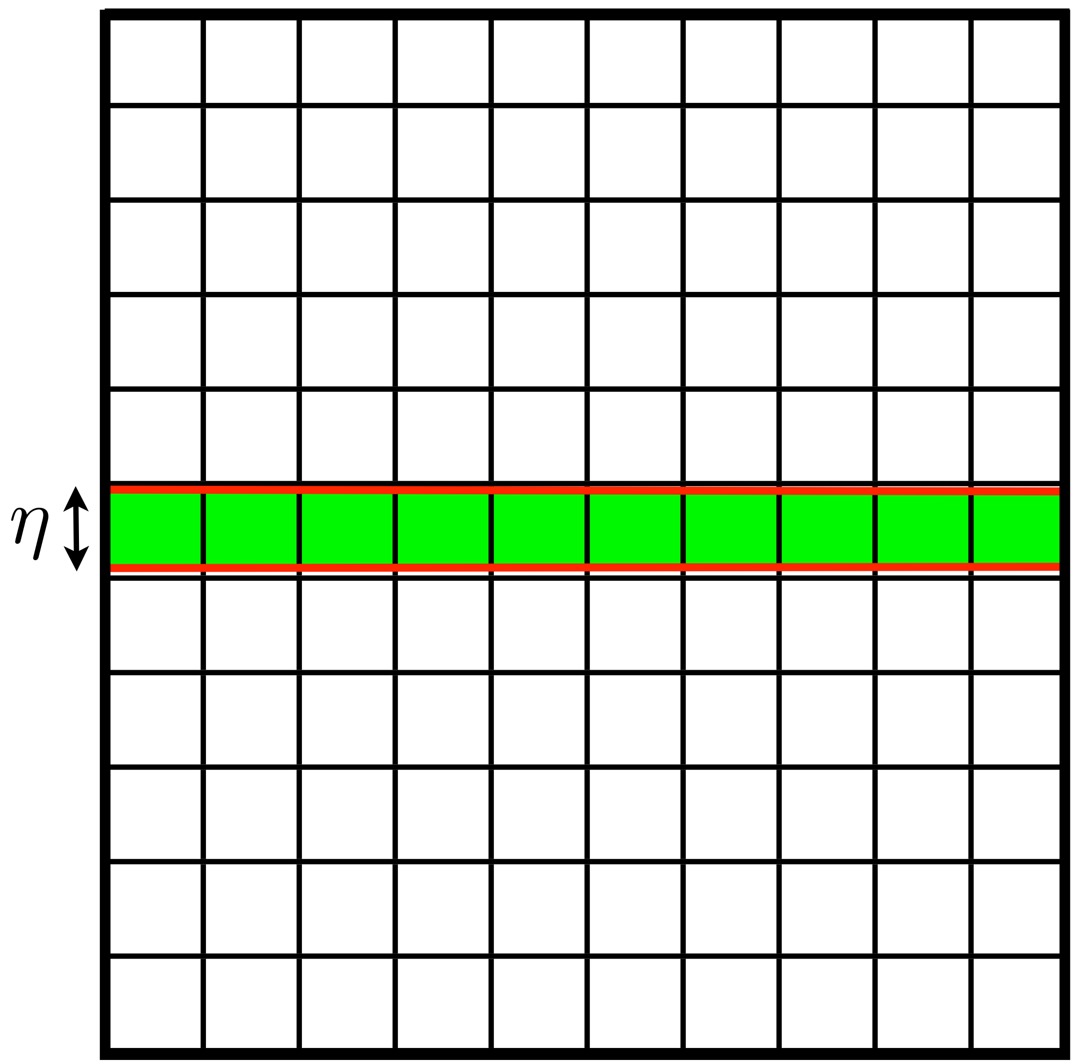

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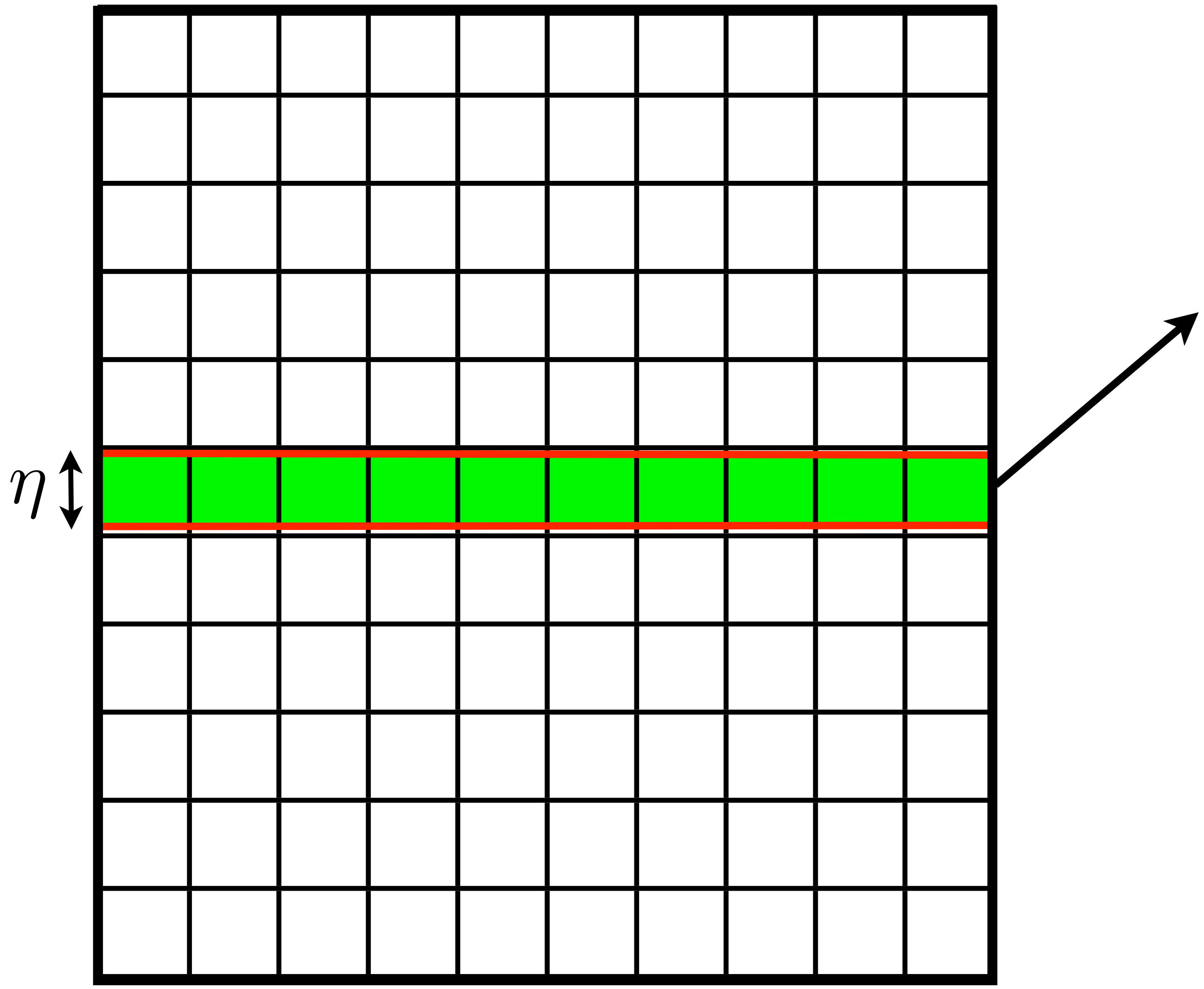

We need more computation points inside the layer if we want to "see" something

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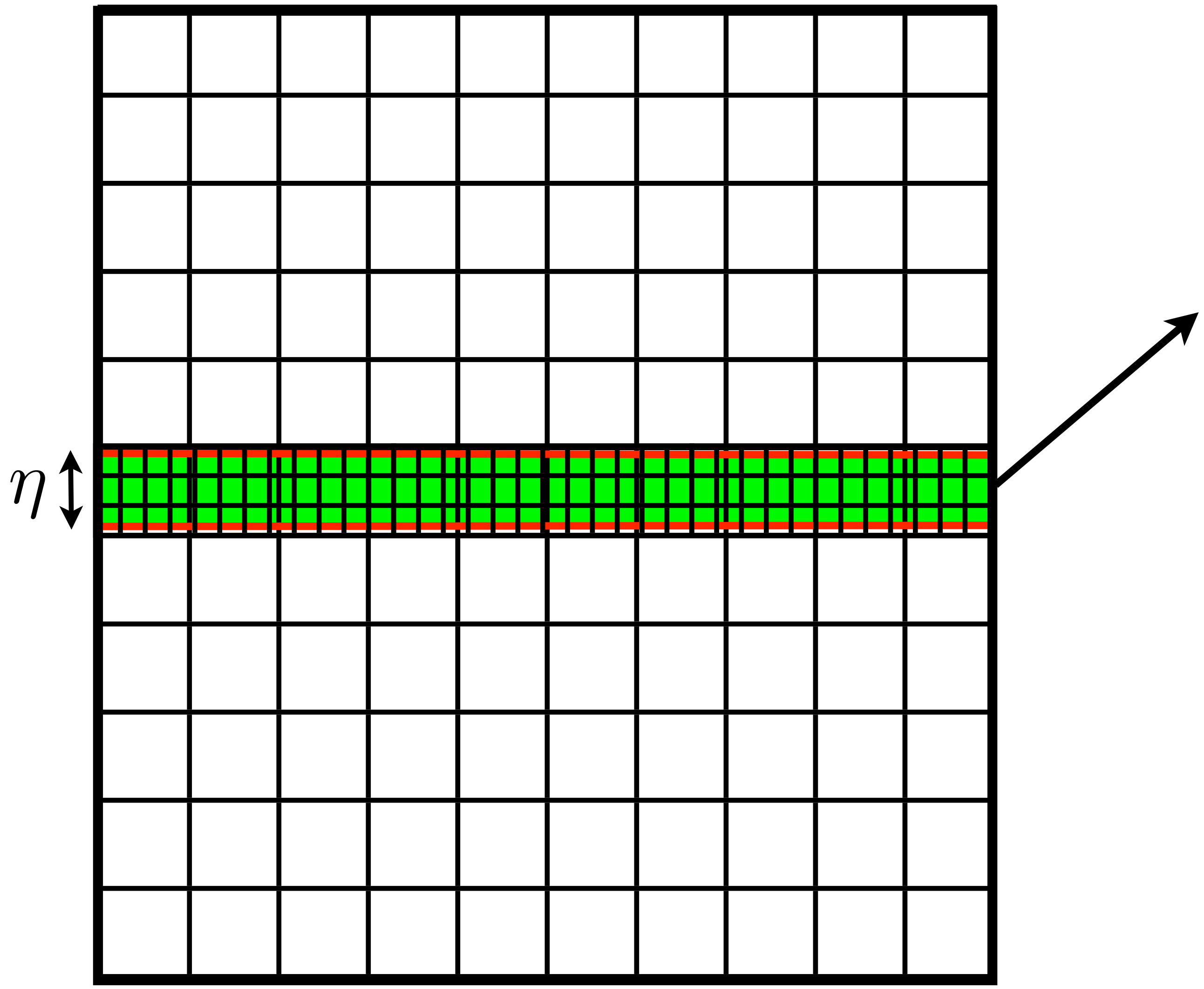

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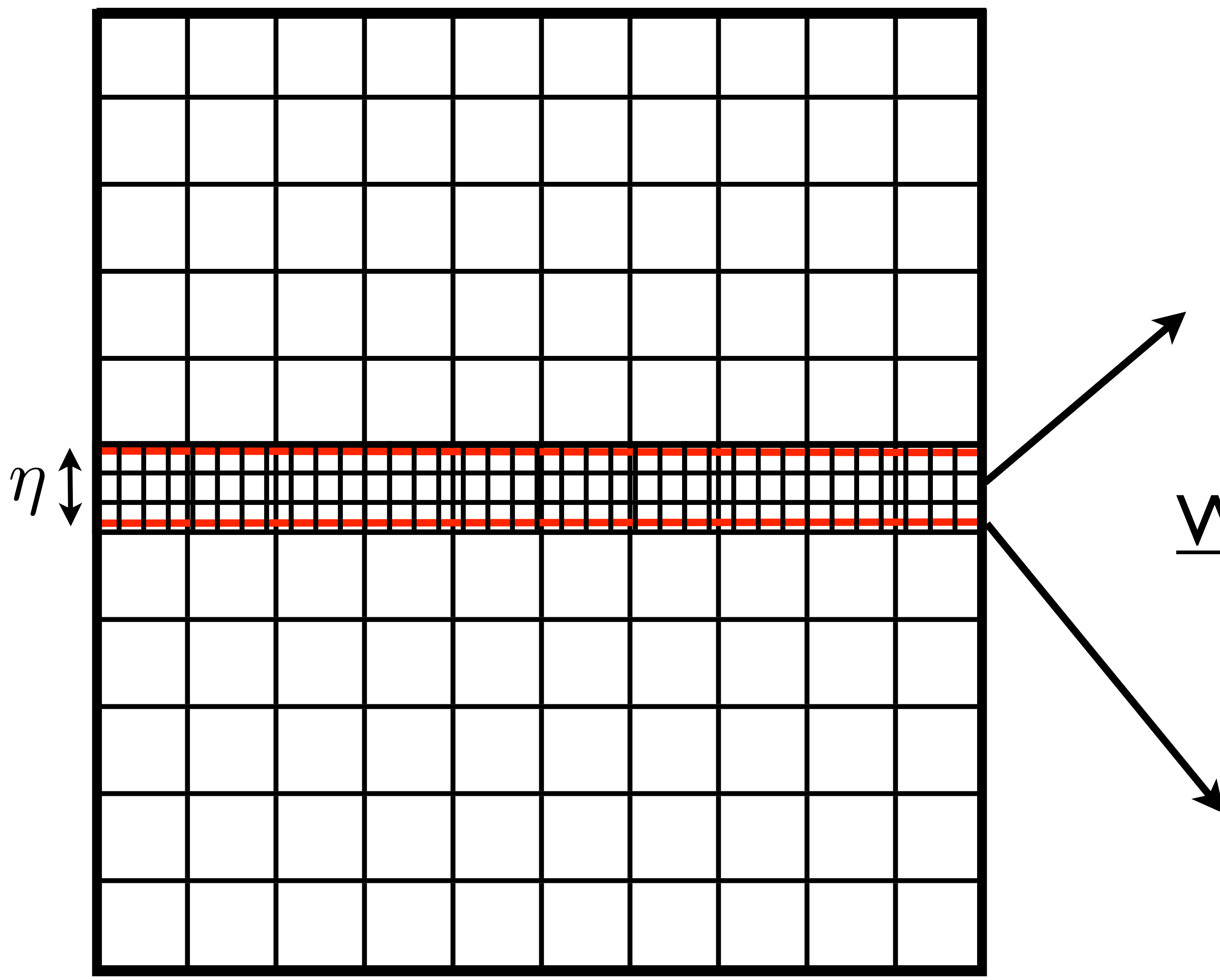
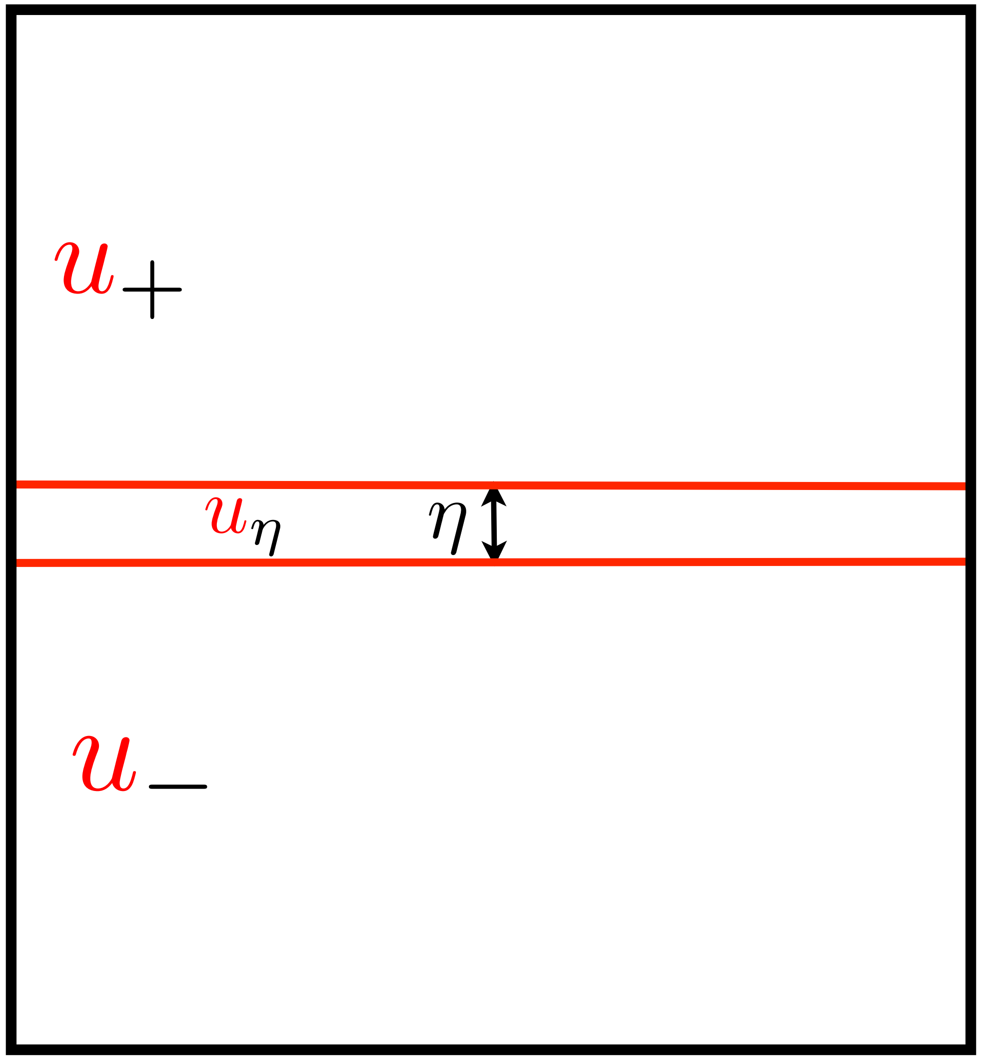


We need more computation points inside the layer if we want to "see" something

We increase the calculation cost

Aim: find relationships between the unknowns on the upper boundary and on the bottom boundary

How do we do that?



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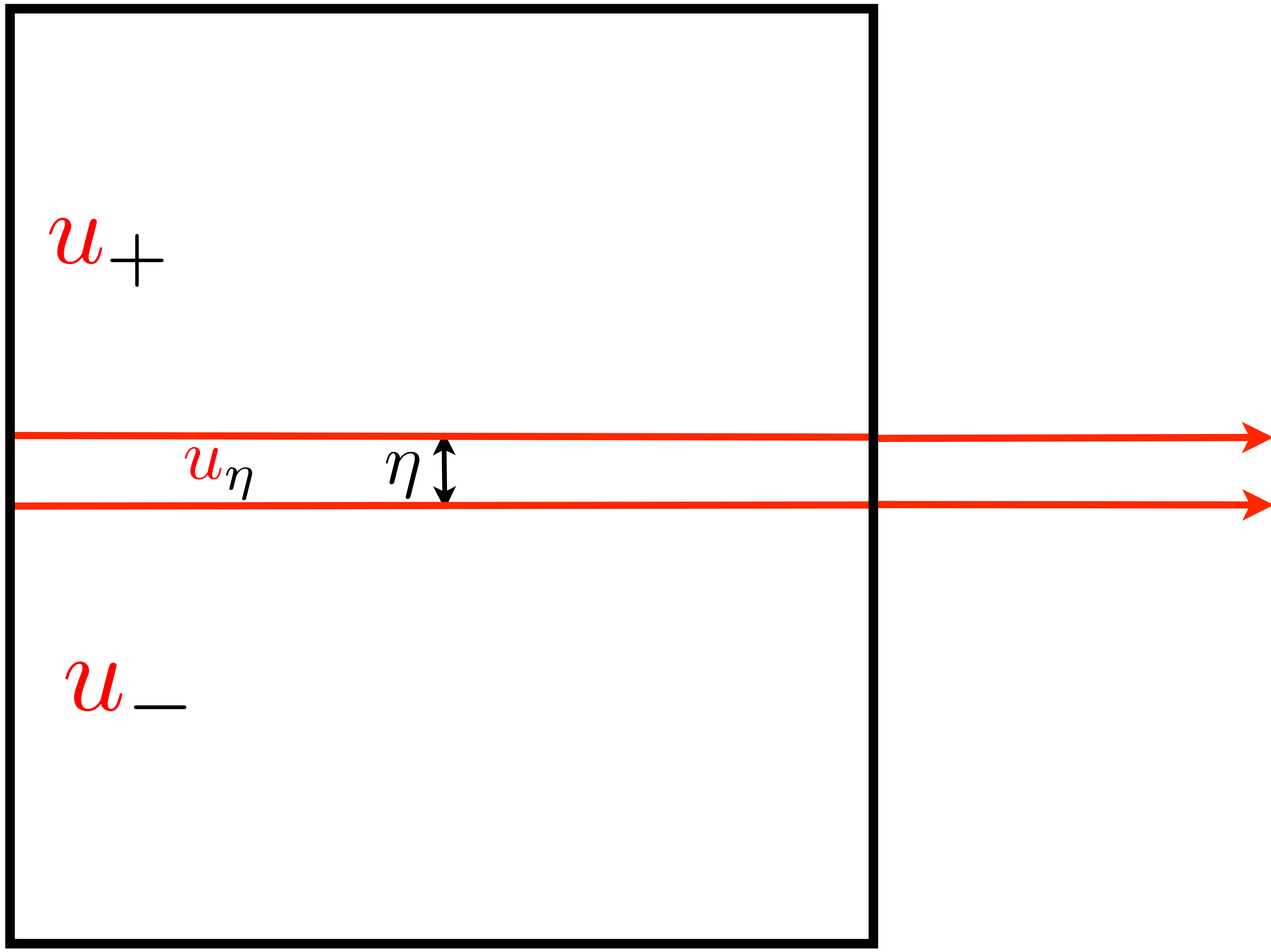

We use the

Interface conditions

$$[u] = 0$$

$$[\sigma(u)\mathbf{n}] = 0$$

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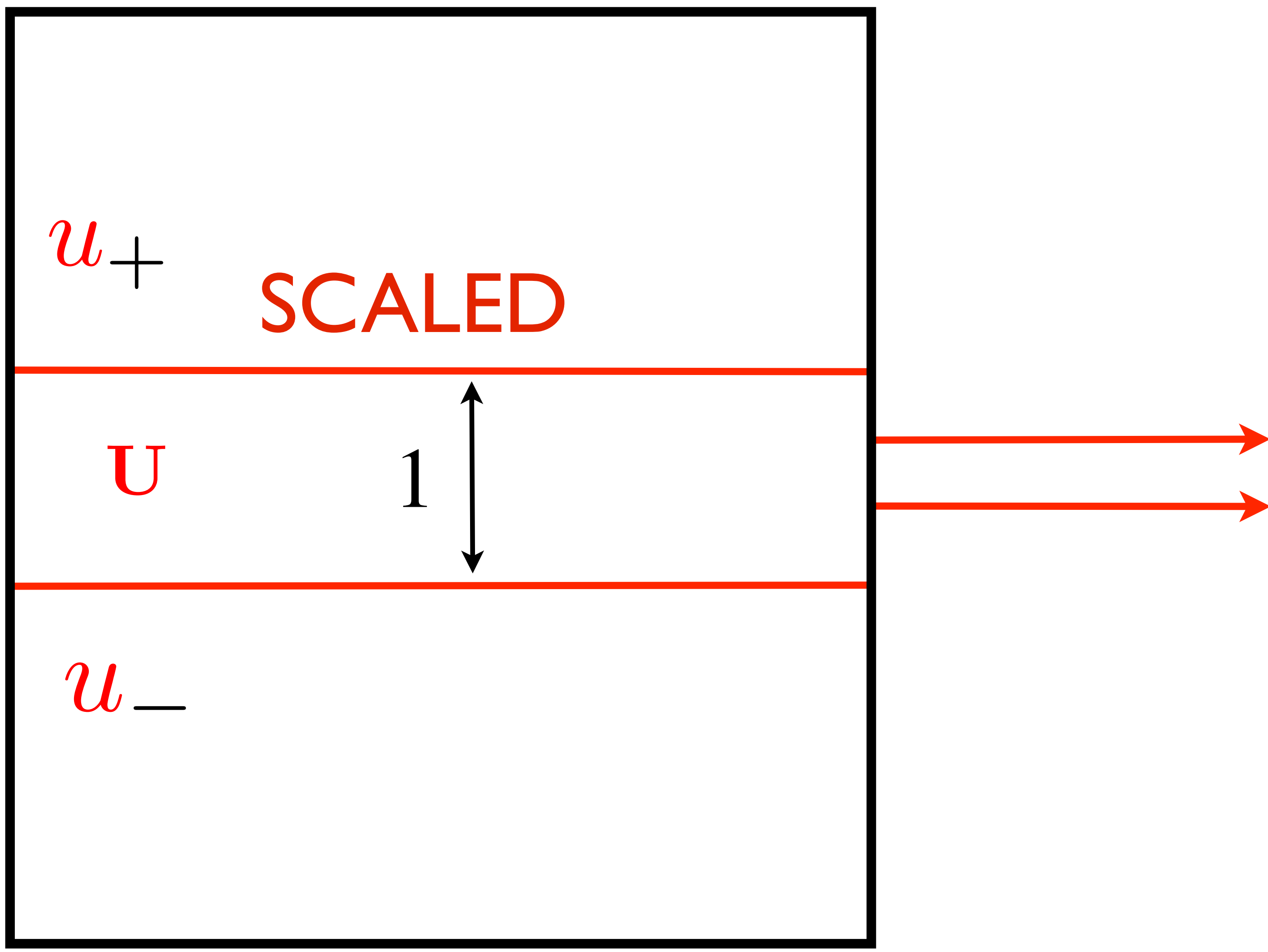

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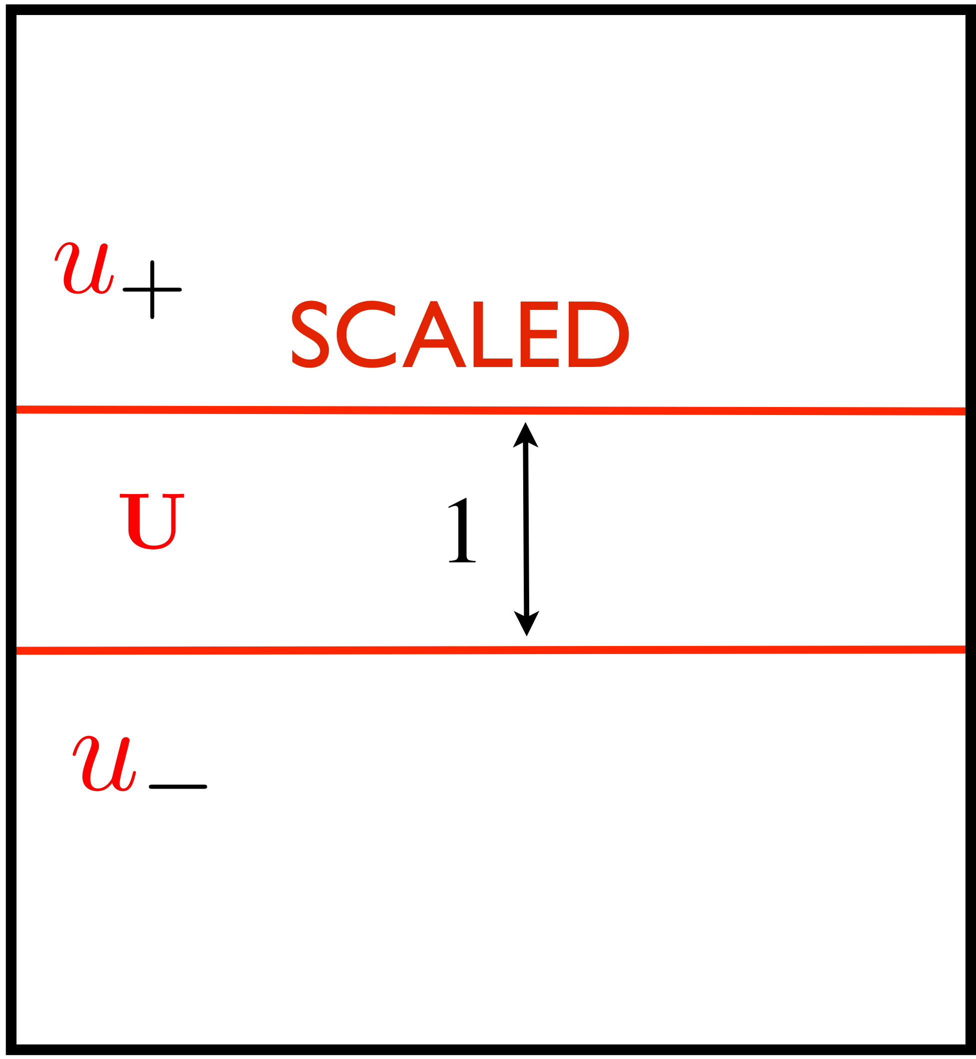

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We develop in η the unknowns $\mathbf{U}$ and $u_\pm$

$$\mathbf{U} = \mathbf{U}^0 + \eta \mathbf{U}^1 + \eta^2 \mathbf{U}^2 + \eta^3 \mathbf{U}^3 \ldots$$

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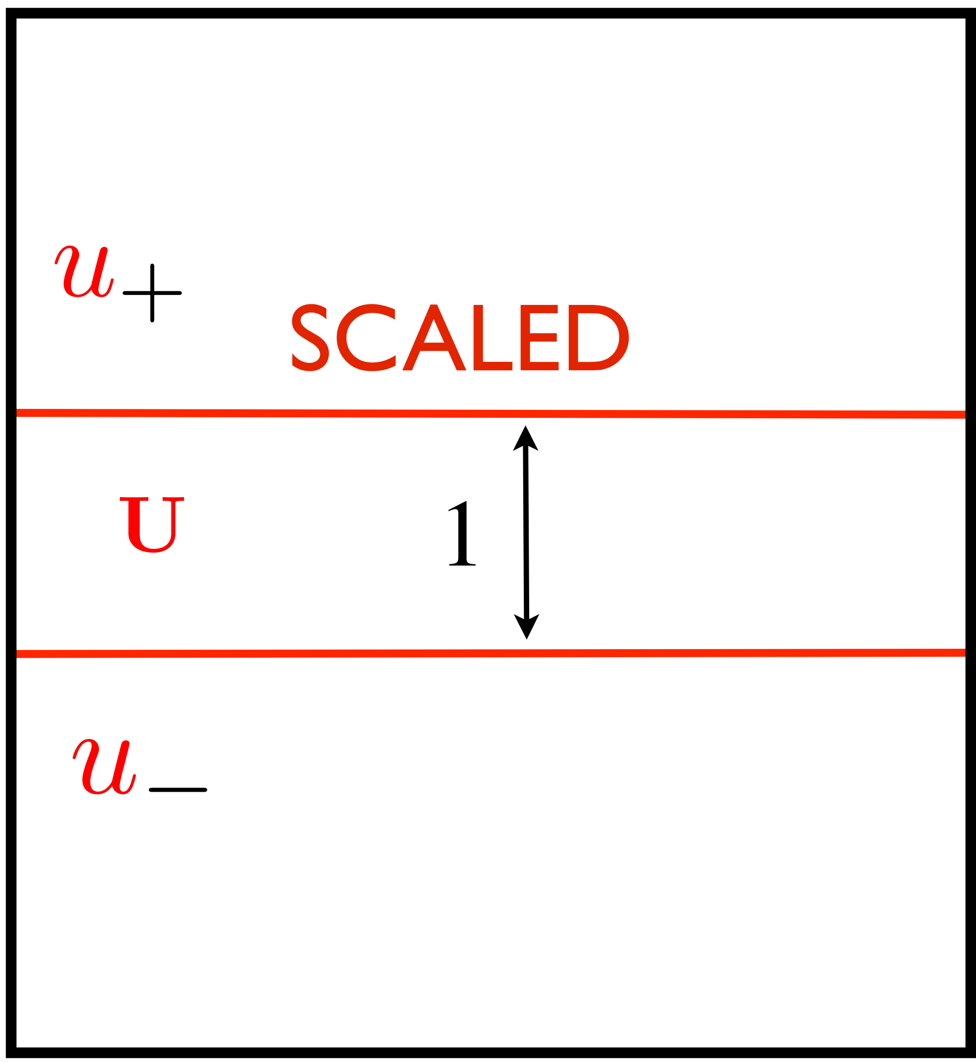

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We use the scaled equation

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2} - (\lambda + 2\mu)\nabla_\eta(\operatorname{div}_\eta \mathbf{U}) + \mu\,\overrightarrow{\operatorname{curl}}_\eta(\operatorname{curl}_\eta \mathbf{U}) = 0$$

How do we do that?

$$[u] = 0$$

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How do we do that?

$$[u] = 0$$

$$[\sigma(u)\mathbf{n}] = 0$$

$$\mu(\partial_1 u_2 + \partial_2 u_1)$$

$$(\lambda + 2\mu)\partial_2 u_2 + \lambda \partial_1 u_1$$

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} - (\lambda + 2\mu)\nabla_\eta(\operatorname{div}_\eta \mathbf{U}) + \mu \operatorname{curl}_\eta(\operatorname{curl}_\eta \mathbf{U}) = 0$$

Tuesday, 26 June 12

How do we do that?

$$[u] = 0$$

$$[\sigma(u)\mathbf{n}] = 0$$

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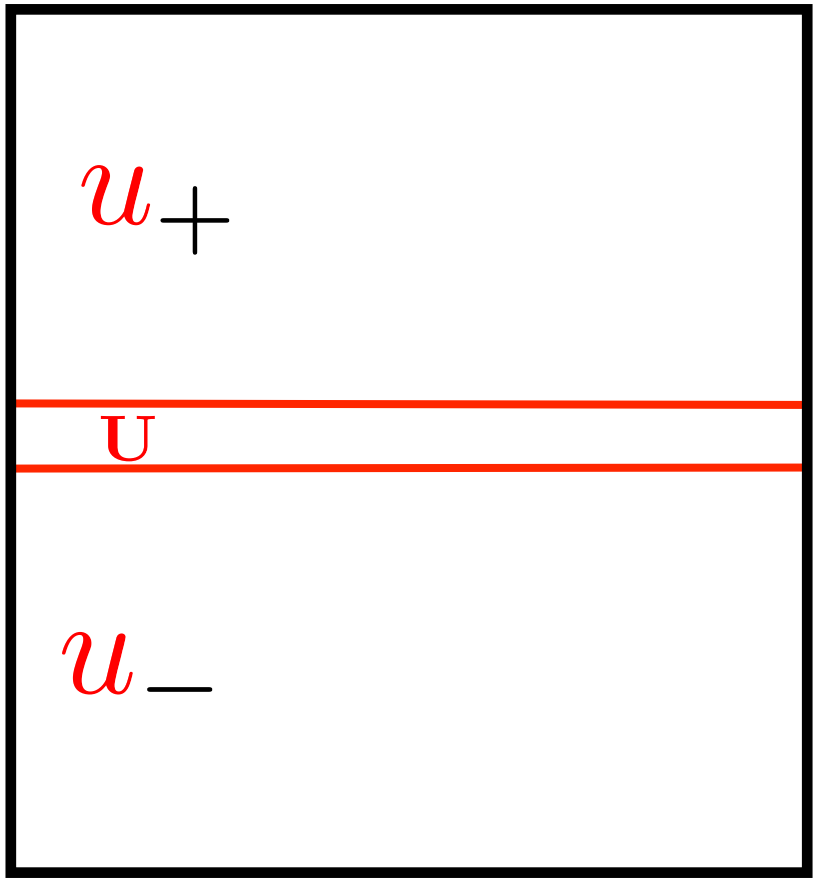
We do quite long calculus...

Effective Interface Conditions



It leads to conditions for each i = 0,1,2,3...

$$[u^i_\pm] = F(u^{i-1}_\pm)$$
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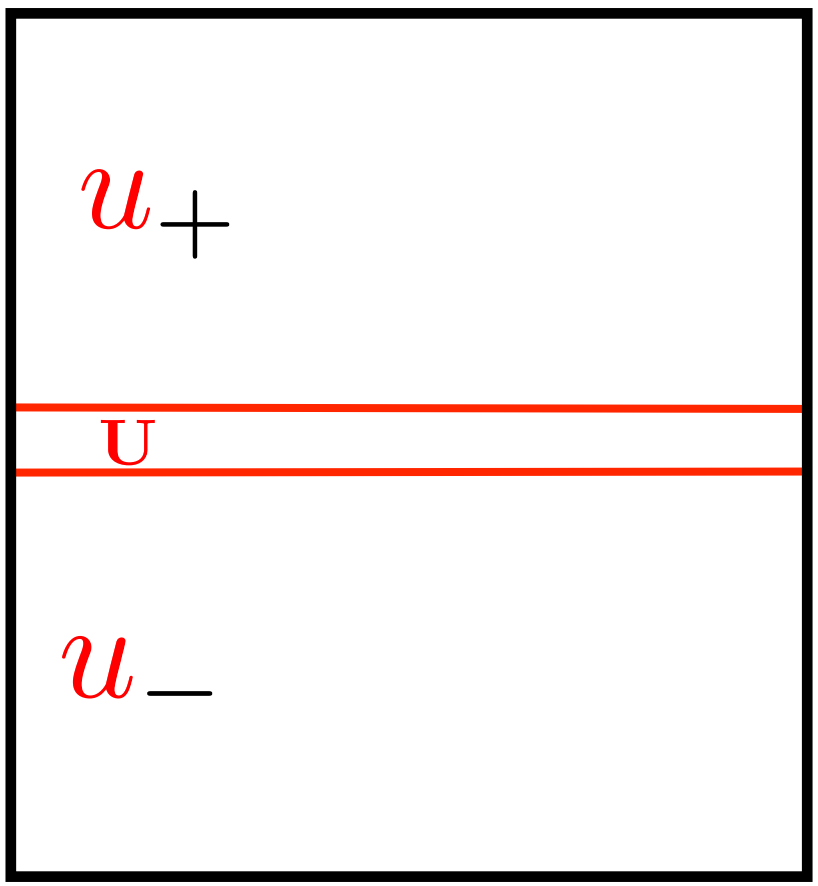

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We reconstruct each jump:

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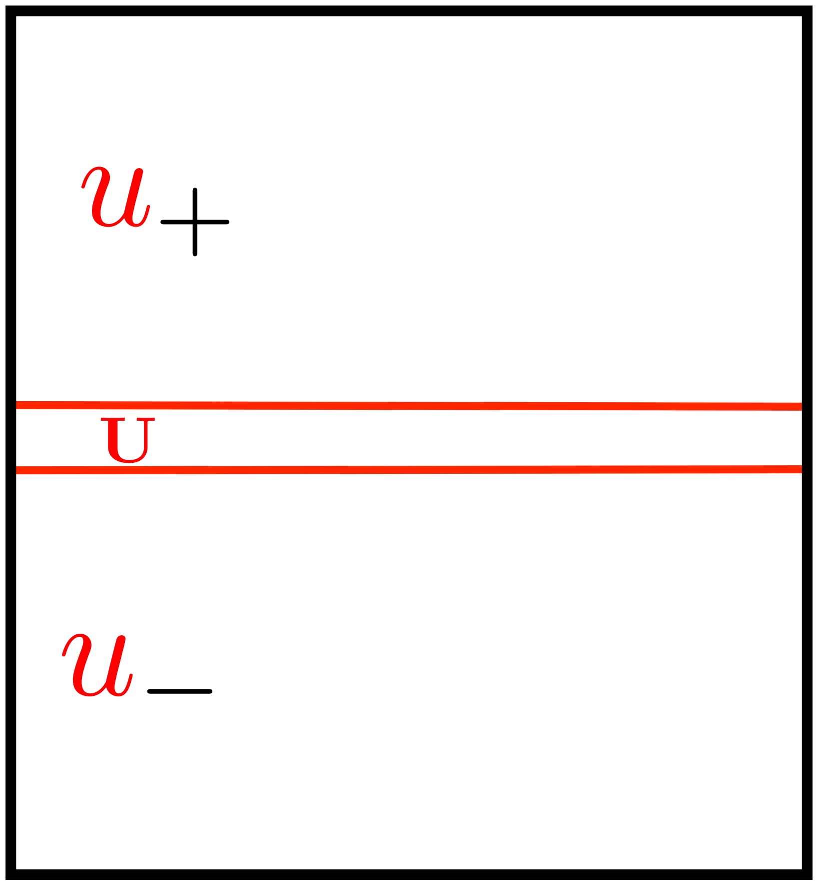

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with the previous conditions.

So that, given an initial condition on the bottom part for instance, if I know every u_-^i on the bottom boundary, I can deduce u_+ on the upper boundary.

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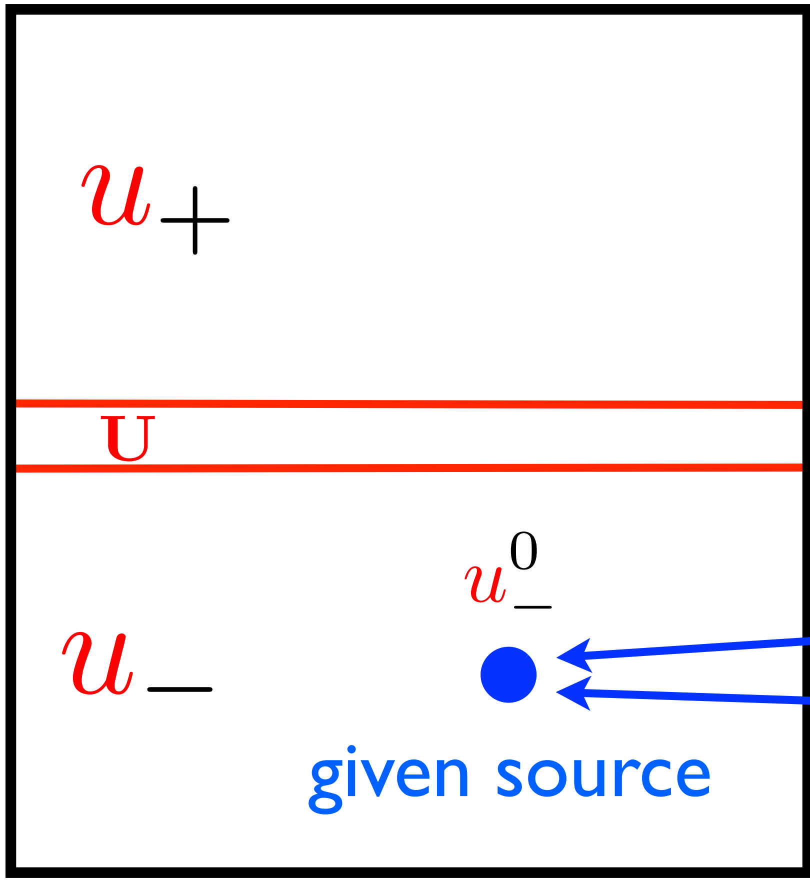

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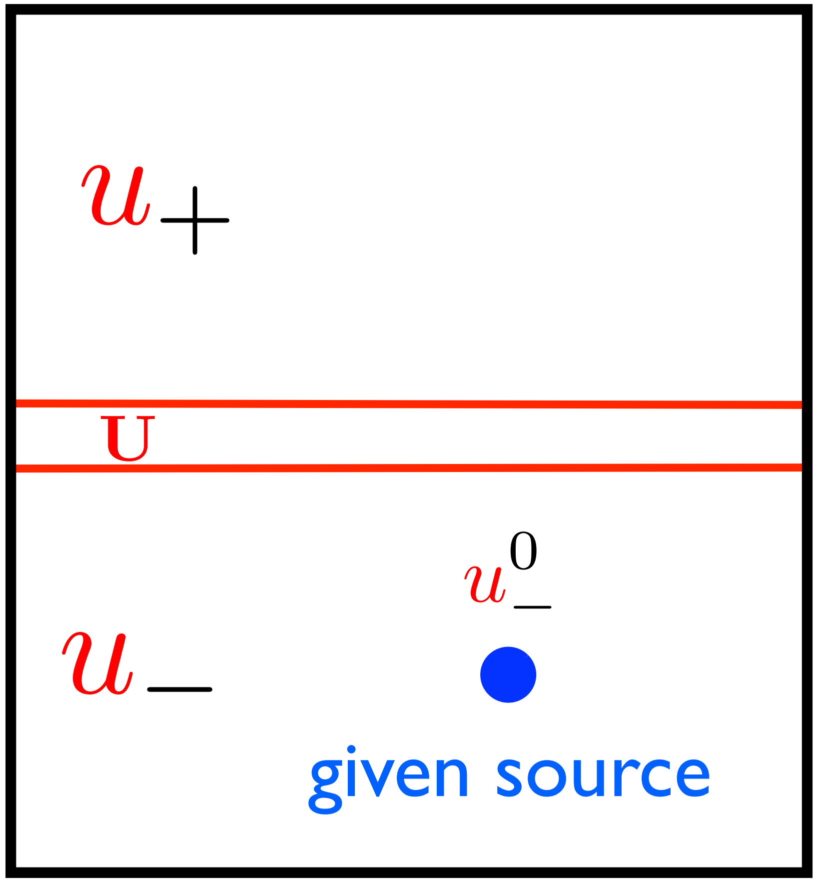

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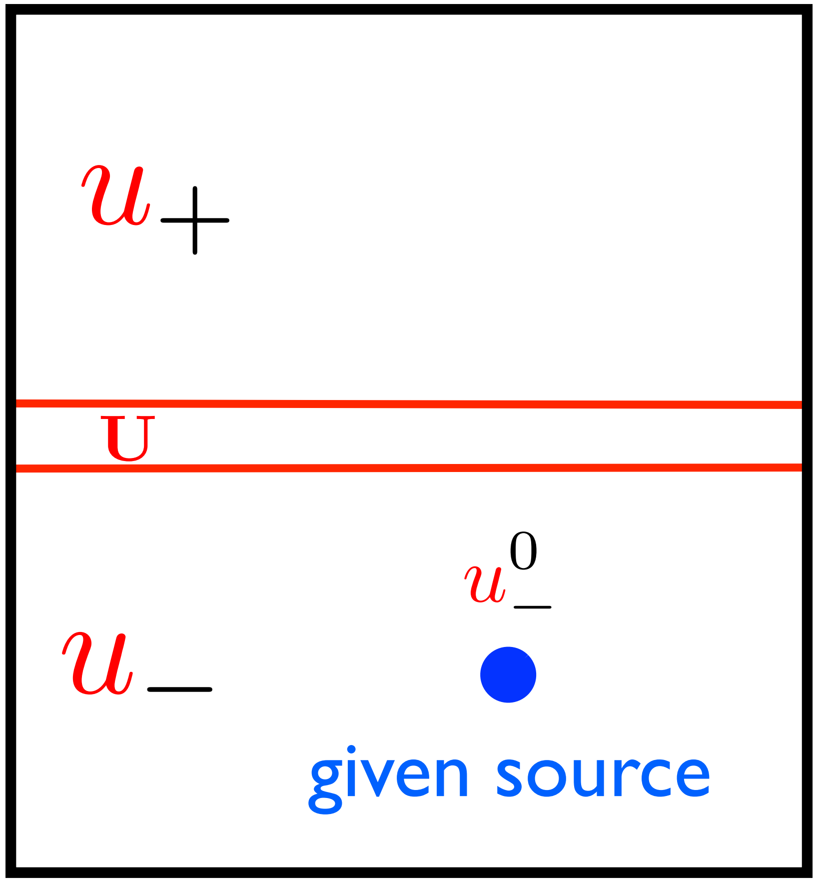

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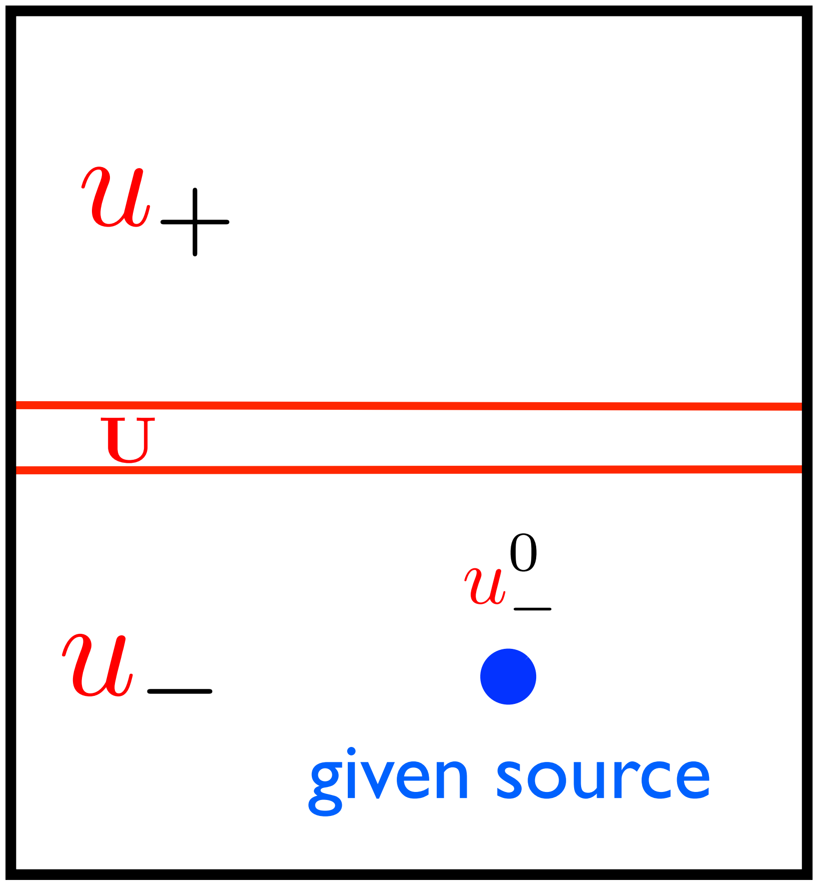

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Effective Interface Conditions

This method allows us to approximate the displacement on the layer interfaces. If I know what's going on on the upper interface, then I know the displacement on the bottom interface.

So if the width of the interface is small enough, we can compute "as if" there were nothing.

Q: are we sure this approximation will lead to stable conditions?

Scaled asymptotic expansion

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Partially

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Q: are we sure this approximation will lead to stable conditions?

Partially

Yes, up to the order 2: this approximation preserves an energy

No, with orders greater than 2, for now

Work in progress...

To be continued...

Part 1

- Dirichlet 3D
- Neumann condition? Unstable in time for the moment
- Transmission? Neumann needed...

Part 2

- Stability of the “high” orders
- 3D
- Numerical scheme + code

Finally: use of these two methods together?

Thank you for your attention