Proof interpretations: what they are and what they are good for

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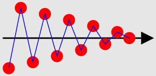
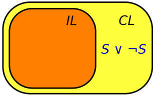
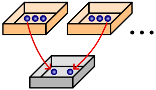
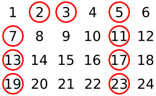
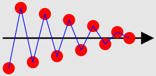
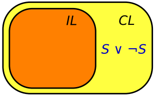
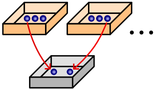
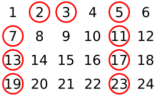
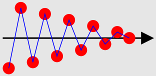
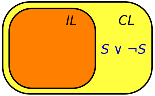
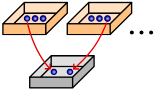
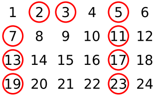
Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T i		
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

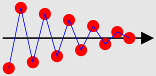
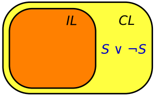
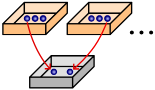
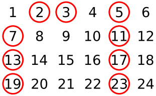
Geometry

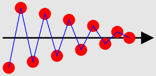
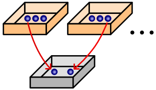
- Talks about points, lines, planes, ...
- Has axioms like "two distinct points determine a line"

Proof interpretation	$I: S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T i		
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b \; S(x)$	nth prime $\leq 2^n$	

Arithmetic

- Talks about $0, 1, 2, \ldots$ and $+, \times$
- Has axioms like $x + y = y + x$

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges $\downarrow$ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

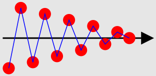
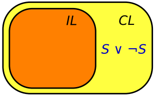
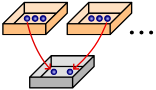
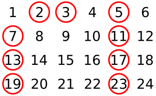
Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T is consistent	CL is c	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

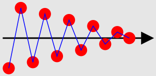
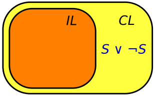
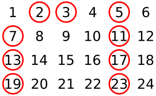
$$\forall \epsilon > 0\ \exists m\ \forall n \geq m\ \ |x_n| < \epsilon$$

↧

$$\exists f\ \forall \epsilon > 0\ \forall n \geq f(\epsilon)\ \ |x_n| < \epsilon$$

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

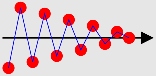
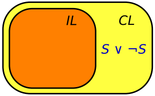
Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doe prove A	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

$T_1 = CL$

$T_2 = IL$

$I =$ negative translation

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges $\downarrow$ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doe prove	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	see grid below

IL proves S $\rightsquigarrow$ program

IL proves $S_1 \wedge (S_1 \rightarrow S_2) \rightarrow S_2$

$\rightsquigarrow$

$x, f \mapsto f(x)$

1	2	3	4	5	6
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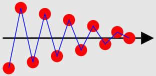
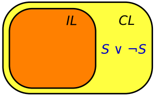
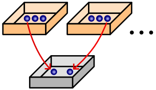
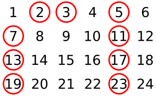
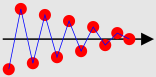
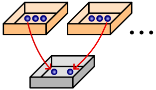
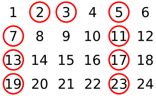
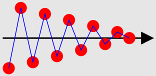
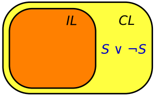
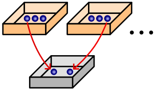
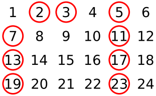
Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges $\downarrow$ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

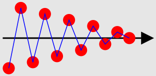
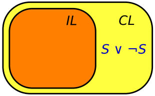
Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T is consistent	CL	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b \; S(x)$	nth prime $\leq 2^n$	

$T_1 = ZF + \exists S$ infinite $S \sim S \times S$

$T_2 = ZF + AC$

$I =$ classical realisability

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges $\downarrow$ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges ↧ speed of convergence	
Consistency	T is consistent	CL is consistent	
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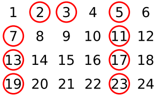
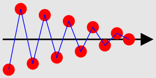
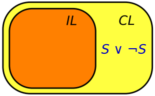
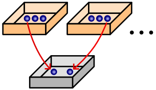
Computational content	T proves $\exists x \leq b\ S(x)$	nth prime $\leq 2^n$	

$T_1 = PA^\omega + \forall n\ \exists p$ prime $n < p \leq 2n$

$T_2 = T_1$

$I =$ monotone functional interpretation after a negative translation

Proof interpretation	I: $S_1 \mapsto S_2$ such that if T_1 proves S_1 then T_2 proves S_2	sequence converges $\downarrow$ speed of convergence	
Consistency	T is consistent	CL is consistent	
Unprovability	T doesn't prove S	ZF doesn't prove AC	
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