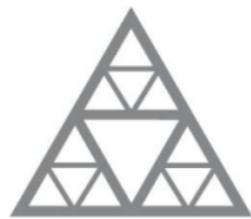


Mathematical analysis of models for aging non-Newtonian fluid: macroscopic and mesoscopic approaches



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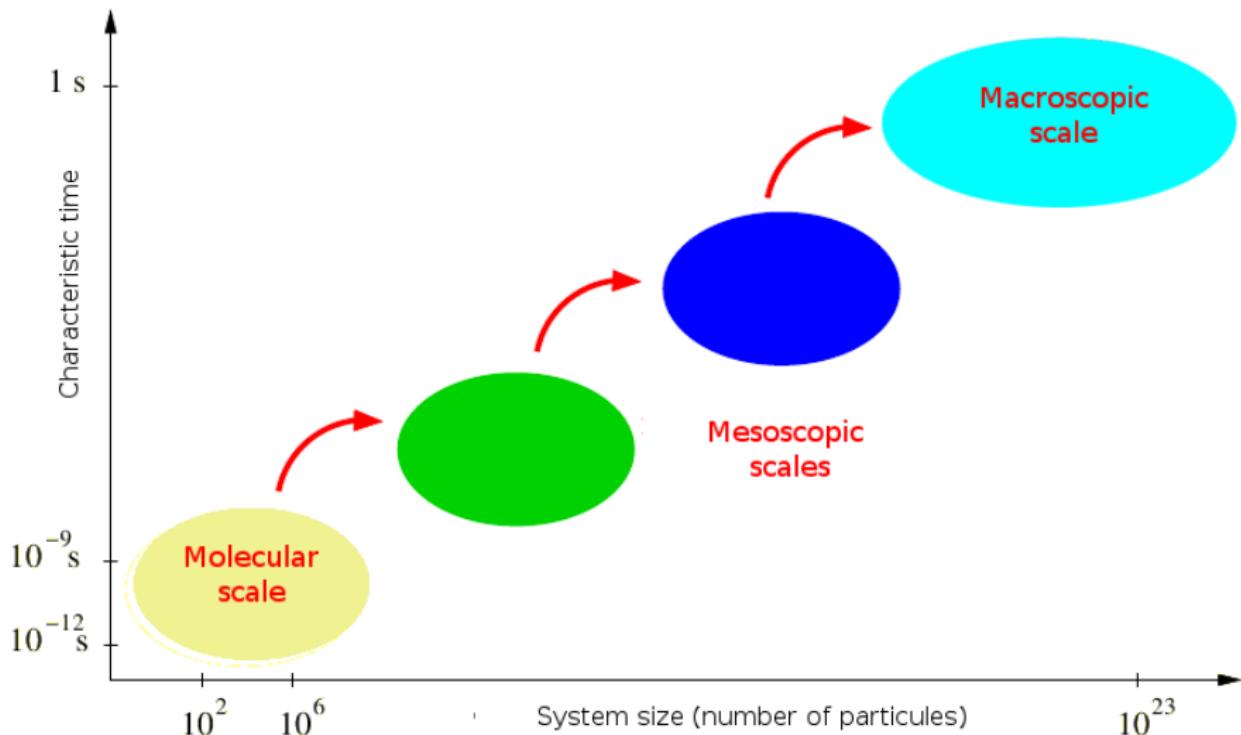
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Joint works with Lingbing He, Claude Le Bris and Tony Lelièvre



From micro to macro

Molecular and Multiscale Modeling



1. Various kinds of fluids
2. Macroscopic description
3. Mesoscopic description
4. Link between both descriptions

Newtonian fluids



- ▶ linear response : stress proportional to the strain rate - the rate of change of its deformation

Non-Newtonian fluids



(a) gels



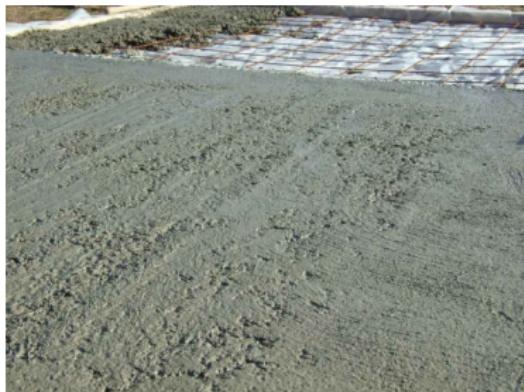
(b) polymer solution

- ▶ some non-linearity
- ▶ presence of some microstructures in the fluid

Specific kind of non-Newtonian fluid



(a) ketchup

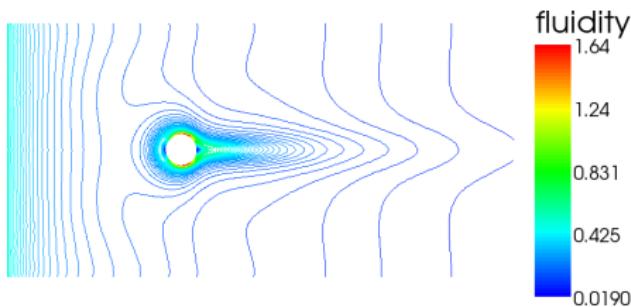
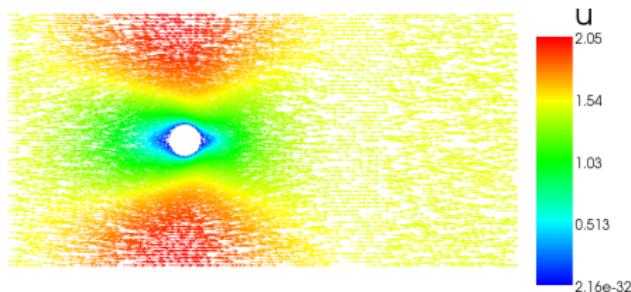


(b) concrete

- ▶ aging
- ▶ flow induced rejuvenation

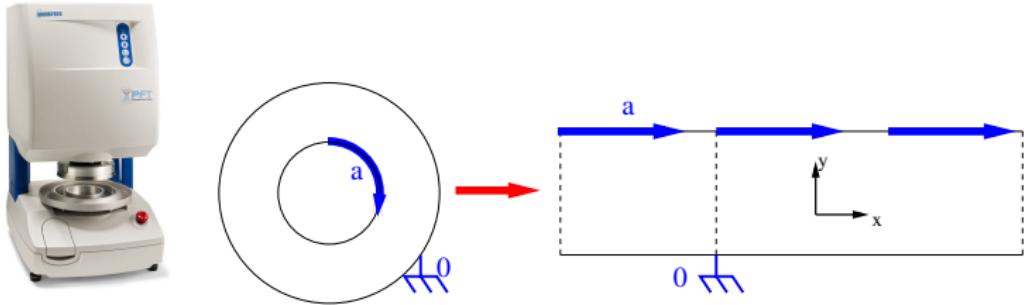
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A test case (2D): flow past a cylinder



- ▶ variables from fluid mechanics function of space x and time t
- ▶ a.e. velocity u
- ▶ additional variable for aging : fluidity f

Another testcase (1D) : Couette flow



Newtonian

$$\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial y} \left(\eta \frac{\partial u}{\partial y} \right)$$

non-Newtonian

$$\begin{cases} \rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial y} \left(\eta \frac{\partial u}{\partial y} + \tau \right), \\ \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y} - \tau, \end{cases}$$

- ▶ boundary conditions $u(t, 0) = 0$ and $u(t, 1) = a \geq 0$ for all $t \in [0, T]$

Macroscopic equations for Couette flow

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \quad (1a)$$

$$\lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y} - f \tau, \quad (1b)$$

$$\frac{\partial f}{\partial t} = \underbrace{-f^2 - \nu f^3}_{\text{aging}} + \underbrace{\xi |\tau| f^2}_{\text{rejuvenation}} \quad (1c)$$

- ▶ C. Derec, F. Lequeux... : λ/f relaxation time of the extra-stress τ :
 $f = 0$ solid
- ▶ empirical equation on f
- ▶ limitations with $f = 0$



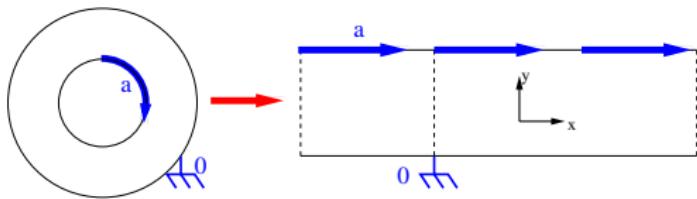
C. Derec, A. Ajdari, and F. Lequeux.

Rheology and aging: a simple approach, *Eur. Phys. J. E*, 4(3):355–361, 2001.

Theoretical results

- ▶ Existence and uniqueness of a global-in-time strong solution in appropriate functional spaces.
Ideas of the existence proof
 - ▶ formal energy estimates
 - ▶ estimates on a sequence of approximate solutions
 - ▶ passage to the limit
- ▶ Longtime behaviour depends on the boundary condition a and the size where the fluidity f vanishes.

Numerical results: evolution of velocity u and fluidity f



Transition

macro

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{A}(u, \tau, f) \\ \frac{\partial \tau}{\partial t} = \mathcal{B}(u, \tau, f) \\ \frac{\partial f}{\partial t} = \mathcal{C}(u, \tau, f) \end{cases}$$

non-linearities in the macroscopic equations

meso

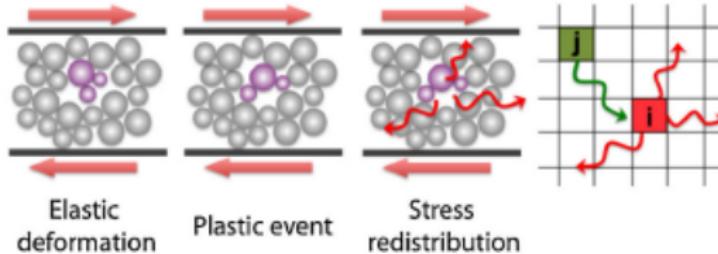
$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{A}(u, \tau, f) \\ \frac{\partial p}{\partial t} = \mathcal{D}(u, p) \\ \tau = \mathcal{E}(p) \\ f = \mathcal{F}(p) \end{cases}$$

average middle-sized description of the microstructures:
 p probability of the stress σ

τ and f are 'moments' of p

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Initial models



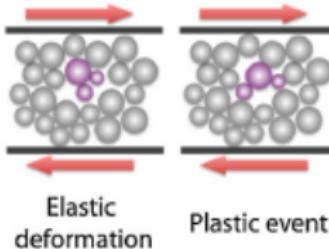
redistribution in the phase space σ :

- P. Hébraud and F. Lequeux,
Mode-coupling theory for the pasty rheology of soft glassy materials,
Phys. Rev. Lett., 81(14):2934–2937, Oct 1998.

redistribution in x and σ :

- L. Bocquet, A. Colin, and A. Ajdari,
Kinetic theory of plastic flow in soft glassy materials,
Phys. Rev. Lett., 103(3):036001, Jul 2009.

Our model : no redistribution



$p(t, \sigma)$ probability to find a stress σ at a time t

$$\frac{\partial p}{\partial t}$$

$$= -\mathbb{1}_{|\sigma| > \sigma_c} p + \underbrace{\left(\int_{|\sigma'| > \sigma_c} p(t, \sigma') d\sigma' \right) \delta_0(\sigma)}_{\text{return to 0 above a threshold } \sigma_c} -$$

$$\underbrace{\gamma(t) \frac{\partial p}{\partial \sigma}}$$

$$\text{elastic deformation } \gamma = \frac{\partial u}{\partial x}$$

Theoretical results

- ▶ Existence and uniqueness of a solution in appropriate functional spaces
- ▶ Longtime behaviour : return to equilibrium at exponential rate

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Averaging

Mesoscopic equation

$$\frac{\partial p}{\partial t} = -\mathbb{1}_{|\sigma| > \sigma_c} p + \left(\int_{|\sigma'| > \sigma_c} p(t, \sigma') d\sigma' \right) \delta_0(\sigma) - \gamma(t) \frac{\partial p}{\partial \sigma} \quad (2)$$

is 'averaged' : $\int (2) \phi(\sigma) d\sigma$

$$\frac{\partial}{\partial t} \int p \phi = \int p(t, \sigma) \{ \phi(0) - \phi(\sigma) \} \mathbb{1}_{|\sigma| > \sigma_c} d\sigma + \gamma(t) \int p(t, \sigma) \frac{\partial \phi}{\partial \sigma}(\sigma) d\sigma.$$

First link

The 'moments' ($\phi(\sigma) = \sigma$, $\mathbb{1}_{|\sigma| > \sigma_c}$)

$$\tau(t) = \int \sigma p(t, \sigma) d\sigma, \quad f(t) = \int \mathbb{1}_{|\sigma| > \sigma_c} p(t, \sigma) d\sigma$$

satisfy

$$\begin{cases} \frac{\partial \tau}{\partial t} = - \int_{|\sigma| > \sigma_c} \sigma p(\cdot, \sigma) d\sigma + \gamma, \\ \frac{\partial f}{\partial t} = -f + \gamma(p(\cdot, \sigma_c) - p(\cdot, -\sigma_c)). \end{cases}$$

- We need a closure.

Second link

Use $\epsilon \rightarrow 0$ and a macroscopic time θ .

Consider p_ϵ the solution of the mesoscopic equation with drift $\gamma_\infty(\epsilon t)$ and its corresponding moments τ_ϵ and f_ϵ .

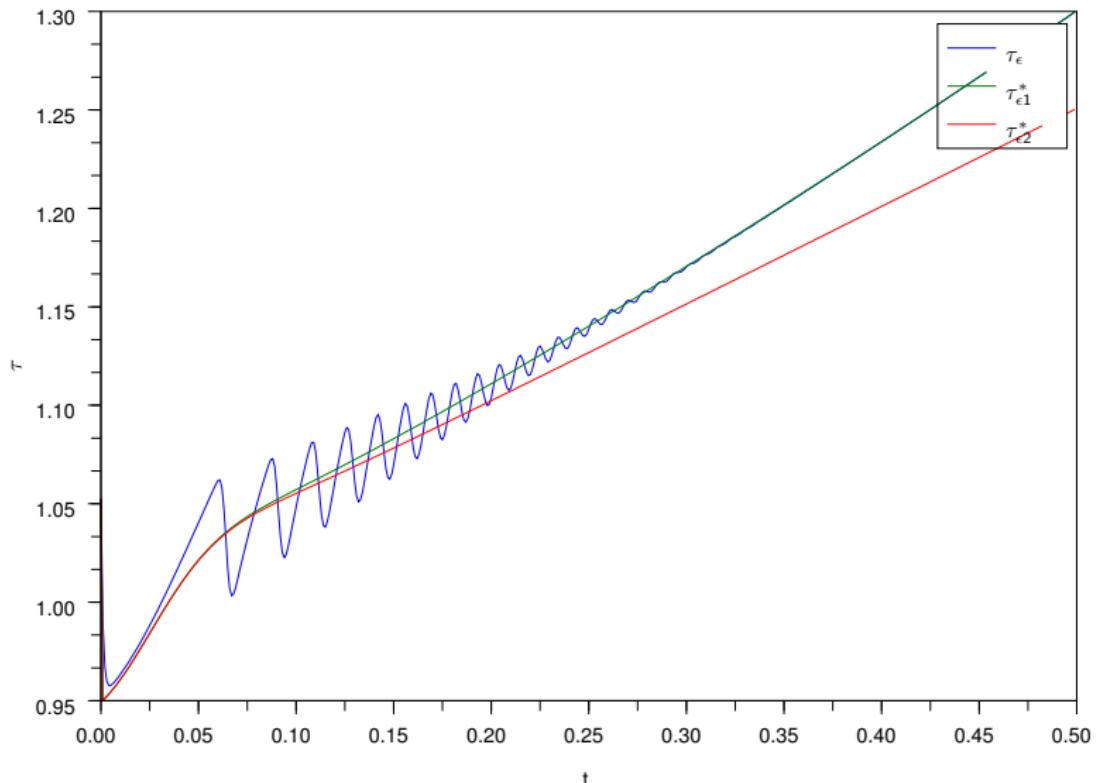
Consider $(\tau_\epsilon^*, f_\epsilon^*)$ solution of

$$\begin{cases} \epsilon \frac{d\tau_\epsilon^*}{d\theta} = -\vartheta f_\epsilon^* \tau_\epsilon^* + \gamma_\infty \\ \epsilon \frac{df_\epsilon^*}{d\theta} = -f_\epsilon^* + \frac{\gamma_\infty}{\sigma_c + \gamma_\infty}. \end{cases}$$

Then,

$$\left| \tau_\epsilon \left(\frac{\theta}{\epsilon} \right) - \tau_\epsilon^*(\theta) \right| + \left| f_\epsilon \left(\frac{\theta}{\epsilon} \right) - f_\epsilon^*(\theta) \right| = O(\epsilon).$$

Numerical results with $\gamma_\infty(s) = s$



Conclusion

- ▶ from a mesoscopic equation on a probability density,
- ▶ derivation of equations on macroscopic variables which are 'moments' of the probability density.



D. Benoit, L. He, C. Le Bris, and T. Lelièvre.

Mathematical analysis of a one-dimensional model for an aging fluid, M3AS, *in press*.

[arXiv:1203.0928](https://arxiv.org/abs/1203.0928).