

Melnikov Theory for Planar Hybrid Systems: Invariant Cones in Piecewise Linear Systems

V. Carmona, S. Fernández-García, E. Freire and F. Torres



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Departamento de Matemática Aplicada II

Inria Project Team: Mycenae (Multiscale dYnamiCs in neuroENdocrine AxEs)

1) MYCENAE Project-Team

Mycenae
Multiscale dYnamiCs in neuroENdocrine AxEs

▼ Presentation ▼ Research News Job offers

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Presentation

Team presentation

Mycenae (Multiscale dYnamiCs in neuroENdocrine AxEs) is a project-team dedicated to mathematical neuroendocrinology and mathematical neurosciences. We are interested in the modeling, analysis and simulation of multiscale in time and/or space dynamics in the fields of neurosciences, endocrinology and physiology.

Mycenae embeds close connections with the Mathematical Neuroscience Laboratory in CIRB (Center for Interdisciplinary Research in Biology, Collège de France) and the Jacques-Louis Lions Laboratory (Pierre & Marie Curie University).

Research themes

- Numerical and theoretical studies of slow-fast systems with complex oscillations
- Non conservative transport equations for cell population dynamics
- Macroscopic limits of stochastic neural networks and neural fields

International and industrial relations

INRA Research Unit on Reproductive Physiology and Behaviors

Boston University (Department of Mathematics and Statistics)

Florida State University (Department of Mathematics)

Pittsburgh University (Department of Mathematics)

Sevilla University (Department of Applied Mathematics)

1) MYCENAE Project-Team: members

The screenshot shows the homepage of the Mycenae project team website. At the top, there is a header with the Inria logo and a search bar. Below the header, there is a navigation menu with links for "Presentation", "Research", "News", and "Job offers". The main content area includes sections for "Presentation", "Team presentation", "Research themes", and "International and industrial relations". The "Research themes" section lists three items: "Numerical and theoretical studies of slow-fast systems with complex oscillations", "Non conservative transport equations for cell population dynamics", and "Macroscopic limits of stochastic neural networks and neural fields". The "International and industrial relations" section lists several research units and universities.

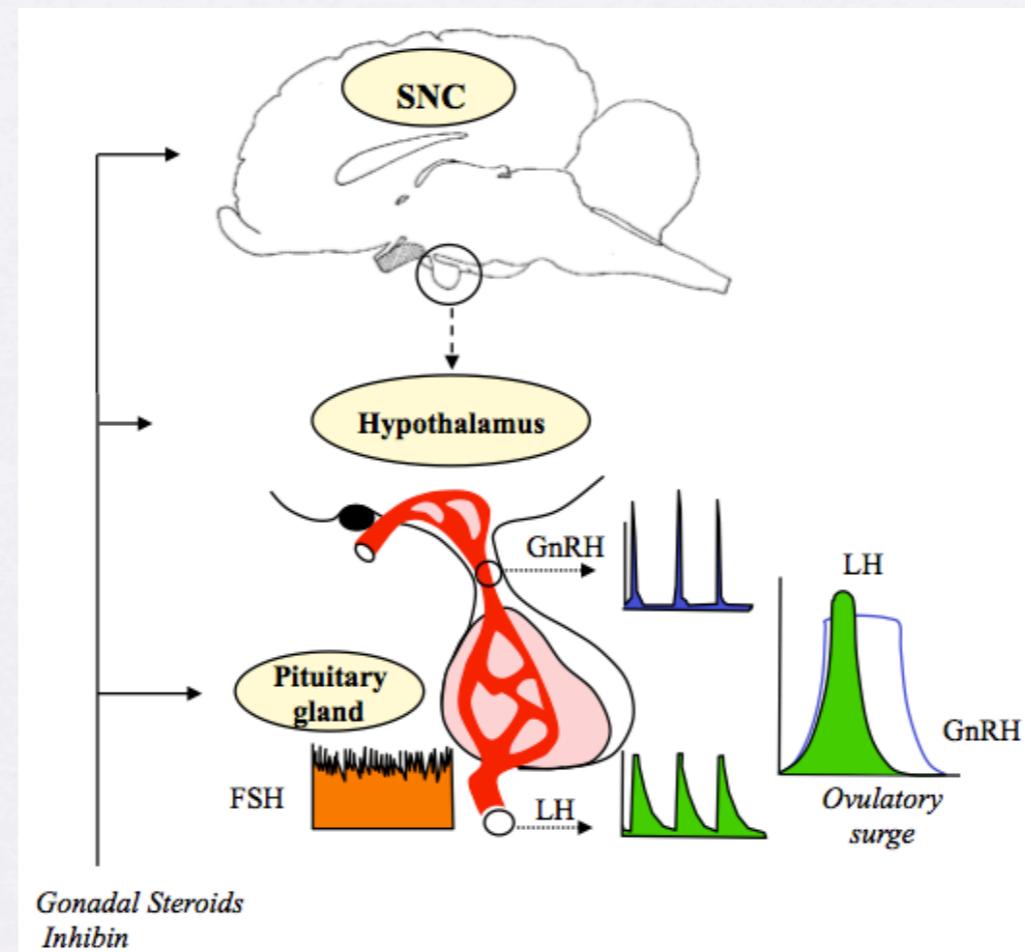
The screenshot shows the "Team members" page from the Mycenae project team website. It is organized into several sections: "Team leader", "Team assistant", "Permanent researchers", "Associate researchers", "PhD students", and "Postdocs". Each section contains a list of names preceded by a double quote icon. The "Team leader" section lists Frédérique Clément. The "Team assistant" section lists Martine Vernerille. The "Permanent researchers" section lists Mathieu Desroches, Maciej Krupa, and Jonathan Touboul. The "Associate researchers" section lists Jean-Pierre Françoise, Marie Postel, and Alexandre Vidal. The "PhD students" section lists Benjamin Aymard, Elif Köksal, and Lucile Megret. The "Postdocs" section lists Albert Granados Corsellas and Soledad Fernández García.

- Team leader**
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1) MYCENAE Project-Team: thematics

Biological background: neuro-endocrinology

- Understanding complex hormonal oscillations (GnRH)



1) MYCENAE Project-Team: thematics

Mathematical tool : multiple timescale dynamical systems

- Core model: GnRH pulse & surge generator
- 4 ODEs with slow and fast variables

Two coupled
FitzHugh-Nagumo

$$\begin{aligned}\varepsilon \delta \frac{dx}{dt} &= -y + f(x) \\ \varepsilon \frac{dy}{dt} &= a_0 x + a_1 y + a_2 + c X \\ \varepsilon \frac{dX}{dt} &= -Y + g(X) \\ \frac{dY}{dt} &= X + b_1 Y + b_2\end{aligned}$$
$$f(x) = -x^3 + 3\lambda x \quad g(X) = -X^3 + 3\mu X$$

3 time scales : $O(1), O(\varepsilon), O(\delta\varepsilon)$

Quasi-stationary regime,
quasi-periodic regime

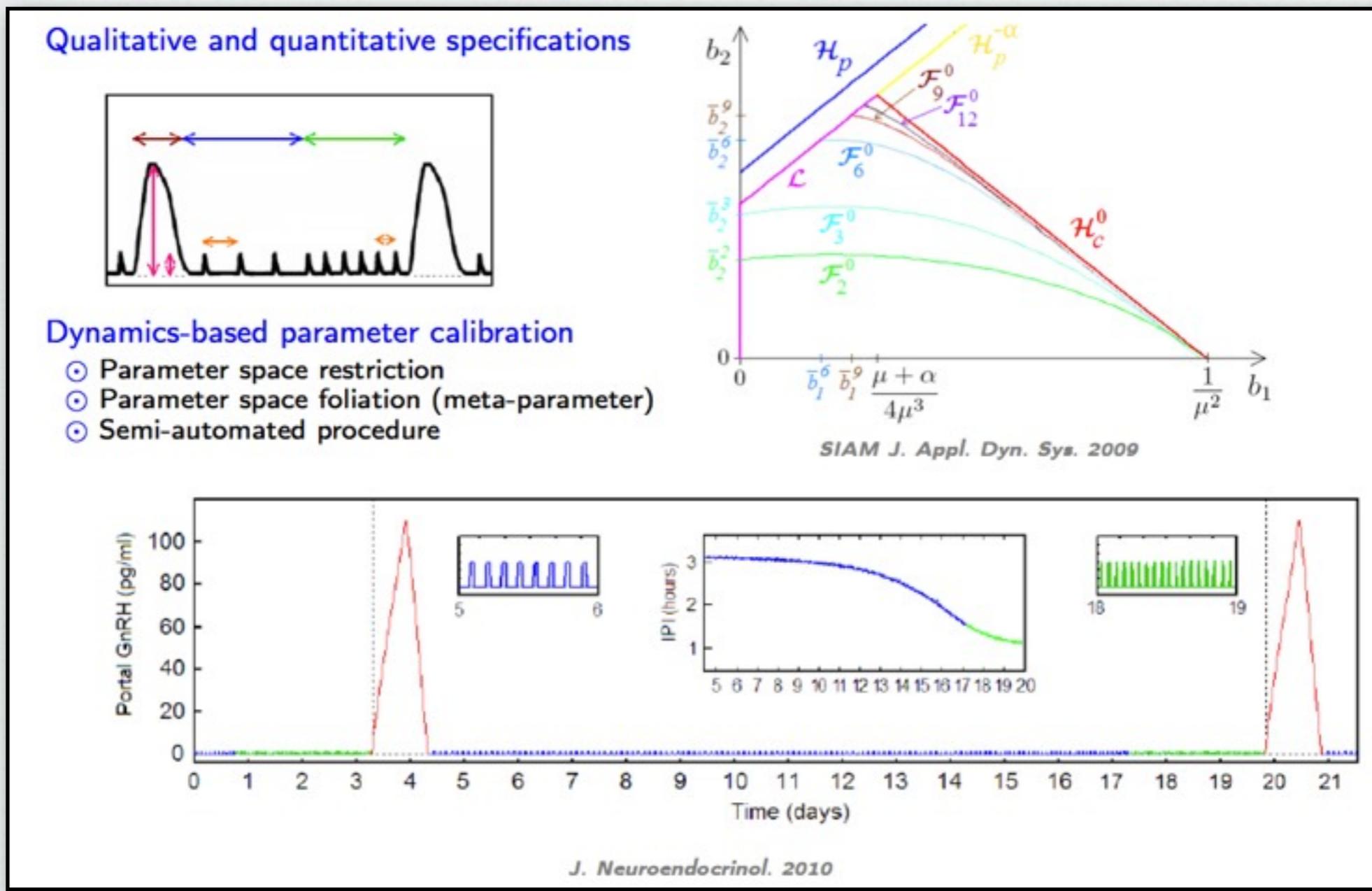
Periodic regime

SIAM J. Appl. Dyn. Syst. 2007

1) MYCENAE Project-Team: thematics

Mathematical tool : multiple timescale dynamical systems

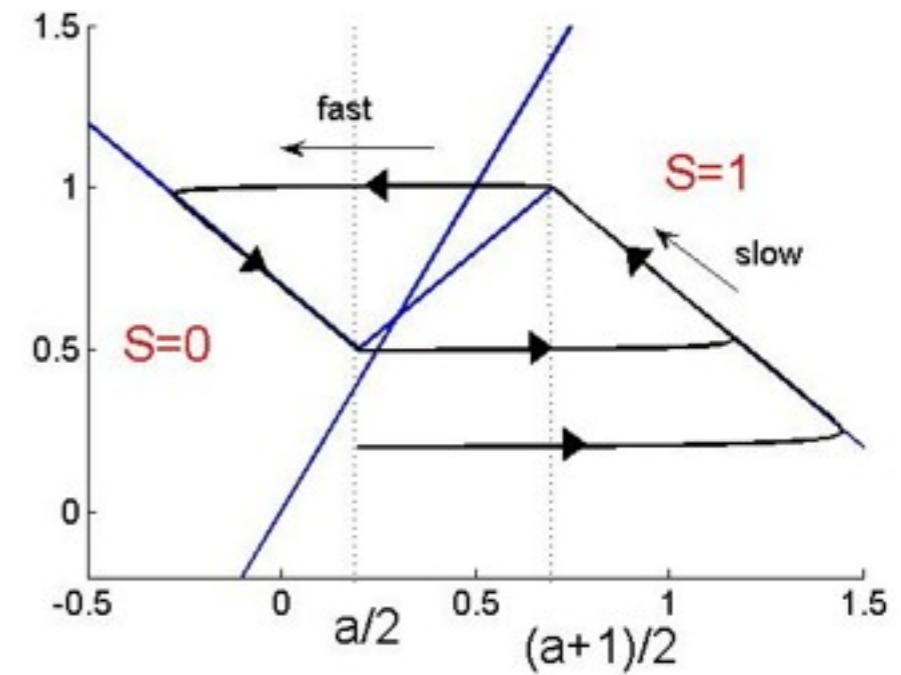
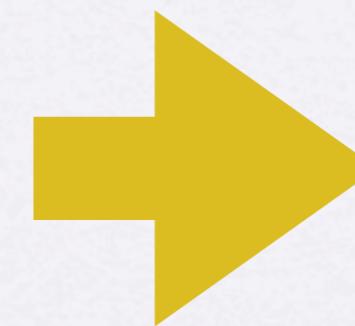
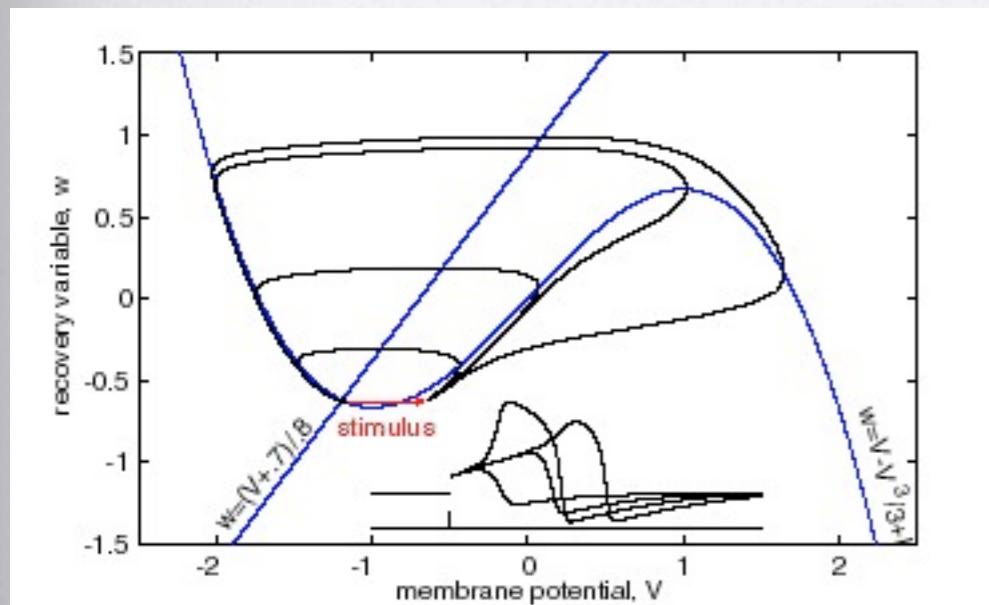
- Phenomenological model with both qualitative & quantitative aspects



1) MYCENAE Project-Team: postdoc

Mathematical tools : multiple timescale dynamics with PieceWise-Linear (PWL) systems

- Idea: replace the FitzHugh-Nagumo model by the PWL equivalent, McKean caricature model



- Goal: keep the richness of the dynamics with better access to quantitative outputs

(source: *scholarpedia*)

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

⌚**Stability of 3D Piecewise Linear Systems**

⌚**Invariant cones in 3D Piecewise Linear Systems**

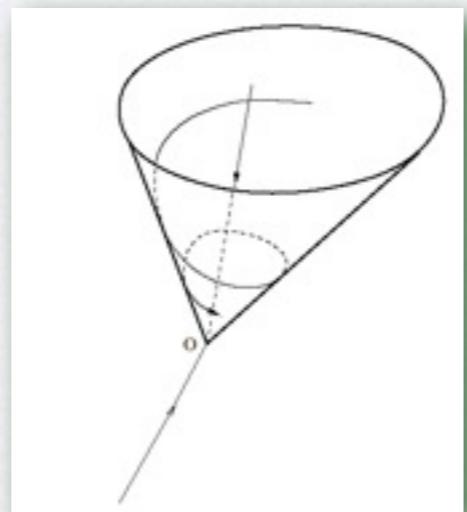
⌚**Periodic Orbits in 2D Hybrid Systems**

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⌚ Stability of 3D Piecewise Linear Systems



⌚ Invariant cones in 3D Piecewise Linear Systems



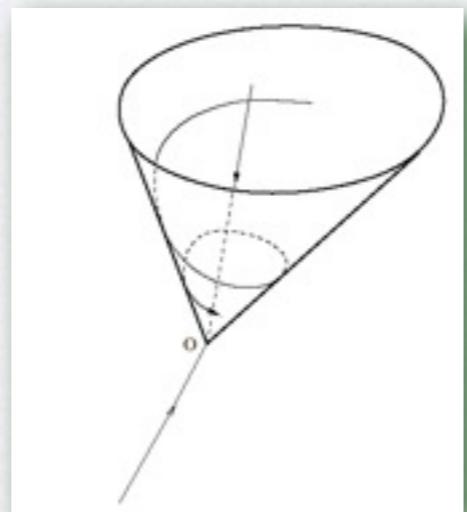
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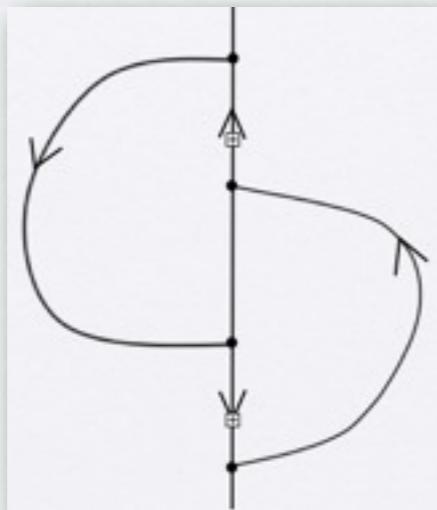


⌚ Invariant cones in 3D Piecewise Linear Systems



V. Carmona, S. F-G, E. Freire, *Saddle-Node Bifurcation of Invariant Cones in 3D Piecewise Linear Systems*. Physica D, 241 (2012) 623–635

⌚ Periodic Orbits in 2D Hybrid Systems



Melnikov Theory

V. Carmona, S. F-G, E. Freire and F. Torres *Melnikov Theory for a Class of Planar Hybrid Systems*. Physica D, 248 (2013) 44-54

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Classical Melnikov Theory

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad \mathbf{f} \in C^1(\mathbb{R}^2)$$

Continuum of periodic orbits with transversal intersection with a section

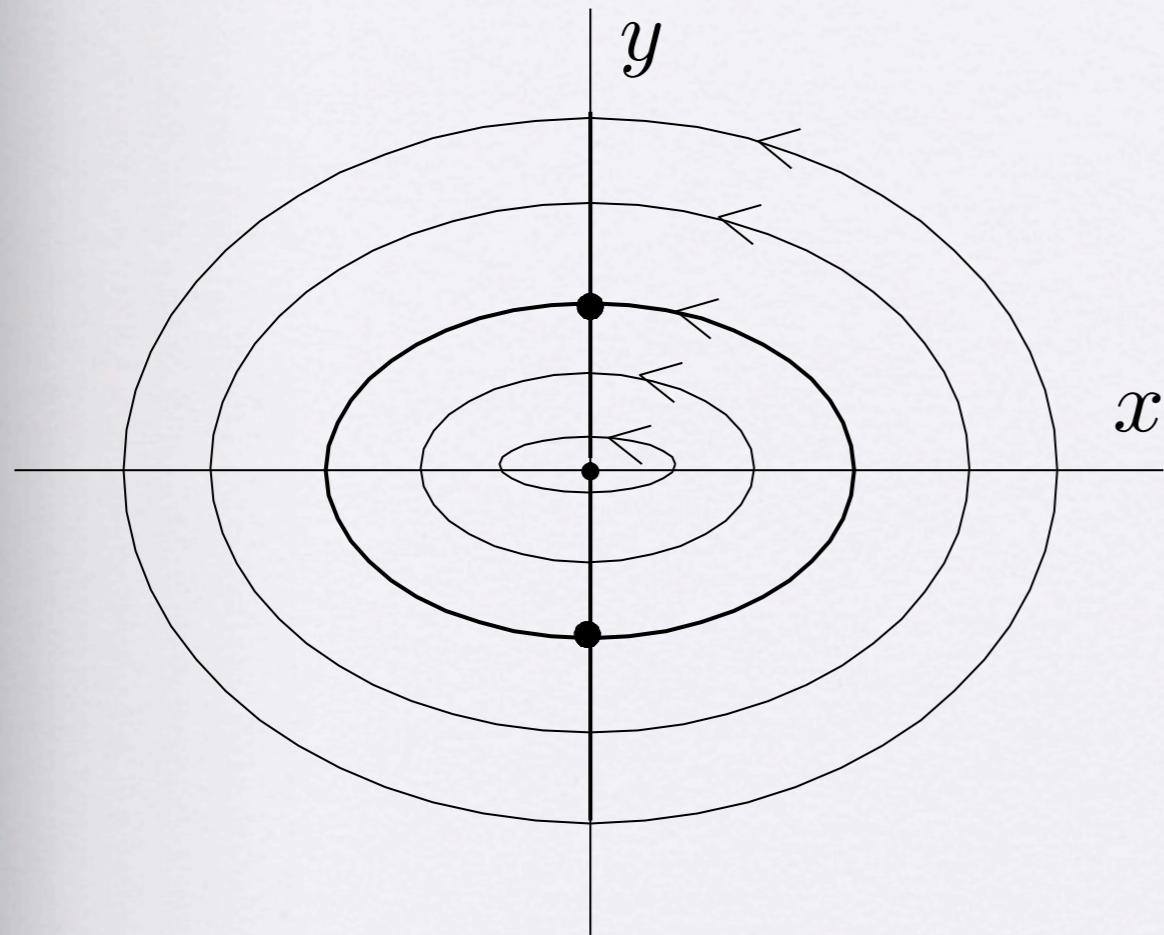
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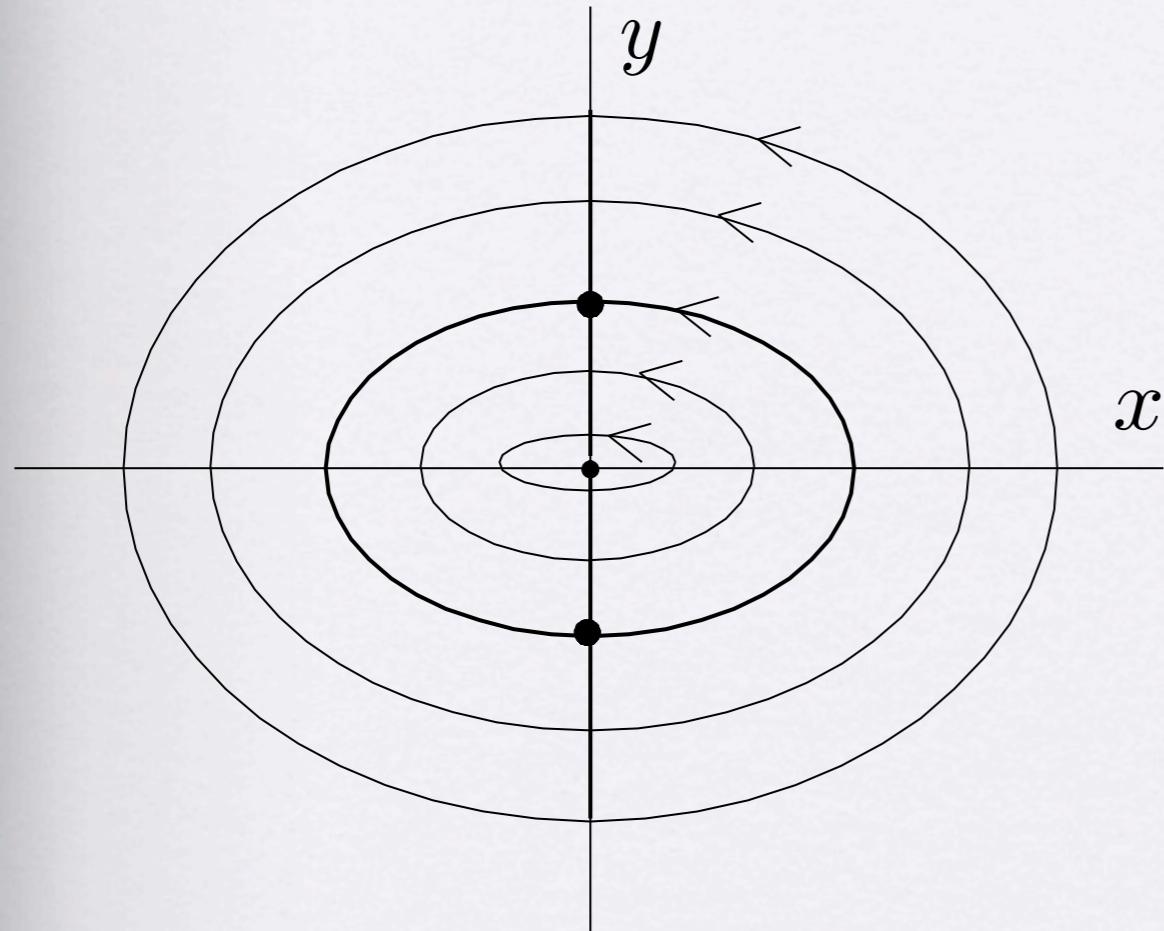
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Continuum of periodic orbits with transversal intersection with a section



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, \varepsilon, \mu)$$

Which periodic orbits
persist after the perturbation?

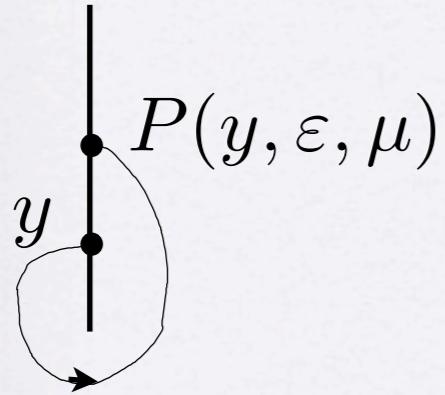
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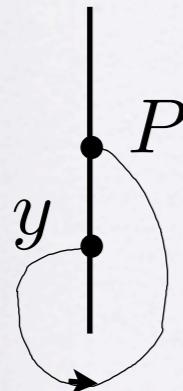
Classical Melnikov Theory



2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Classical Melnikov Theory

Periodic orbits = Fixed points of the **Poincaré map**

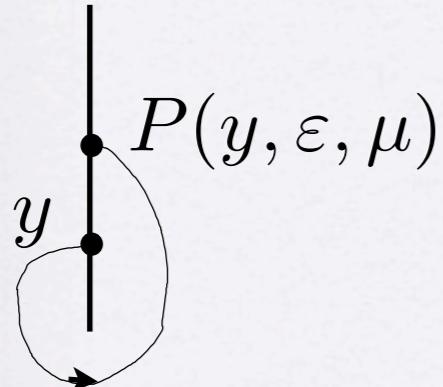


$$P(y, \varepsilon, \mu) = y$$

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Classical Melnikov Theory

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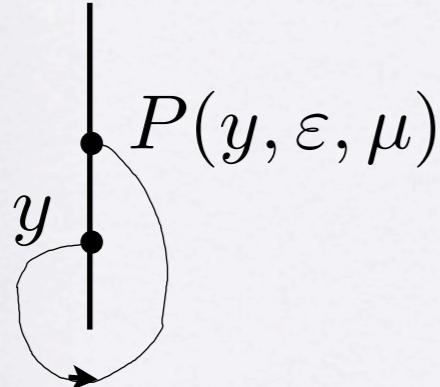
$$d(y, \varepsilon, \mu) = P(y, \varepsilon, \mu) - y = 0$$

Periodic orbits = Zeros of the **displacement function**

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Classical Melnikov Theory

Periodic orbits = Fixed points of the **Poincaré map**



$$P(y, \varepsilon, \mu) = y$$

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Periodic orbits = Zeros of the **displacement function**

$$d(y, 0, \mu) = 0 \longrightarrow d(y, \varepsilon, \mu) = \varepsilon D(y, \varepsilon, \mu)$$

Melnikov function

$$D(y, 0, \mu) = \frac{\partial d}{\partial \varepsilon}(y, 0, \mu)$$

Objective: extending the Melnikov Theory to our Hybrid Systems

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

The unperturbed system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))^T$$

$$x < 0 \qquad \qquad x = 0 \qquad \qquad x > 0$$

$$\mathbf{f}^-(\mathbf{x}) = (f_1^-(\mathbf{x}), f_2^-(\mathbf{x})) \quad \quad \quad \mathbf{f}^+(\mathbf{x}) = (f_1^+(\mathbf{x}), f_2^+(\mathbf{x}))$$

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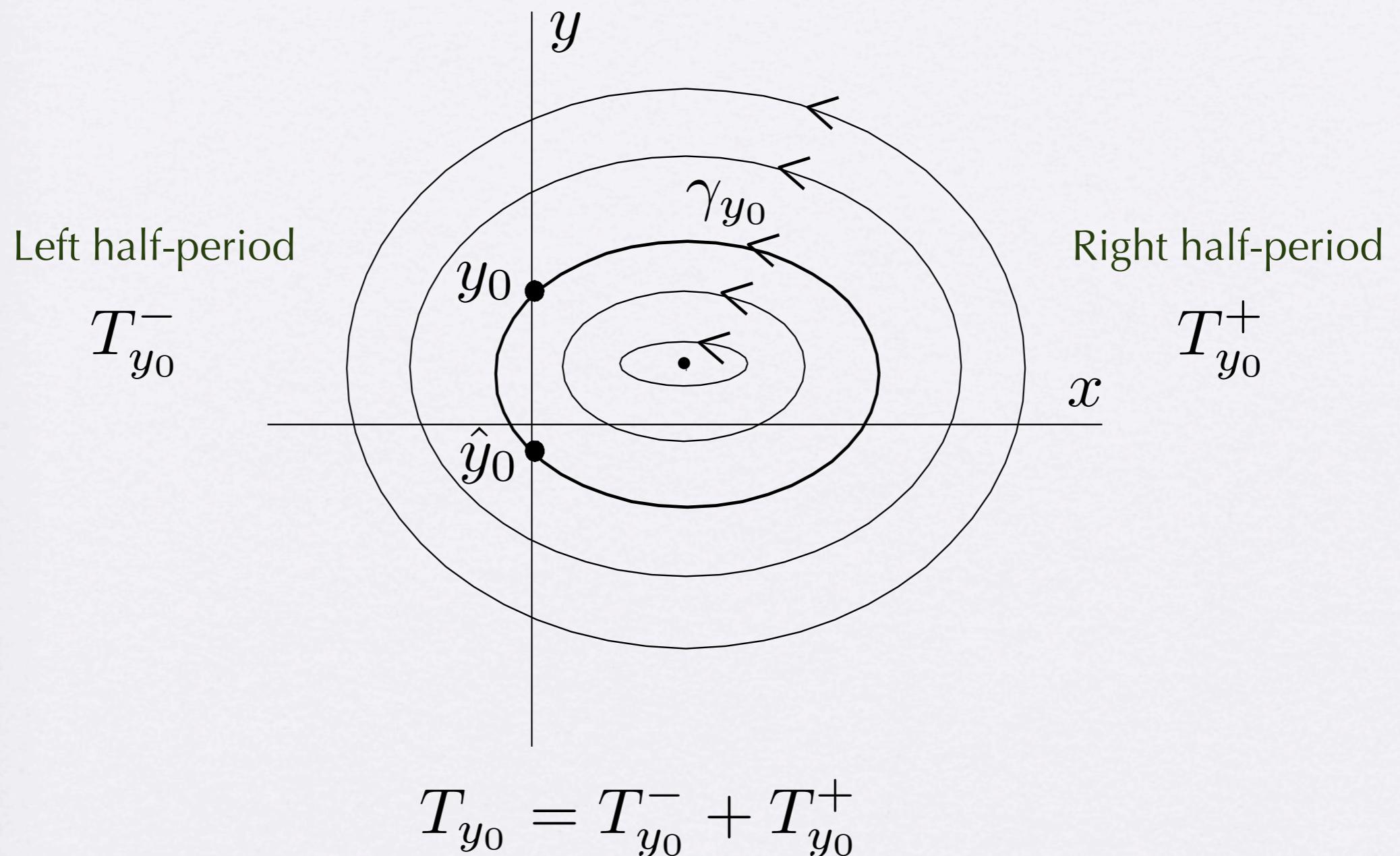
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$$\mathbf{f}^-, \mathbf{f}^+ \in \mathcal{C}^r(\mathbb{R}^2), r \geq 1$$

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

The unperturbed system

Continuum of periodic orbits crossing the separation line transversally



2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

The perturbed system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, \varepsilon, \mu) \quad \begin{array}{ccc} \eta : \mathbb{R}^{k+2} & \longrightarrow & \mathbb{R} \\ (y_0, \varepsilon, \mu) & \longmapsto & \eta(y_0, \varepsilon, \mu) \end{array}$$

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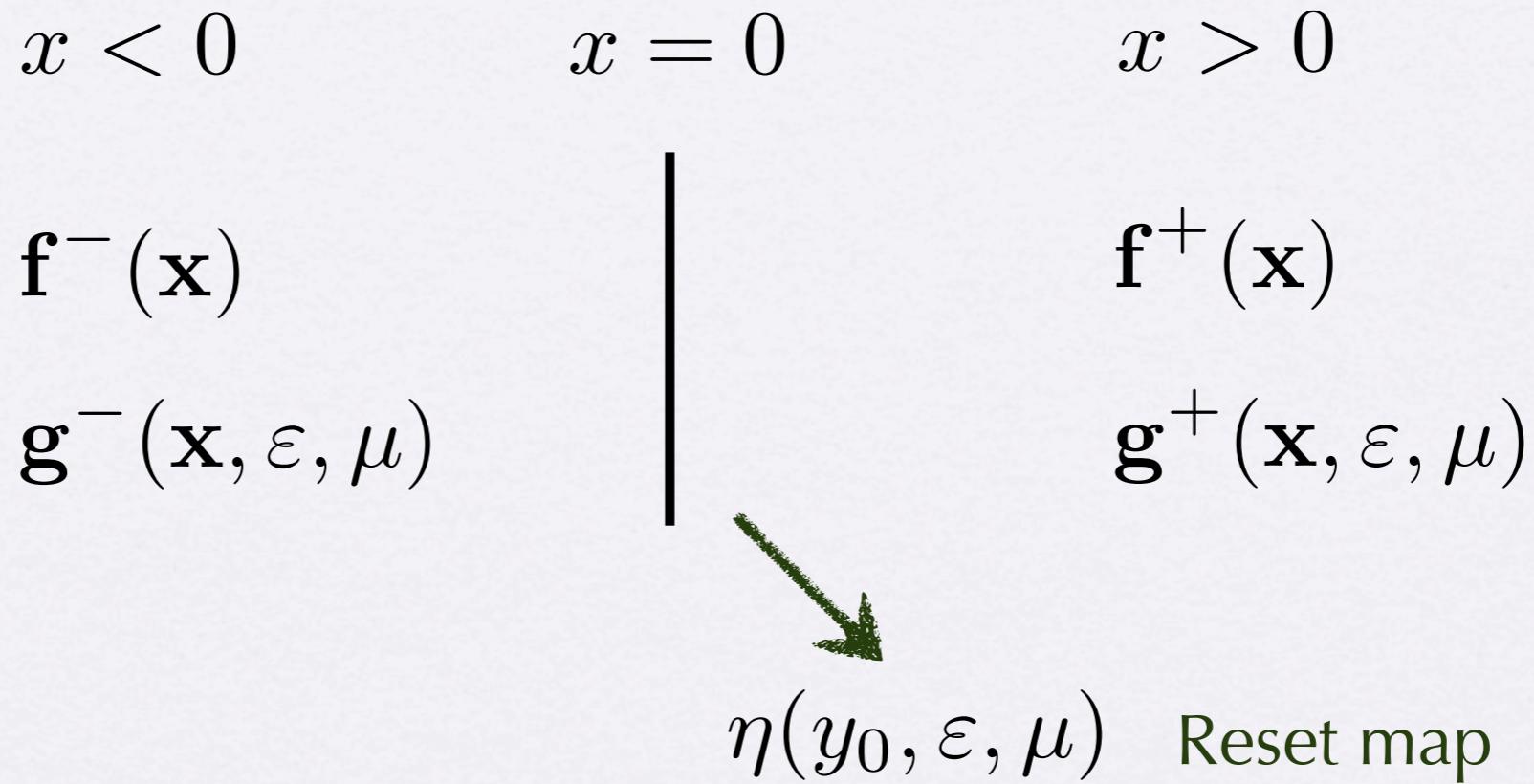
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$$\mathbf{g}^+, \mathbf{g}^- \in \mathcal{C}^r(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^k), r \geq 1, k \in \mathbb{N}$$

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$$\eta \in \mathcal{C}^r(\mathbb{R}^{k+2})$$

$$\eta(y_0, 0, \mu) = y_0 \in \mathbb{R}, \quad f_1(0, y_0) \cdot f_1(0, \eta(y_0, \varepsilon, \mu)) > 0$$

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Poincaré map and Displacement function

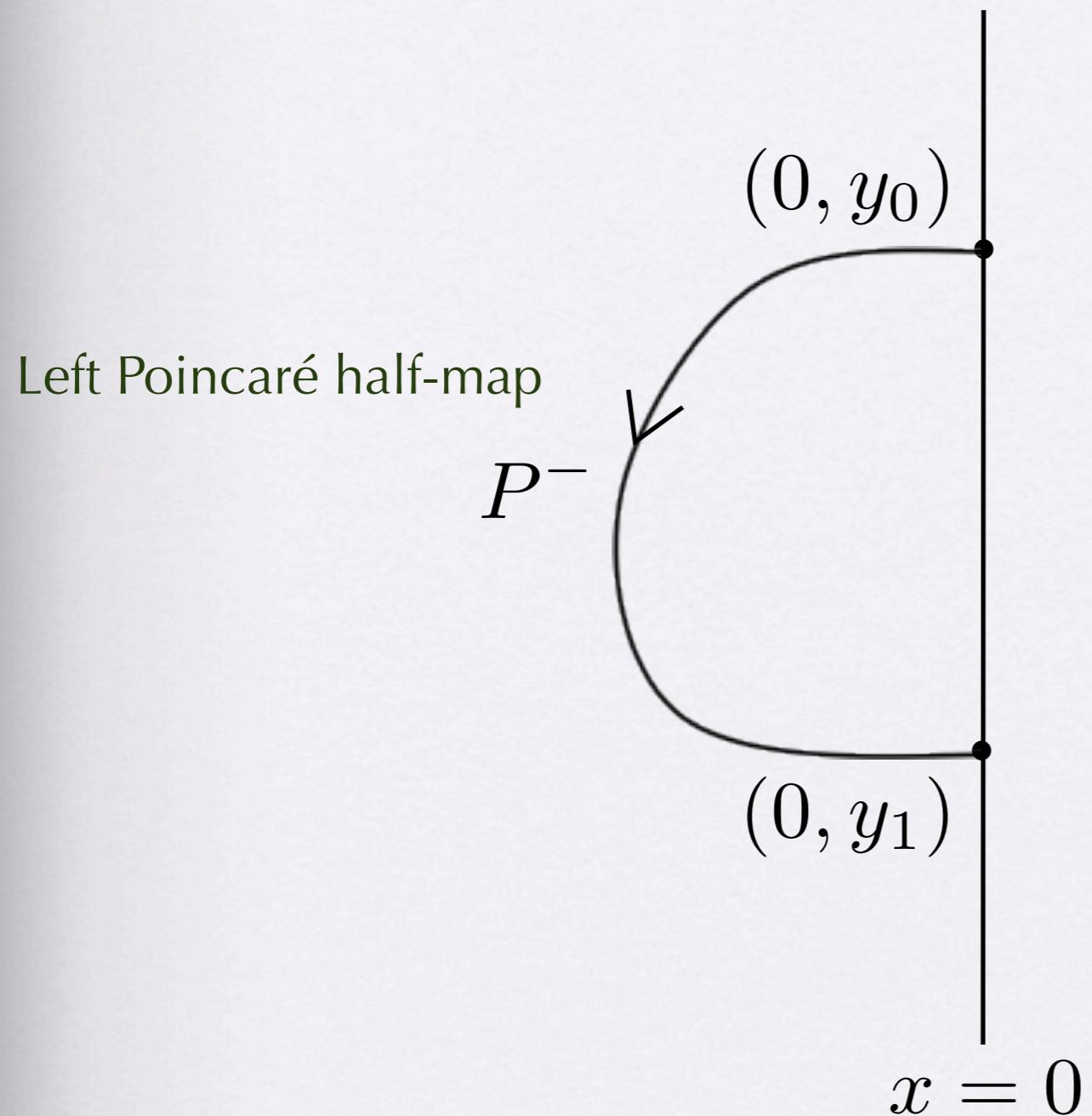
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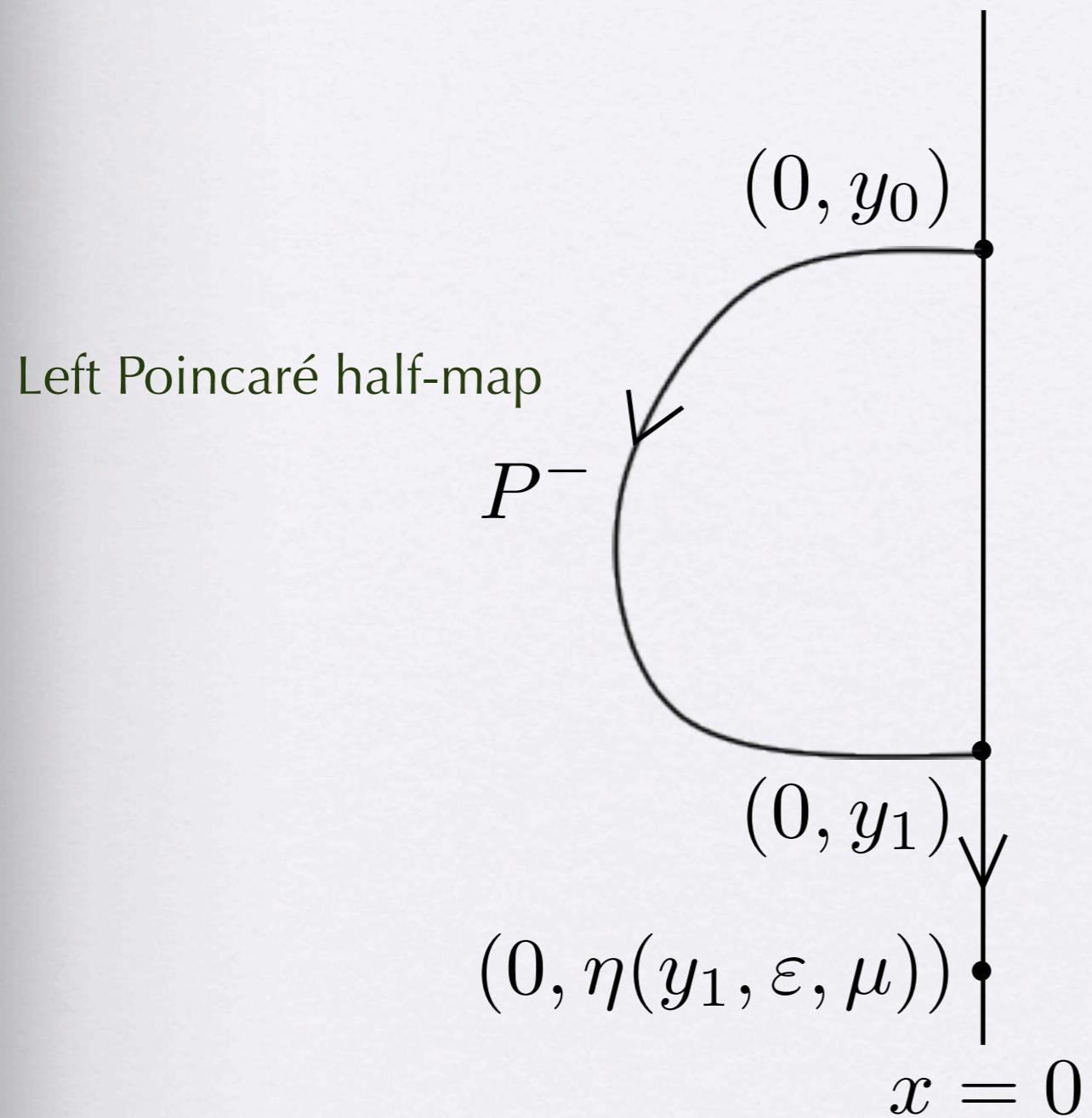
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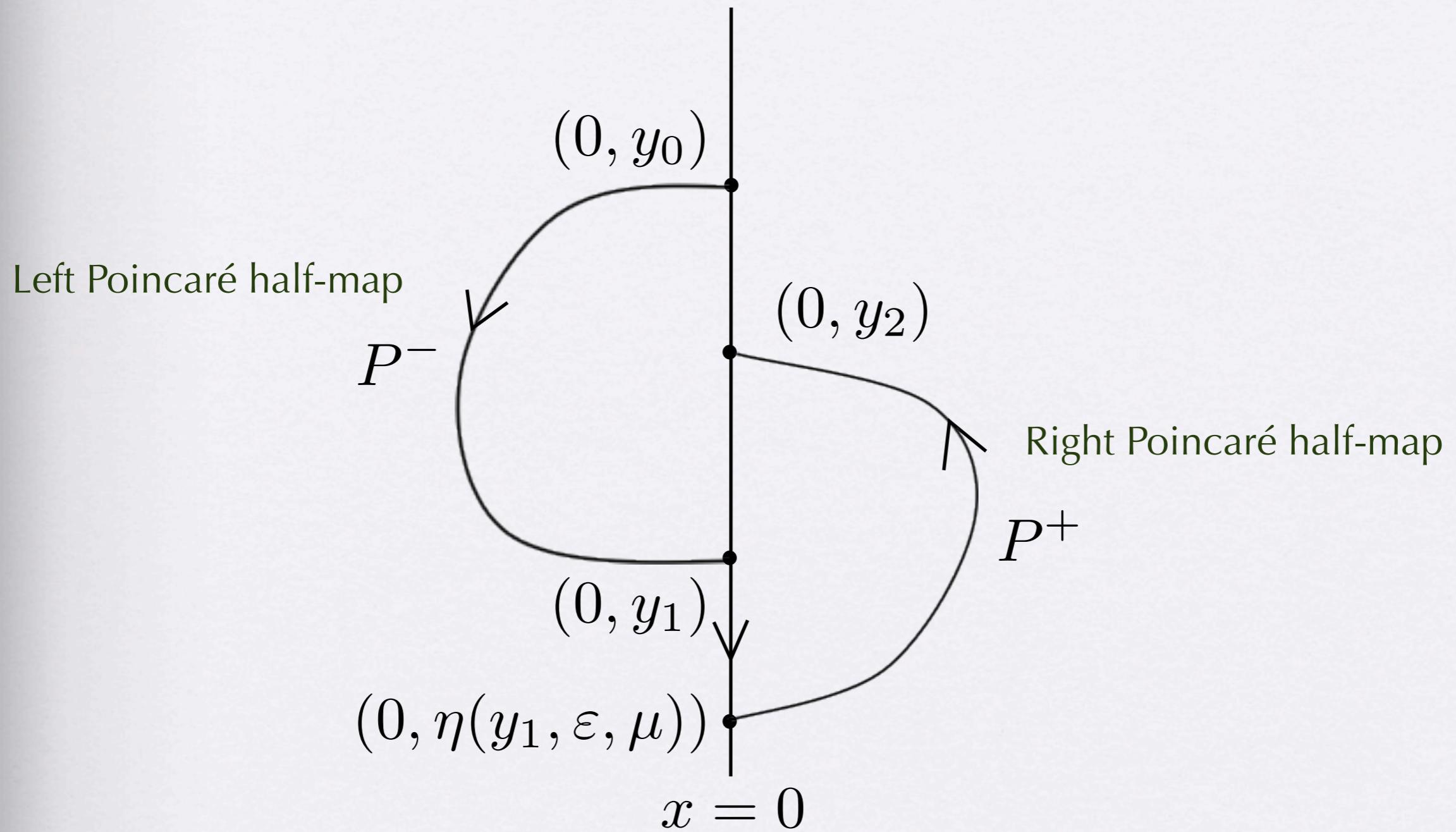
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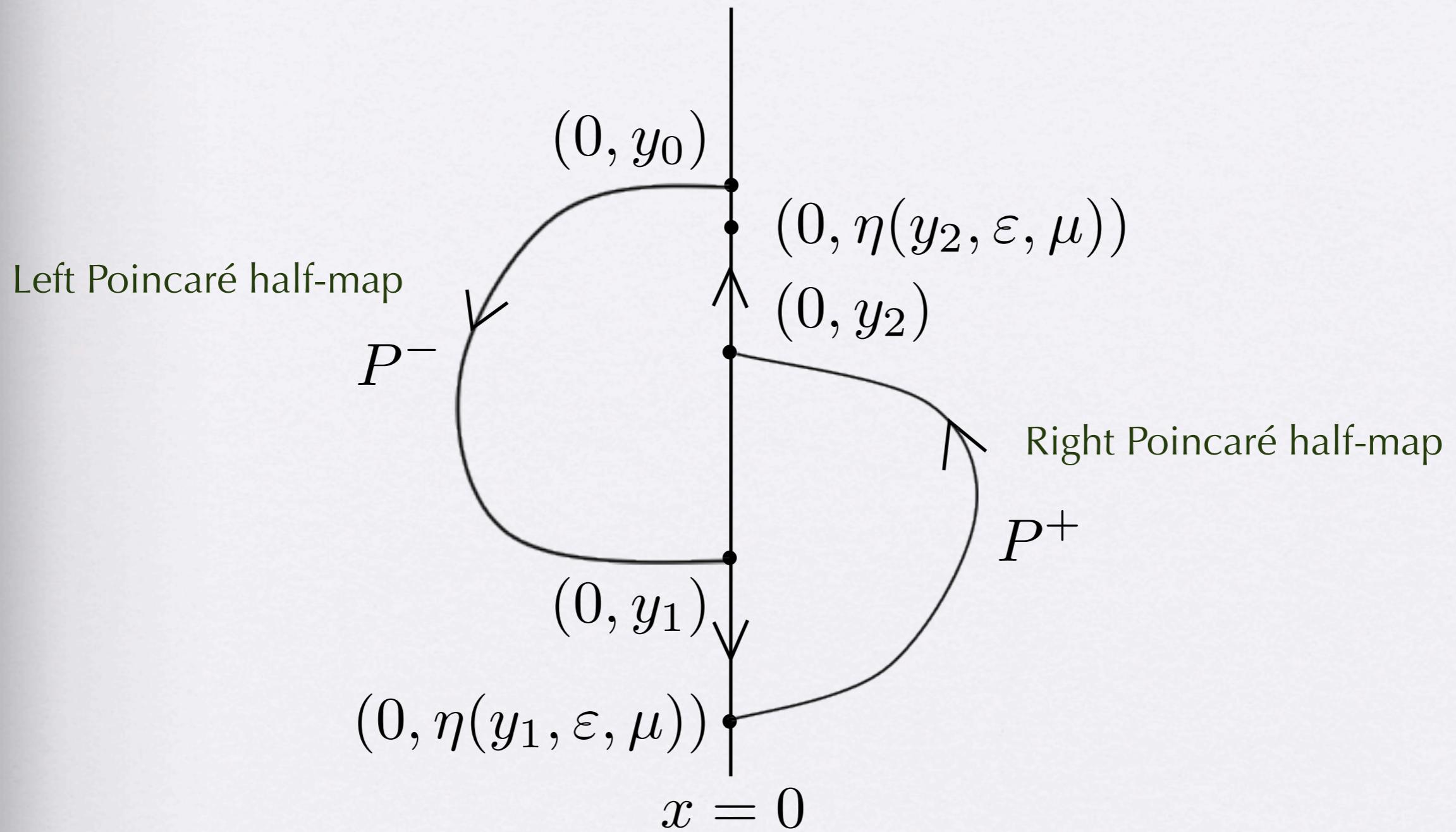
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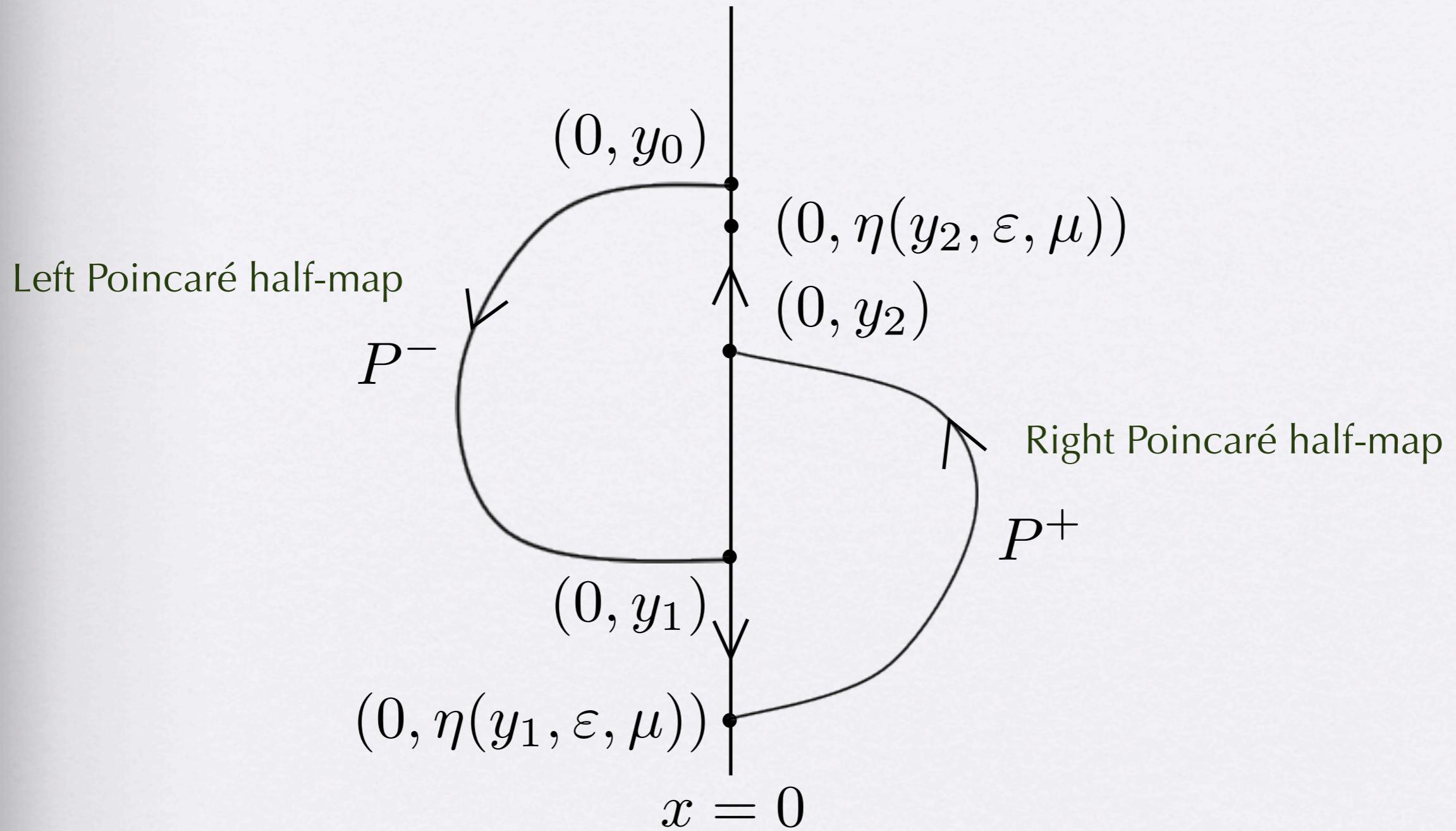
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2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Poincaré map and Displacement function

$$P(y_0, \varepsilon, \mu) = \eta(P^+(\eta(P^-(y_0, \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu)$$

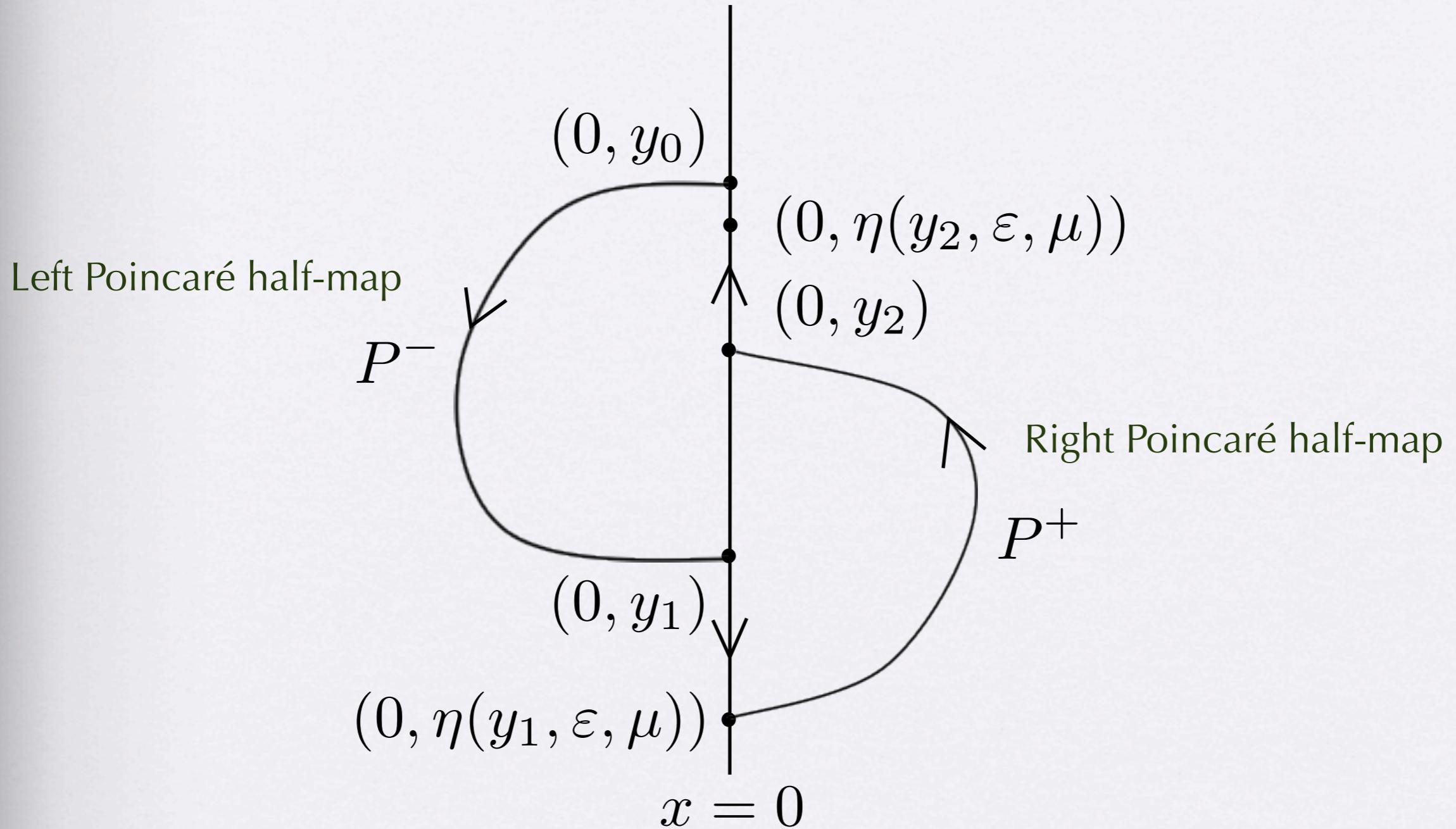


2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Poincaré map and Displacement function

$$P(y_0, \varepsilon, \mu) = \eta(P^+(\eta(P^-(y_0, \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu)$$

$$d(y_0, \varepsilon, \mu) = P(y_0, \varepsilon, \mu) - y_0$$



2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

The Melnikov function for Planar Hybrid Systems

$$M(y_0; \mu) = \frac{\partial d}{\partial \varepsilon}(y_0, 0, \mu)$$

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The Melnikov function for Planar Hybrid Systems

$$M(y_0; \mu) = \frac{\partial d}{\partial \varepsilon}(y_0, 0, \mu)$$

$$\begin{aligned} M(y_0; \mu) &= \frac{\partial \eta}{\partial \varepsilon}(y_0, 0, \mu) + \frac{\rho^+(T_{y_0}^+)}{f_1^+(0, y_0)} \frac{\partial \eta}{\partial y_0}(y_0, 0, \mu) \cdot \\ &\cdot \left[\frac{f_1^+(0, \hat{y}_0)}{f_1^-(0, \hat{y}_0)} \rho^-(T_{y_0}^-) \frac{\partial \eta}{\partial y_0}(\hat{y}_0, 0, \mu) \int_0^{T_{y_0}^-} \frac{\mathbf{f}^-(\gamma_{y_0}(t)) \wedge \mathbf{g}^-(\gamma_{y_0}(t), 0, \mu)}{\rho^-(t)} dt + \right. \\ &\left. \frac{\partial \eta}{\partial \varepsilon}(\hat{y}_0, 0, \mu) f_1^+(0, \hat{y}_0) + \int_0^{T_{y_0}^+} \frac{\mathbf{f}^+(\gamma_{y_0}(t)) \wedge \mathbf{g}^+(\gamma_{y_0}(t), 0, \mu)}{\rho^+(t)} dt \right], \end{aligned}$$

$$\hat{y}_0 = P^-(y_0, 0, \mu) \quad \mathbf{x} \wedge \mathbf{y} = x_1 y_2 - x_2 y_1$$

$$\rho^-(t) = \exp \left(\int_0^t \operatorname{div} \mathbf{f}^-(\gamma_{y_0}(\tau)) d\tau \right), \quad \rho^+(t) = \exp \left(\int_0^t \operatorname{div} \mathbf{f}^+(\gamma_{\hat{y}_0}(\tau)) d\tau \right)$$

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If the first component of the unperturbed system is continuous and the reset map is the identity, then the function M reduces to the classical Melnikov function.

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Applications: 1. Periodic Orbits in Discontinuous Systems

$$\begin{cases} \dot{x} = 2(\varepsilon\omega - \alpha)x - y, \\ \dot{y} = ((\varepsilon\omega - \alpha)^2 + 1)x + \varepsilon A_-, \end{cases} \quad x < 0,$$

$$\begin{cases} \dot{x} = 2\alpha x - y + \varepsilon B, \\ \dot{y} = (\alpha^2 + 1)x + \varepsilon A_+, \end{cases} \quad x > 0,$$

$$\begin{cases} \dot{x} = 2\sigma\varepsilon x - y, \\ \dot{y} = x - a, \end{cases} \quad x < 0,$$

$$\begin{cases} \dot{x} = 2\sigma\varepsilon x - y - \varepsilon B, \\ \dot{y} = x - |a|, \end{cases} \quad x > 0.$$

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

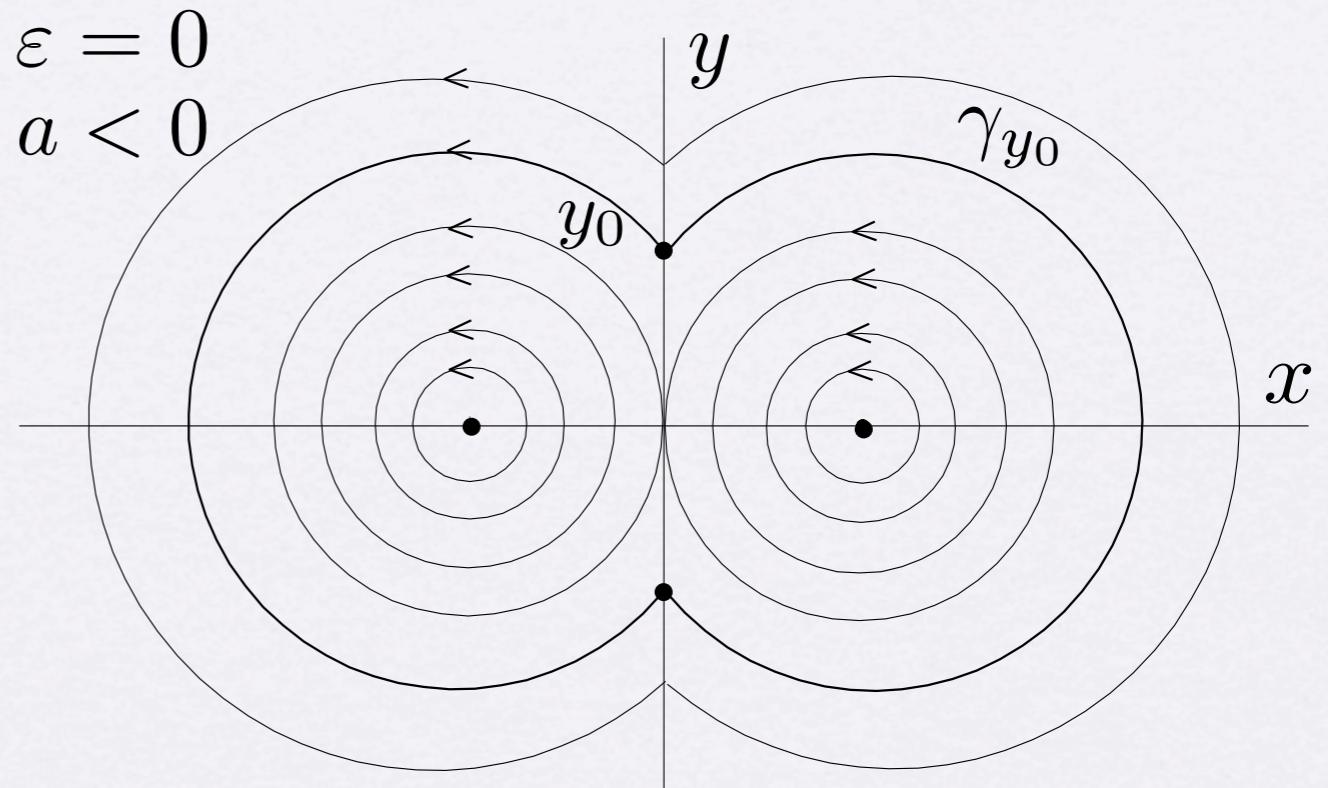
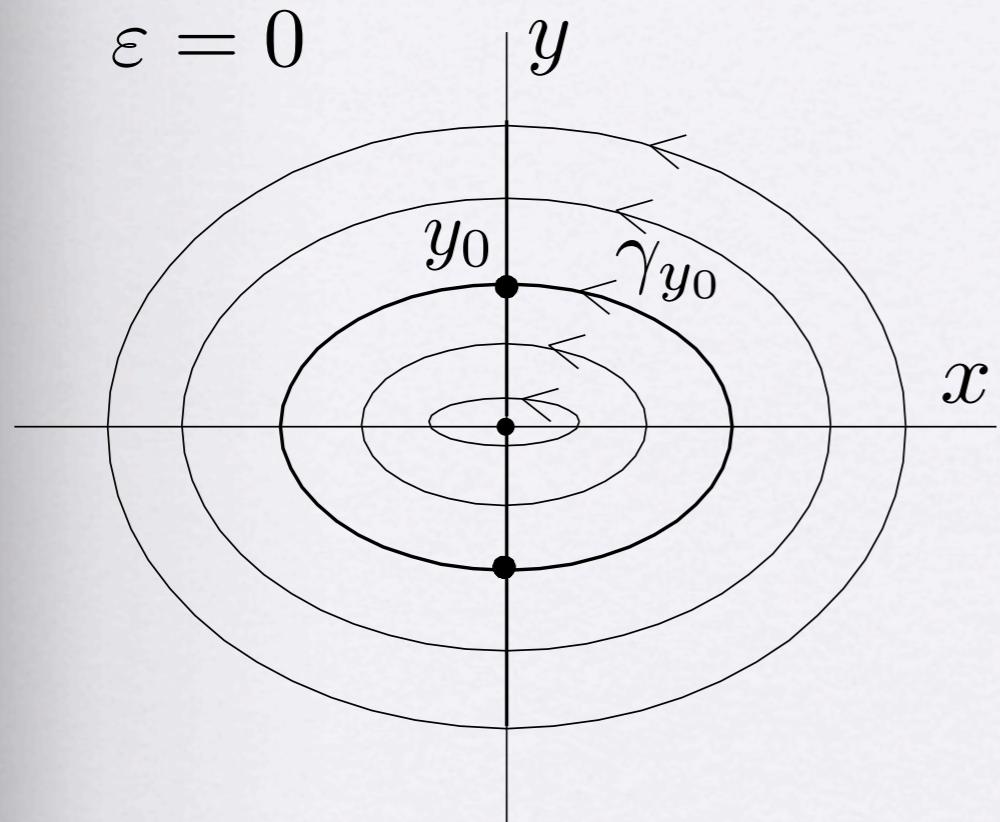
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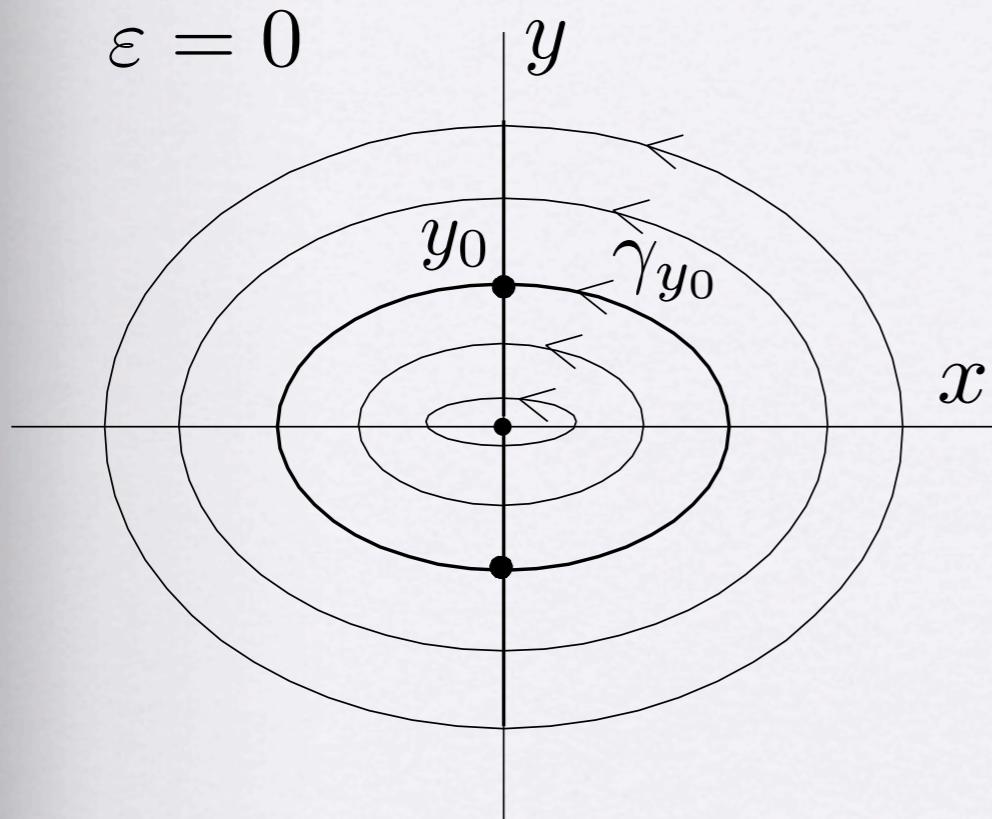
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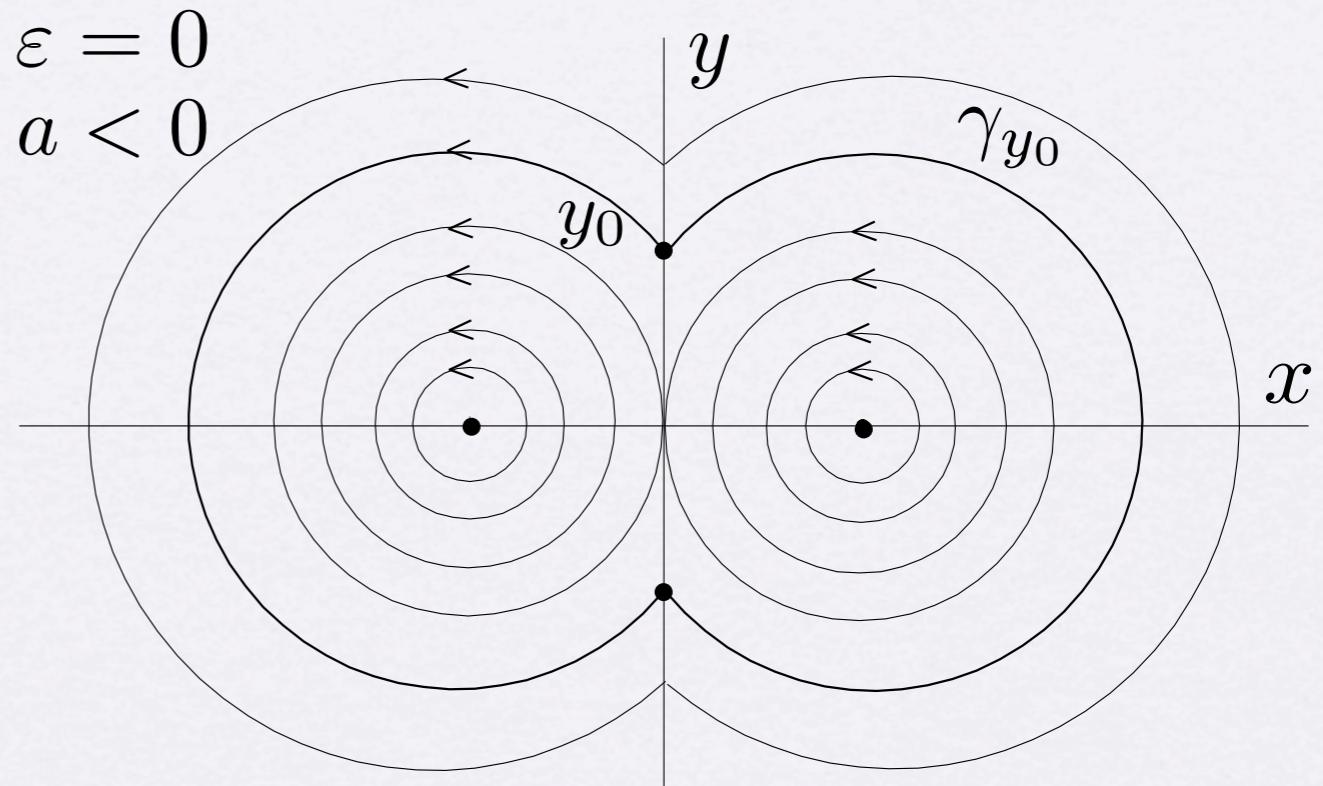
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$$\begin{aligned} \omega \neq 0, \\ \operatorname{sgn}(\omega) = \operatorname{sgn}(2\alpha(A_+ + A_-) - B(\alpha^2 + 1)) \end{aligned}$$

There exists one periodic orbit



$$\begin{aligned} a < 0, \sigma \neq 0 \\ B = -2a\sigma(1 + 2\pi) \end{aligned}$$

Saddle-node bifurcation of two-zonal periodic orbits

2) Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

Applications: 2. Periodic Orbits in Continuous Systems

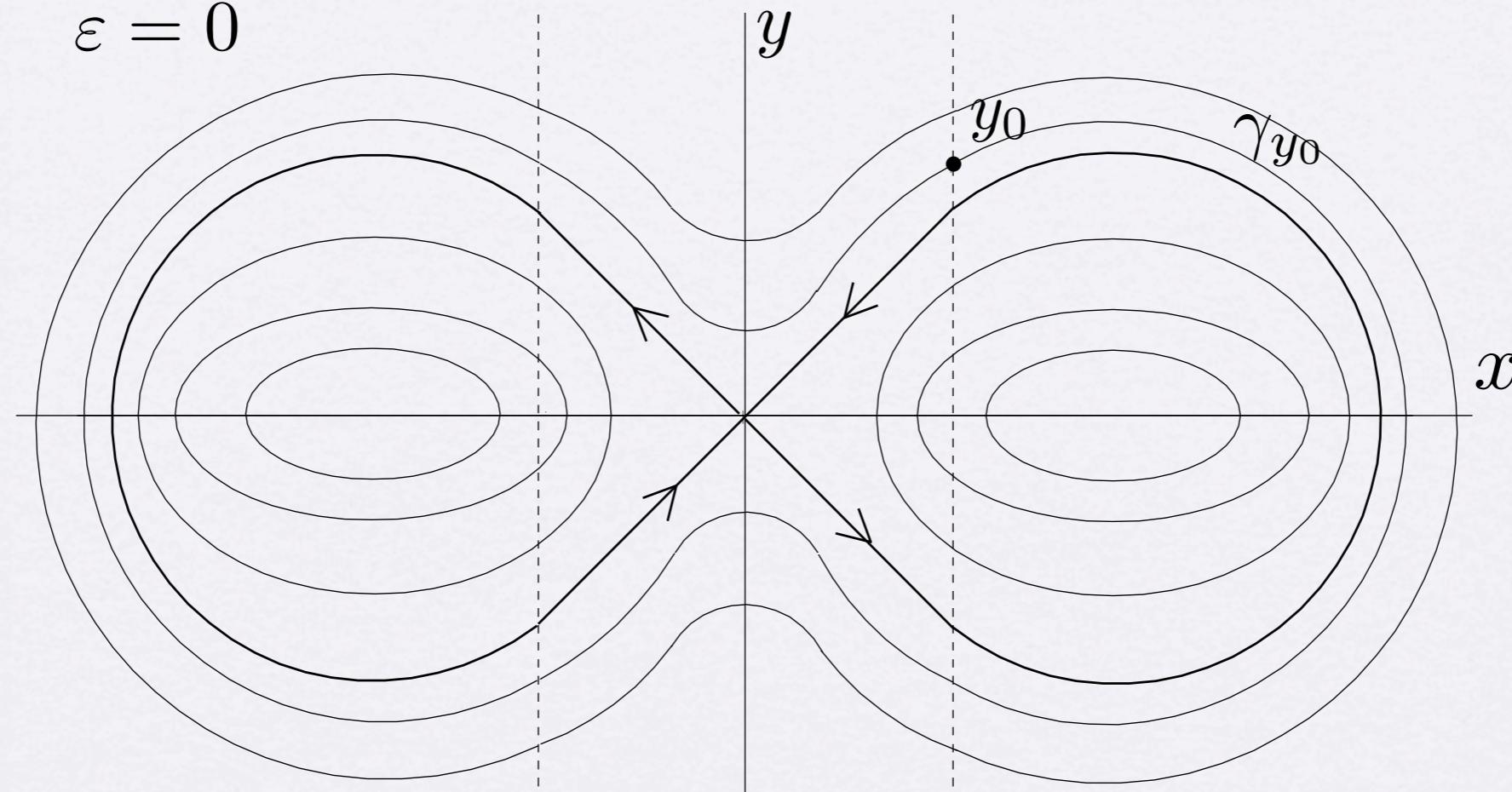
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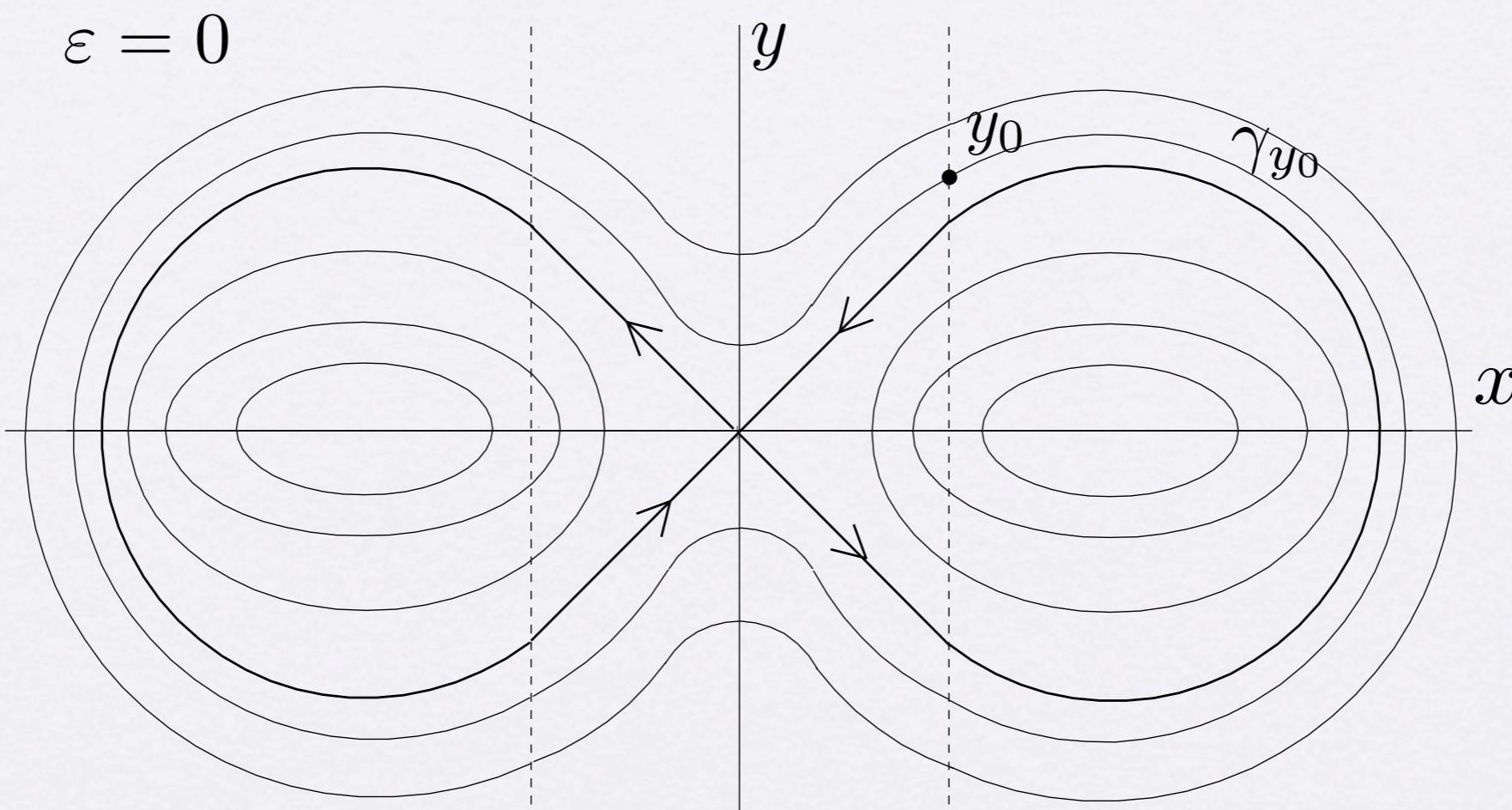
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Saddle-node bifurcation of three-zonal periodic orbits

V. Carmona, S. F-G, E. Freire and F. Torres *Melnikov Theory for a Class of Planar Hybrid Systems*. Physica D, 248 (2013) 44-54

3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory

Why Invariant Cones?

With discontinuity, the matching of stable linear systems can be unstable.

M. S. Branicky, *Multiple Lyapunov functions and other analysis tools for switched and hybrid systems*. IEEE Trans. Automat. Contr., 43, 4 (1998) 475-482

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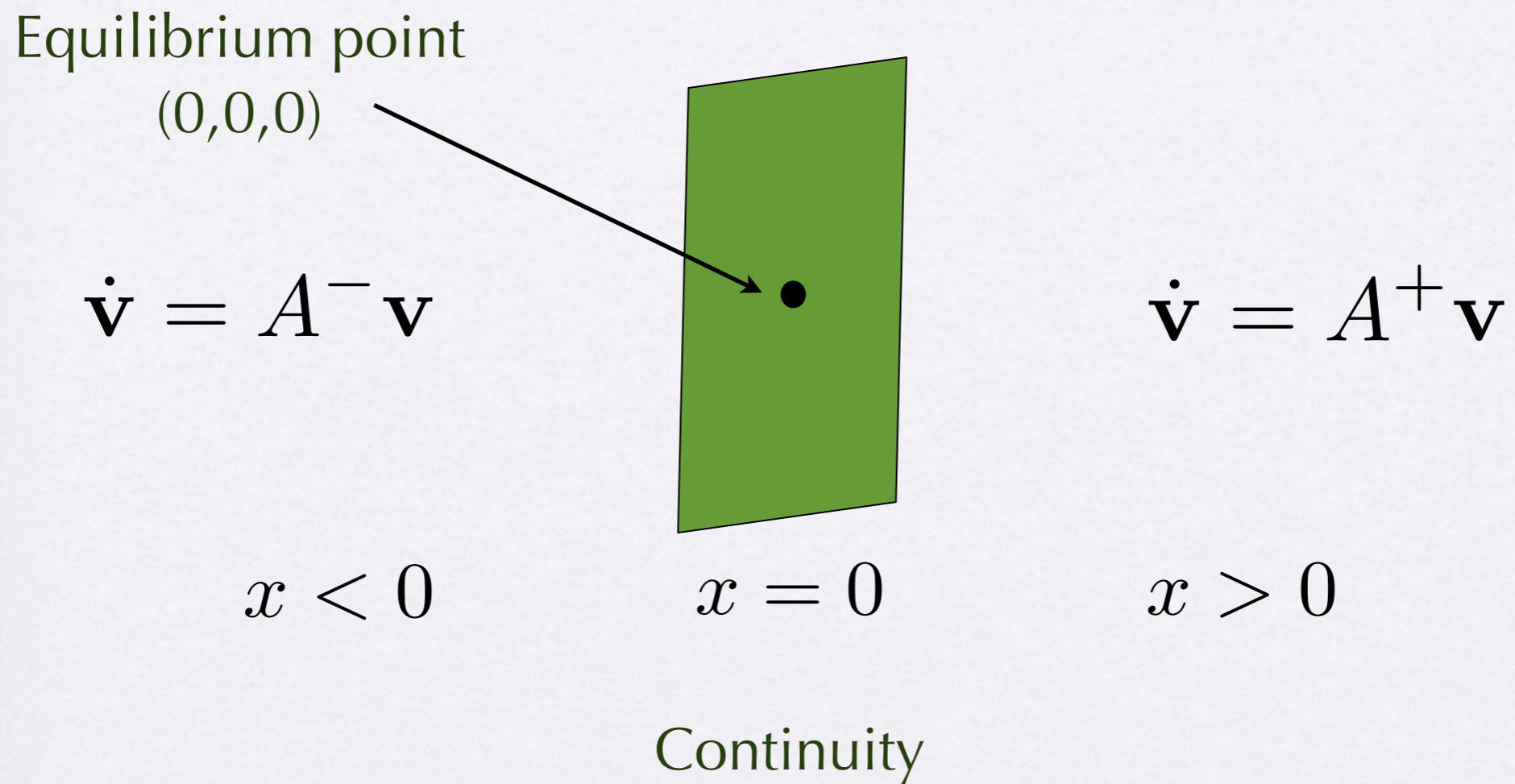
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Which geometric and dynamic elements arise?
How they affect the stability/instability?

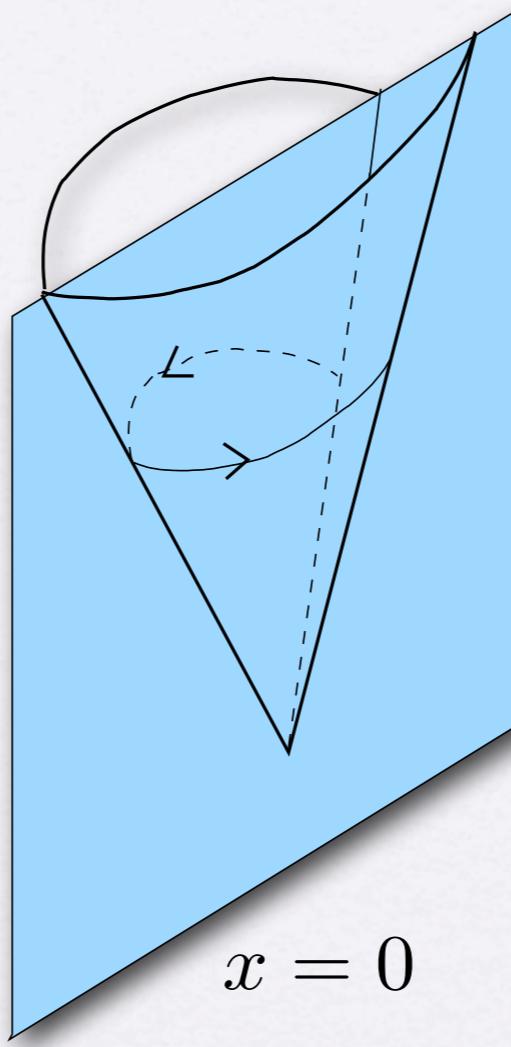
3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory

Homogeneous 2CPWL3 Systems

$$\dot{\mathbf{v}} = A^{\pm} \mathbf{v} \quad \mathbf{v} = (x, y, z)^T$$



3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory



- 📌 The system transforms straight half-lines contained in $x=0$ passing through the origin into straight half-lines contained in $x=0$ passing through the origin.
- 📌 An *invariant straight half-line* provides a **two-zonal invariant cone**.

3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory

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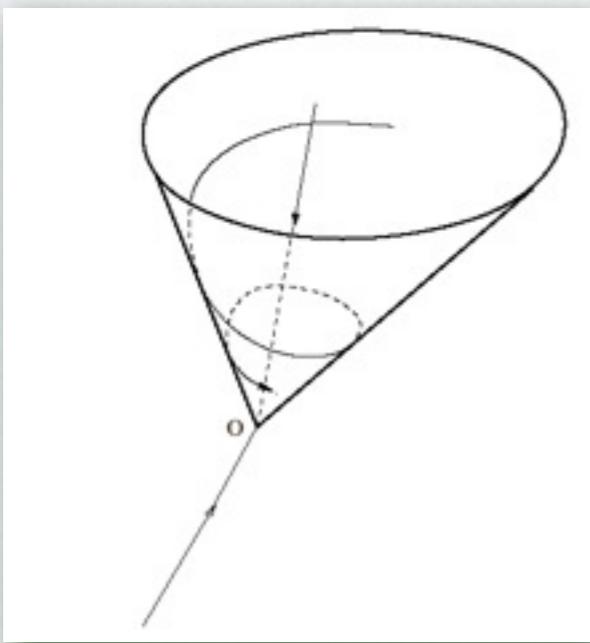
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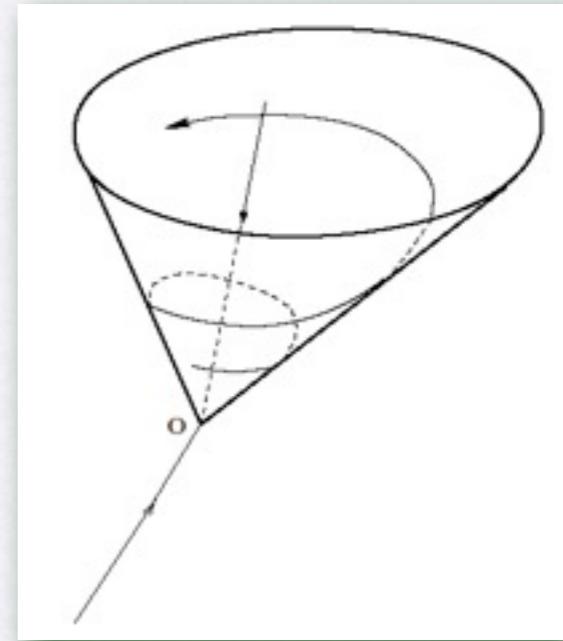
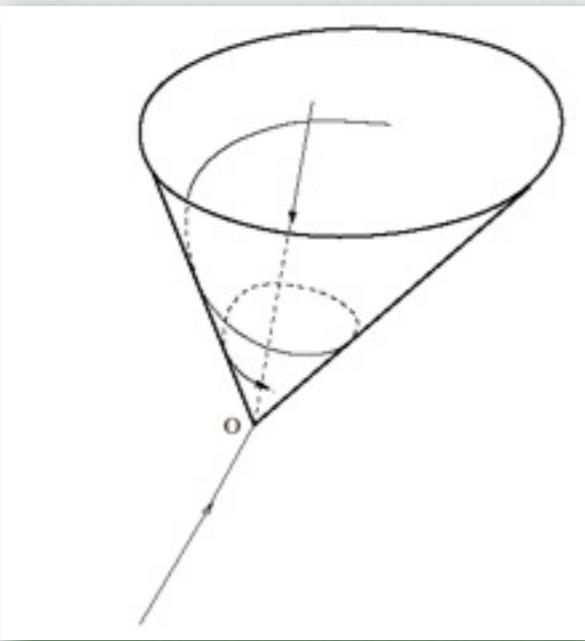


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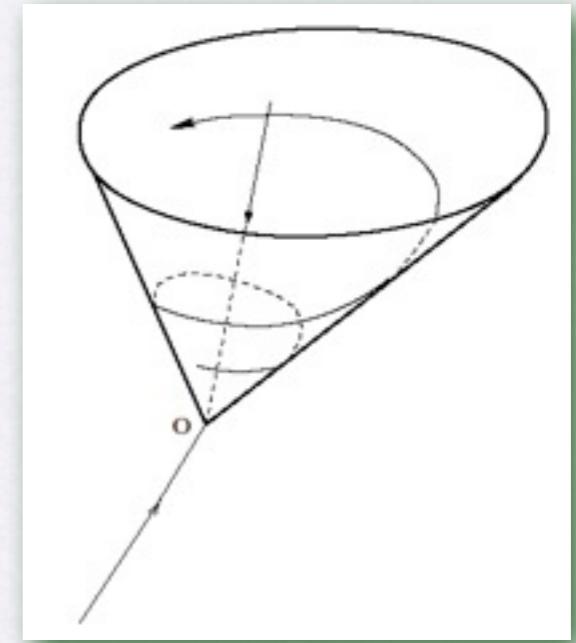
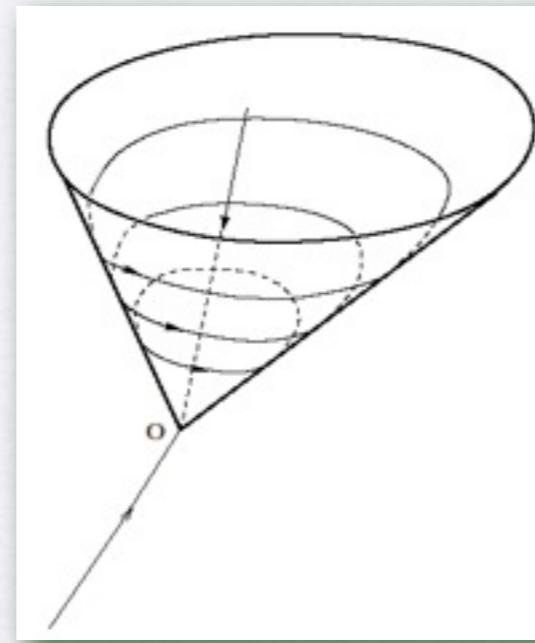
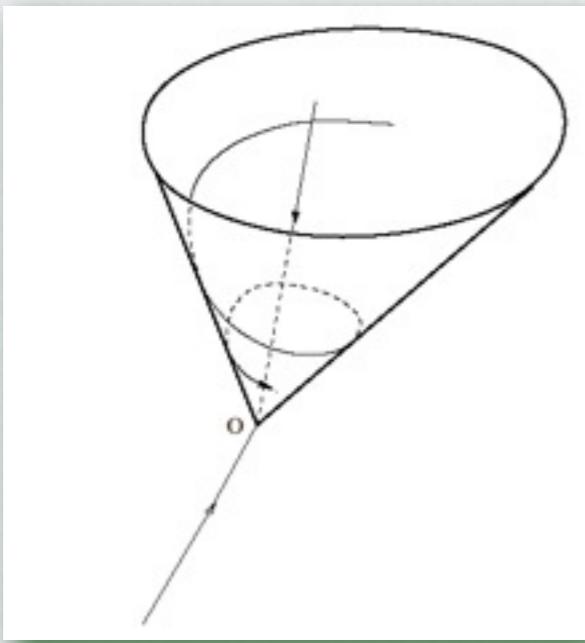


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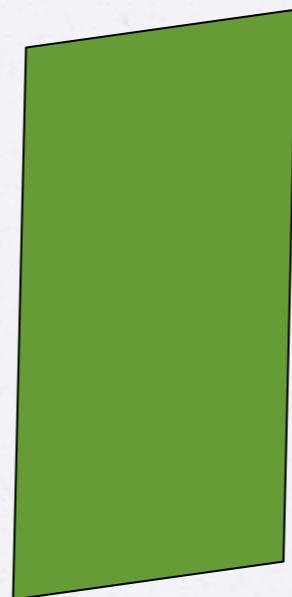


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3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} t^- & -1 & 0 \\ m^- & 0 & -1 \\ d^- & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$x < 0$



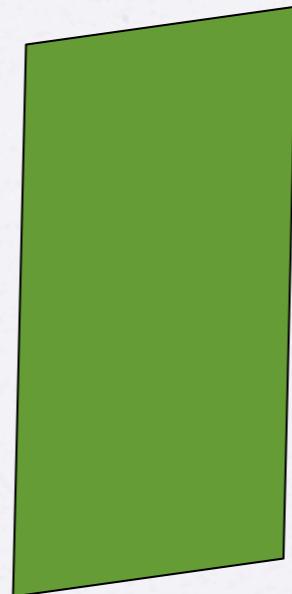
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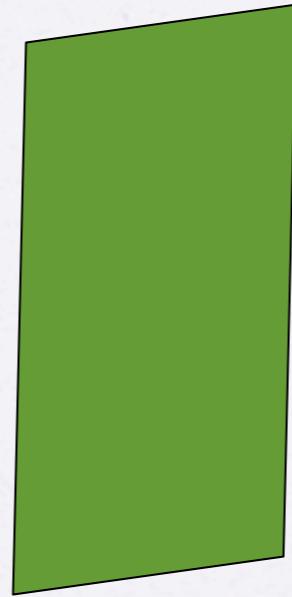
Discontinuous change of variables

$$Z = (\lambda^-)^2 x - \lambda^- y + z$$

$$Z = (\lambda^+)^2 x - \lambda^+ y + z$$

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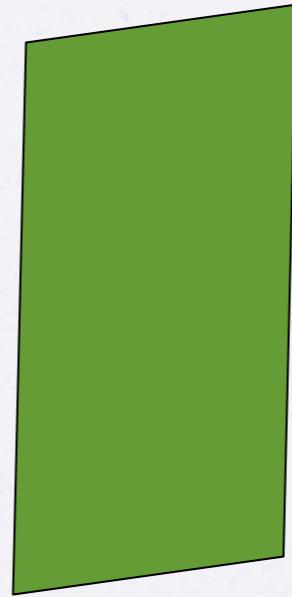
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Reset map

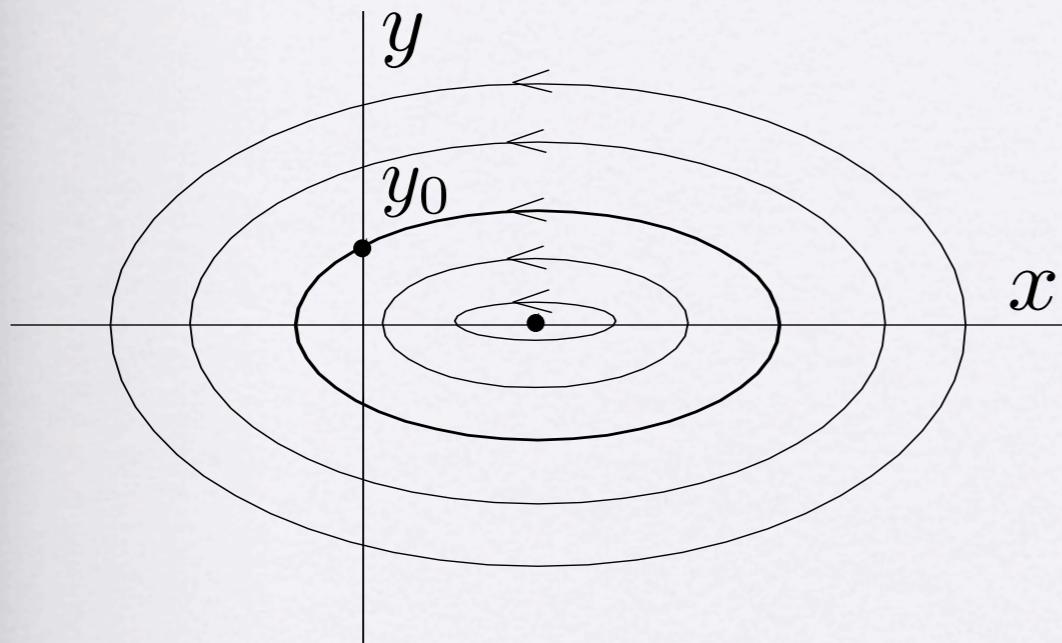
$$U = 0 \quad \delta(y) = \begin{cases} \frac{y}{1 - (\lambda^+ - \lambda^-)y} & \text{if } y \leq 0, \\ \frac{y}{1 + (\lambda^+ - \lambda^-)y} & \text{if } y > 0. \end{cases}$$

3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory

$$\alpha^- = \lambda^- = \alpha^+ = \lambda^+$$

$$\left\{ \begin{array}{l} \dot{x} = -y \\ \dot{y} = (\beta^-)^2 x - 1 \end{array} \right. \quad \left| \quad \left\{ \begin{array}{l} \dot{x} = -y \\ \dot{y} = (\beta^+)^2 x - 1 \end{array} \right. \right.$$

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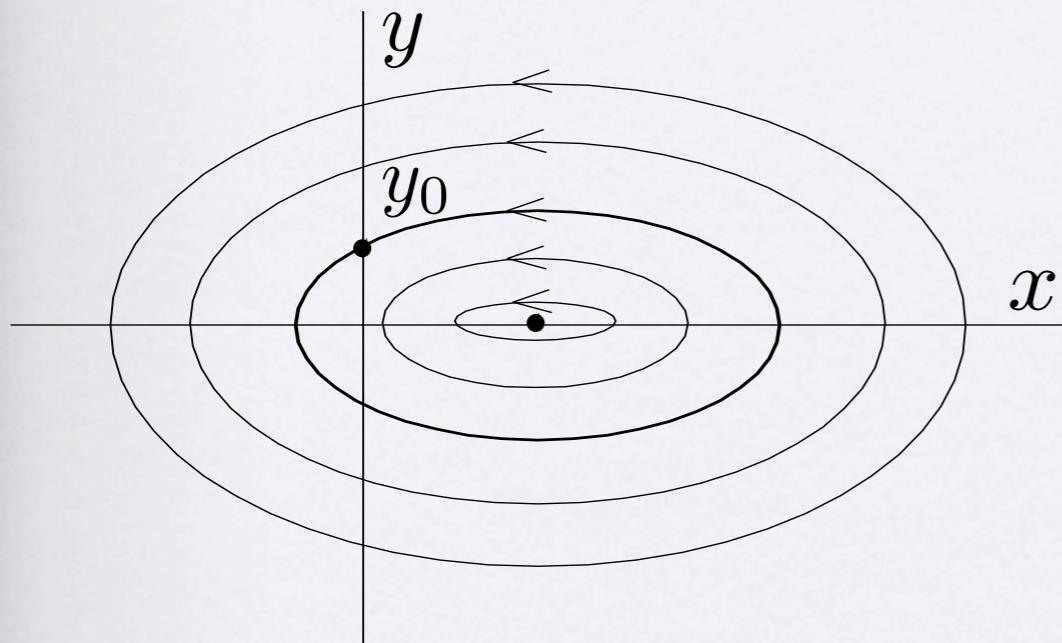


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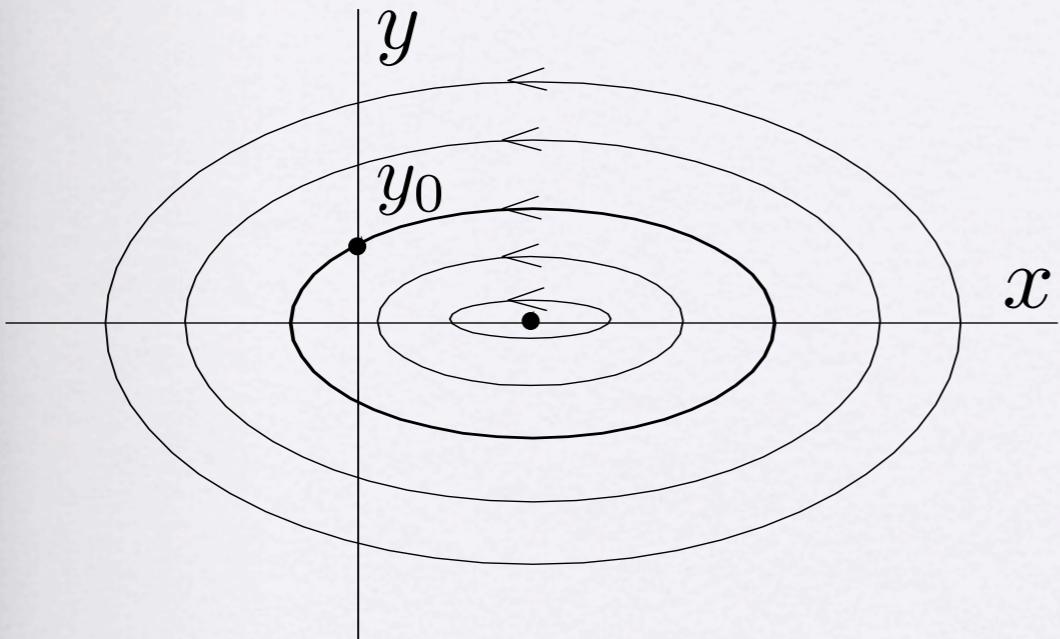
$$\left| \quad \left. \begin{array}{l} \lambda^-, (\lambda^- + \varepsilon\sigma^-) \pm i\beta^- \\ x = 0 \end{array} \right. \quad \left. \begin{array}{l} \lambda^- + \varepsilon\Lambda, (\lambda^- + \varepsilon\sigma^+) \pm i\beta^+ \end{array} \right. \right.$$

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$$\left\{ \begin{array}{l} \dot{x} = 2\varepsilon\sigma^- x - y \\ \dot{y} = (\varepsilon^2(\sigma^-)^2 + (\beta^-)^2)x - 1 \end{array} \right. \quad \left| \quad \left\{ \begin{array}{l} \dot{x} = 2\varepsilon(\sigma^+ - \Lambda)x - y \\ \dot{y} = (\varepsilon^2(\sigma^+ - \Lambda)^2 + (\beta^+)^2)x - 1 \end{array} \right. \right.$$

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3) Invariant Cones in Observable 2CPWL3 Systems via Melnikov Theory

$0 \neq |\varepsilon| \ll 1$ *Results of existence and saddle-node bifurcation*

1. If $\Lambda \cdot \sigma^- < 0$, $\Lambda(\sigma^+ - \Lambda) > 0$

There exist **exactly two** two-zonal invariant cones, **one above and one below** the focal planes

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There exists **one** two-zonal invariant cone **above** the focal planes

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4. If $\beta^- = \beta^+, \Lambda + \sigma^- - \sigma^+ \neq 0, \Lambda(2(\sigma^- - \sigma^+) - \Lambda) > 0$

Saddle-node bifurcation of two-zonal invariant cones **above** the focal planes

Saddle-node bifurcation of two-zonal invariant cones **below** the focal planes

The attractiveness is also studied

V. Carmona, S. F-G, E. Freire, *Saddle-Node Bifurcation of Invariant Cones in 3D Piecewise Linear Systems*. Physica D, 241 (2012) 623–635

Melnikov Theory for Planar Hybrid Systems: Invariant Cones in Piecewise Linear Systems

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UNIVERSIDAD DE SEVILLA

Departamento de Matemática Aplicada II

Inria Project Team: Mycenae (Multiscale dYnamiCs in neuroENdocrine AxEs)