REDUCED ORDER MODEL IN CARDIAC ELECTROPHYSIOLOGY

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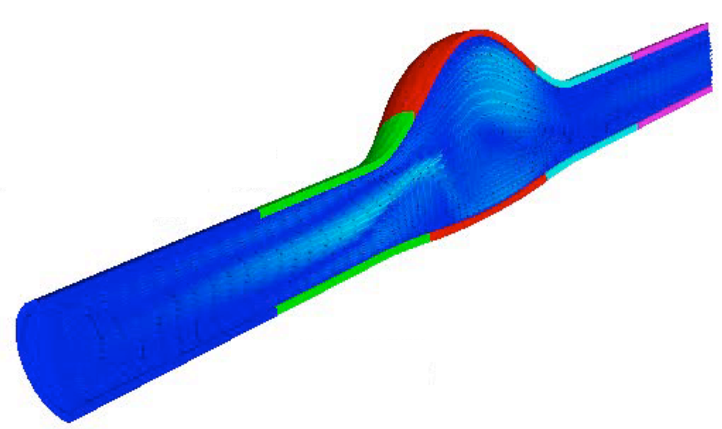
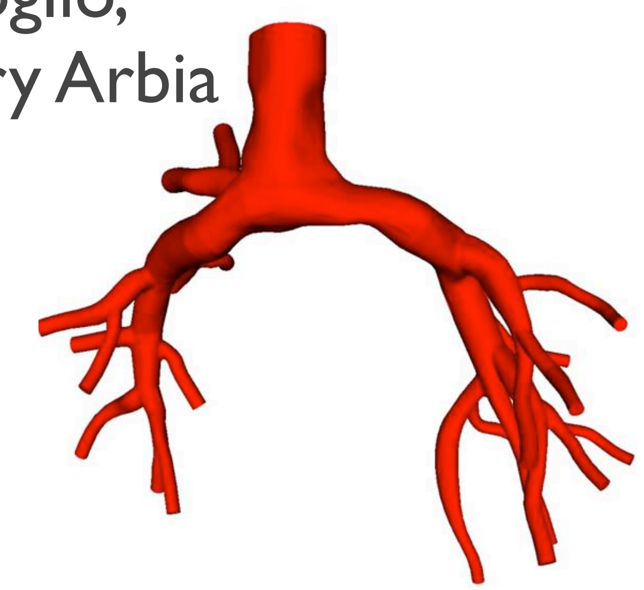
reo team

Numerical simulation of biological flows

Blood flow modelling

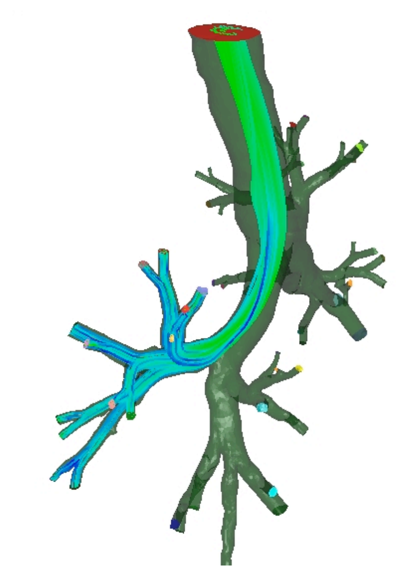
Past talks: Cristobal Bertoglio,
Saverio Smaldone, Gregory Arbia

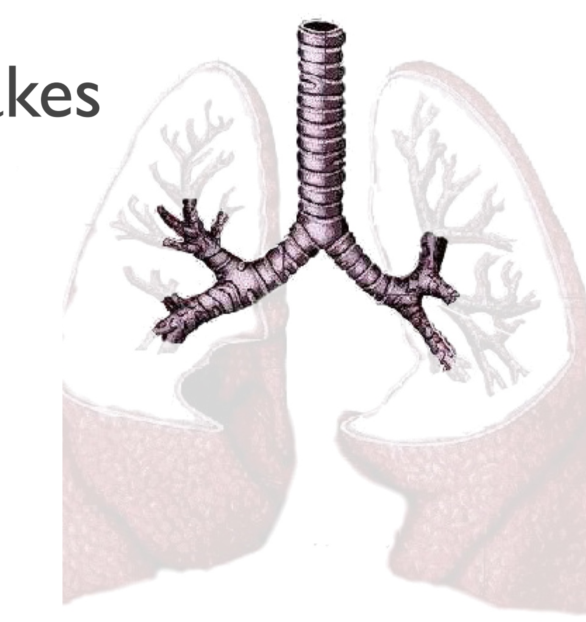




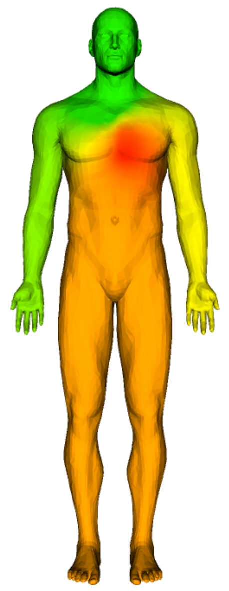
Respiration modelling

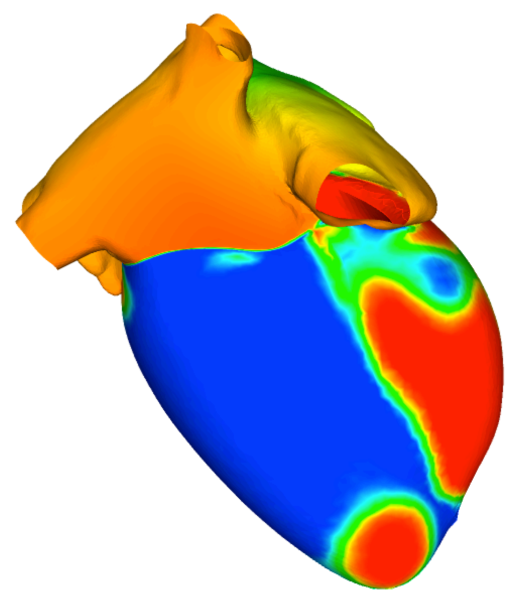
Past talks: Jessica Oakes

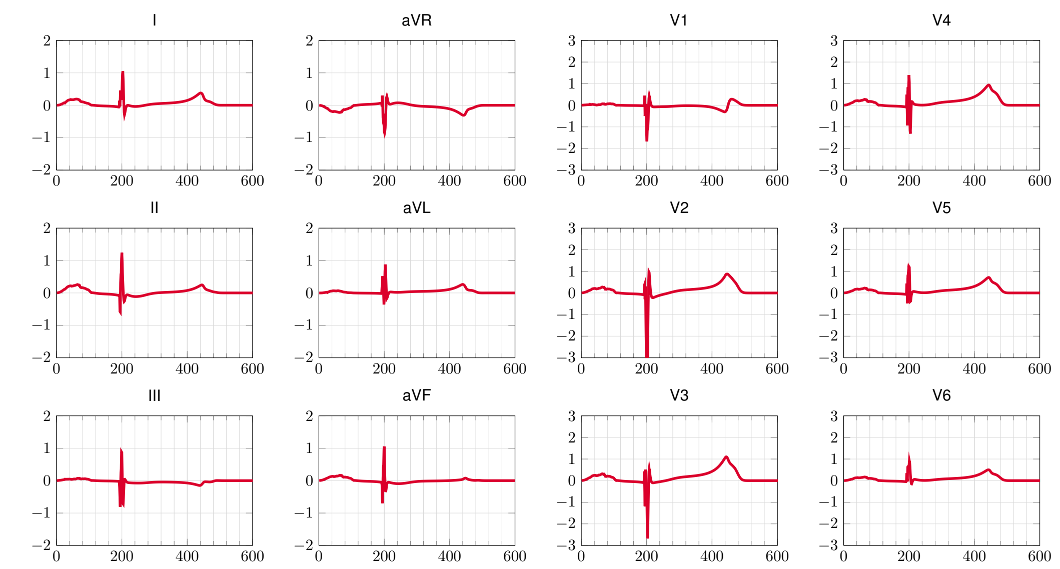




Cardiac electrophysiology







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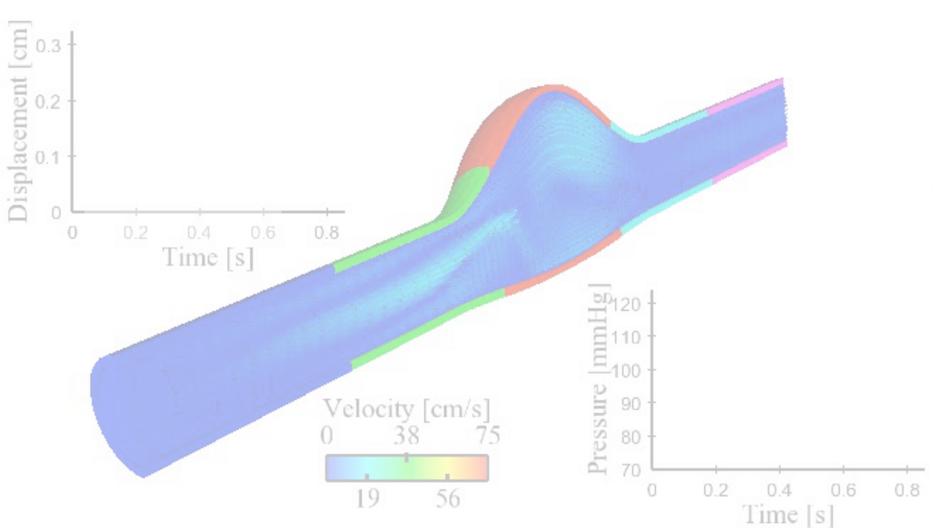

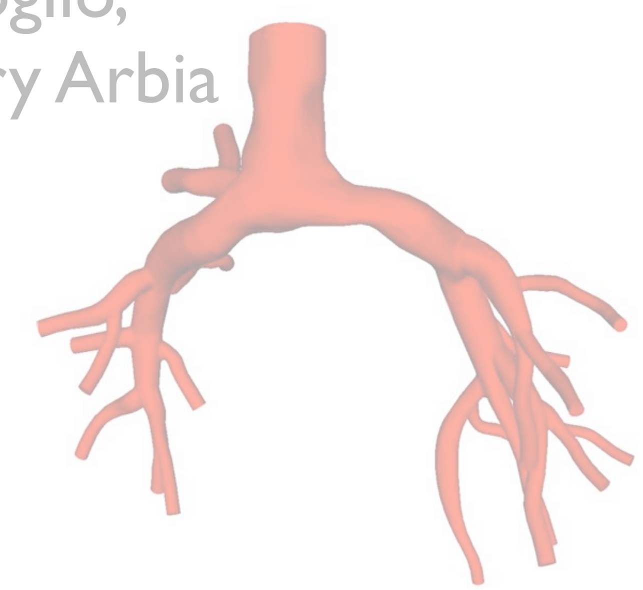

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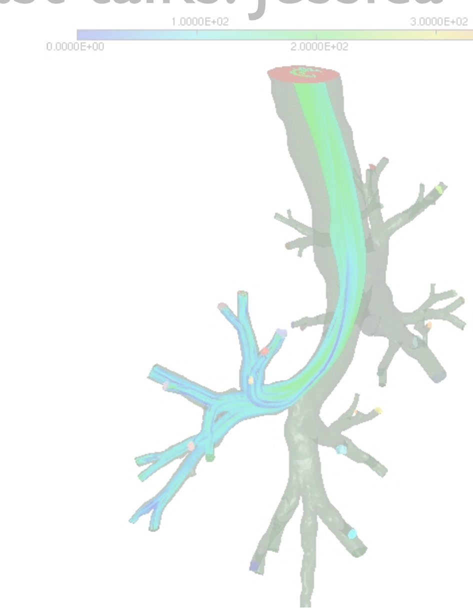

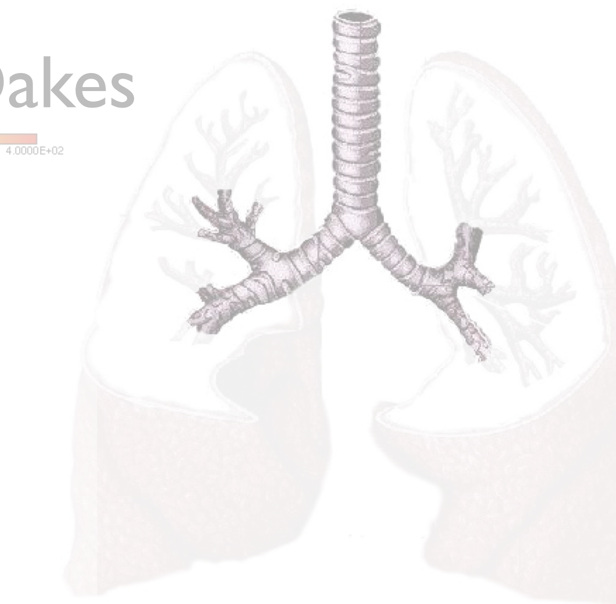

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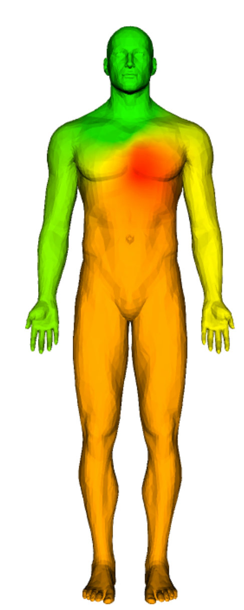

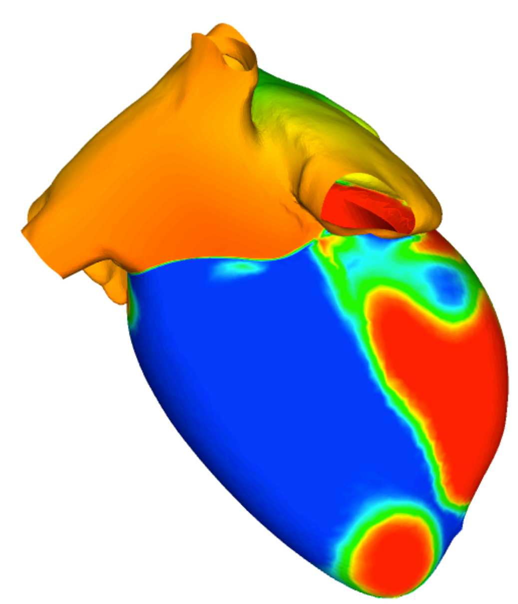

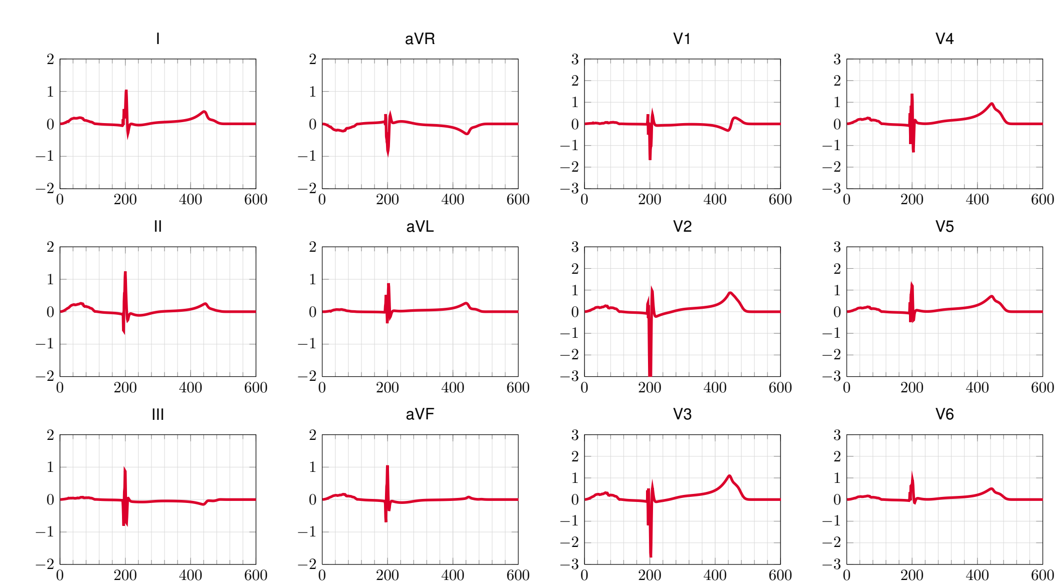

PhD project:
Inverse problems and reduced models in cardiac electrophysiology

- ✓ Supervisors:

  Muriel BOULAKIA and Jean-Frédéric GERBEAU

- ✓ Subjects:

  - ➡ Simulation of realistic Electrocardiograms (with Annabelle Collin)
  - ➡ Parameters estimation
  - ➡ Reduced order models and inverse problems with POD
  - ➡ Reduced order models with Approximated Lax Pairs (with Damiano Lombardi)
  - ➡ Inverse problems

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Summary

- ✓ Bidomain Model equations
- ✓ PDEs numerical approximation
  - ➡ Finite Element Methods (overview)
  - ➡ Reduced Order Methods
- ✓ Numerical Results
- ✓ Conclusions and Perspectives

Cardiac electrophysiology model

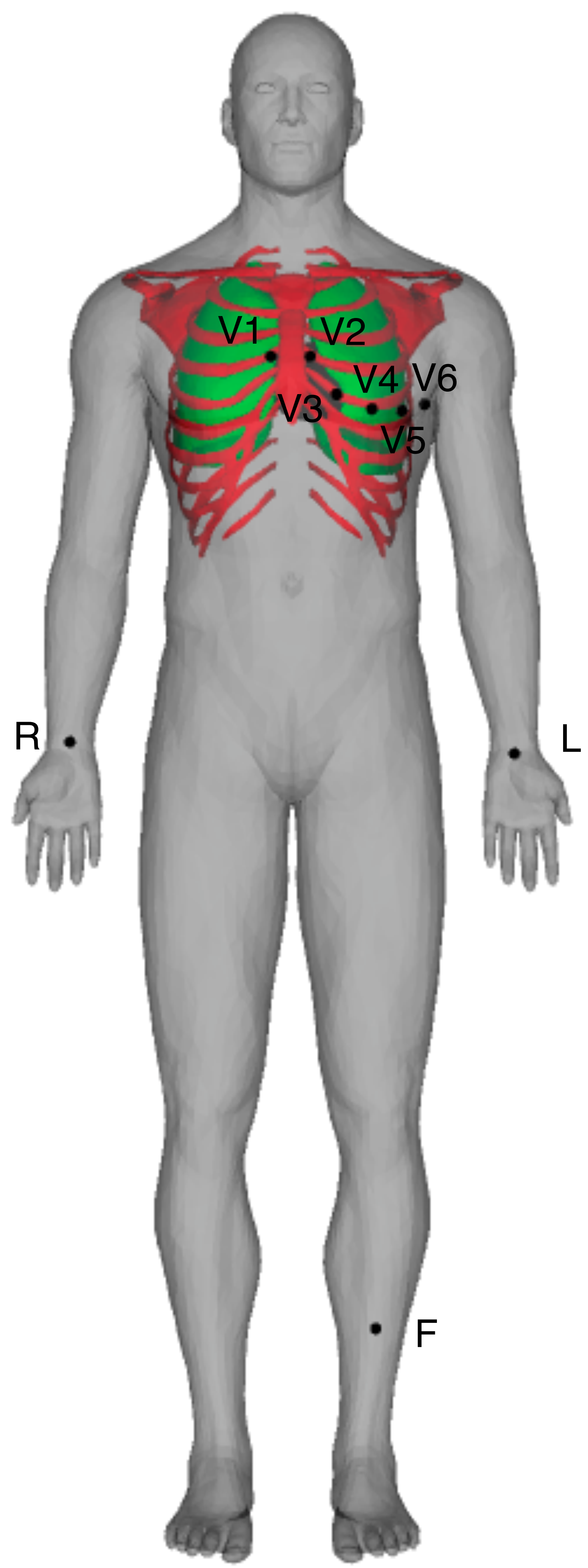
Bidomain equations

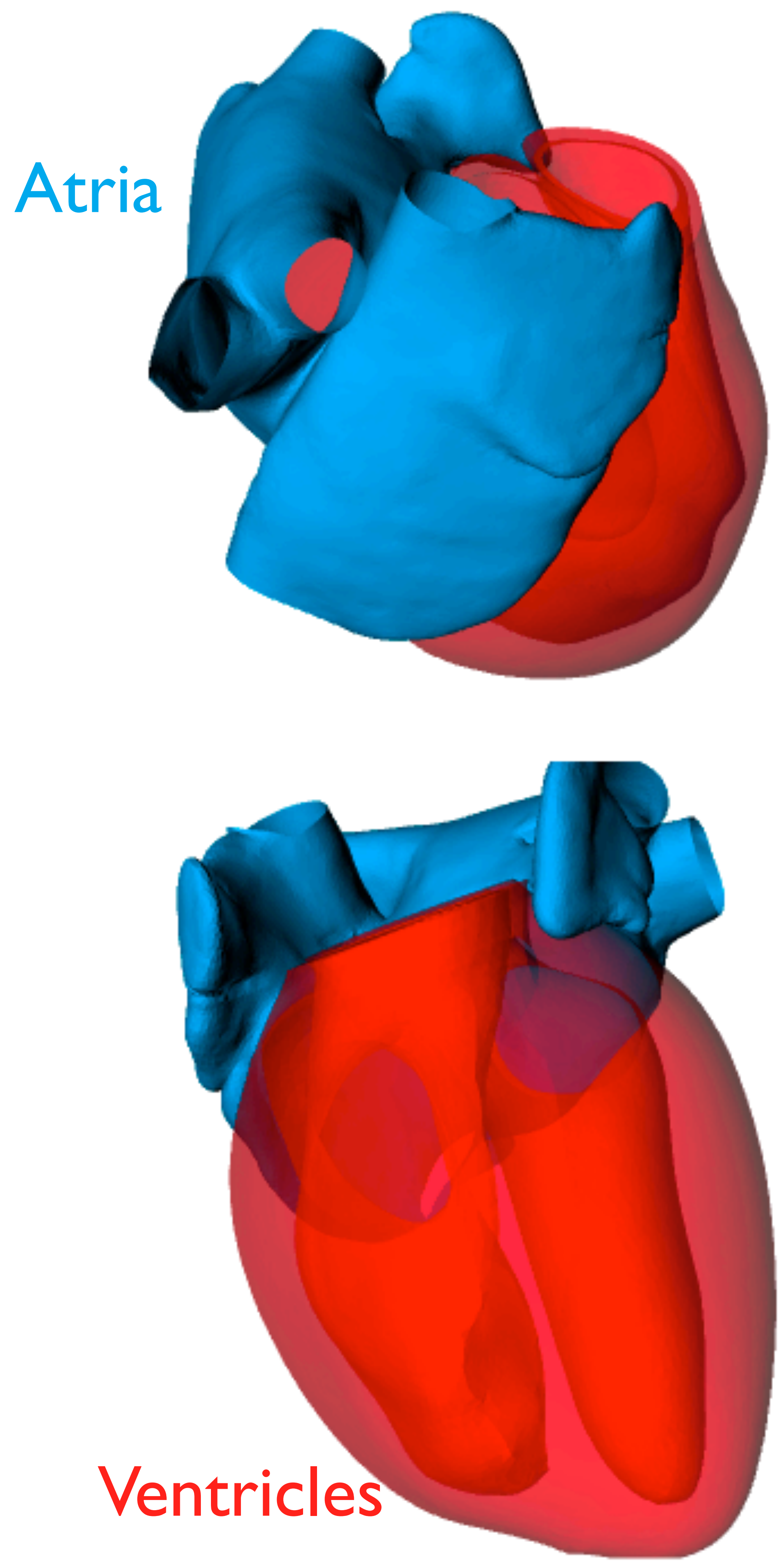
From 9 measures to 12 leads

$$\mathrm{I} = u_T(L) - u_T(R) \qquad \mathrm{aVR} = 1.5(u_T(R) - u_w) \qquad \mathrm{V1} = u_T(V_1) - u_w$$

$$\mathrm{II} = u_T(F) - u_T(R) \qquad \mathrm{aVL} = 1.5(u_T(L) - u_w) \qquad \vdots$$

$$\mathrm{III} = u_T(F) - u_T(L) \qquad \mathrm{aVF} = 1.5(u_T(F) - u_w) \qquad \mathrm{V6} = u_T(V_6) - u_w$$

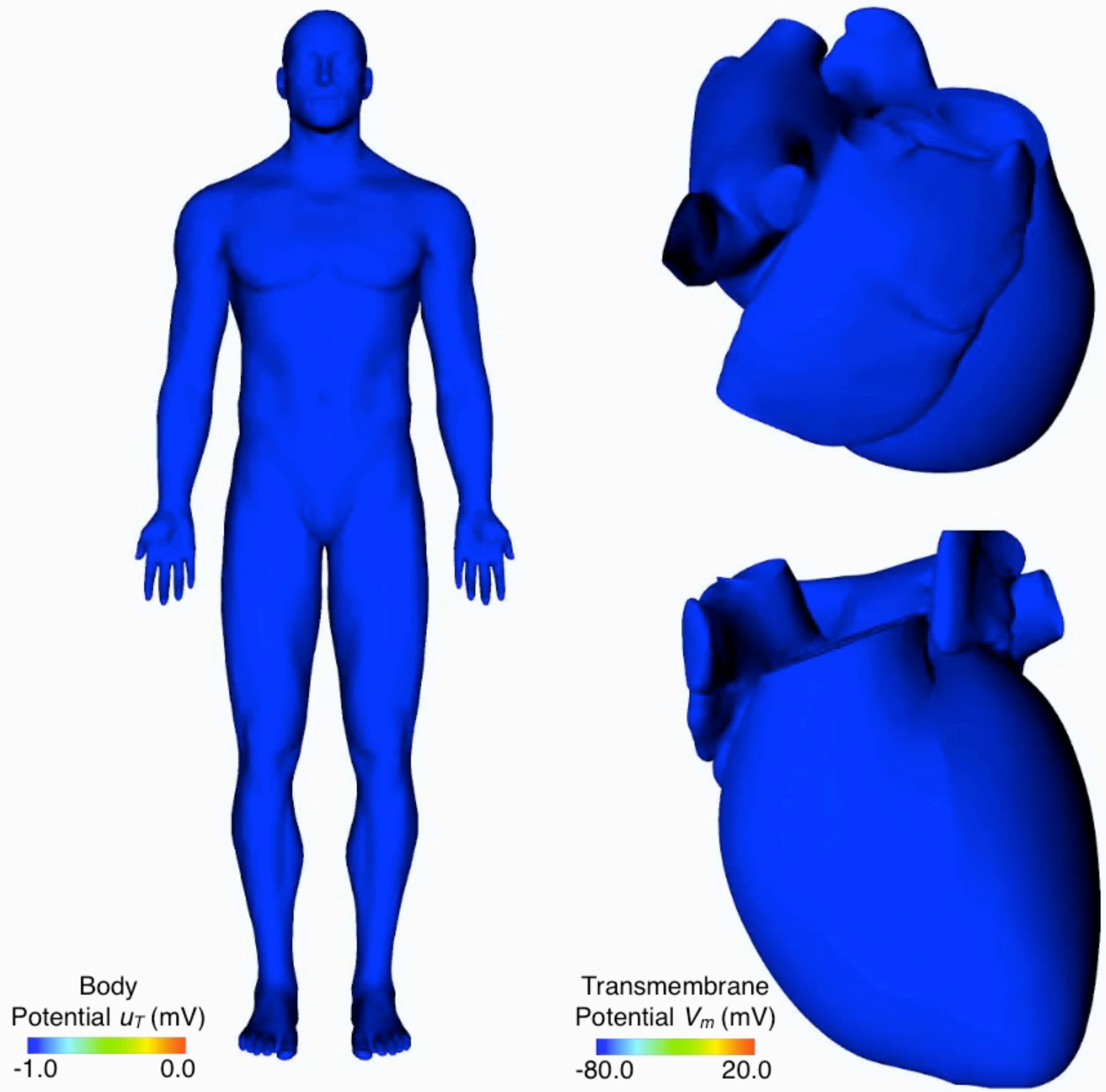




[Collin, Gerbeau, Schenone - 2014]

Cardiac electrophysiology model

Bidomain equations



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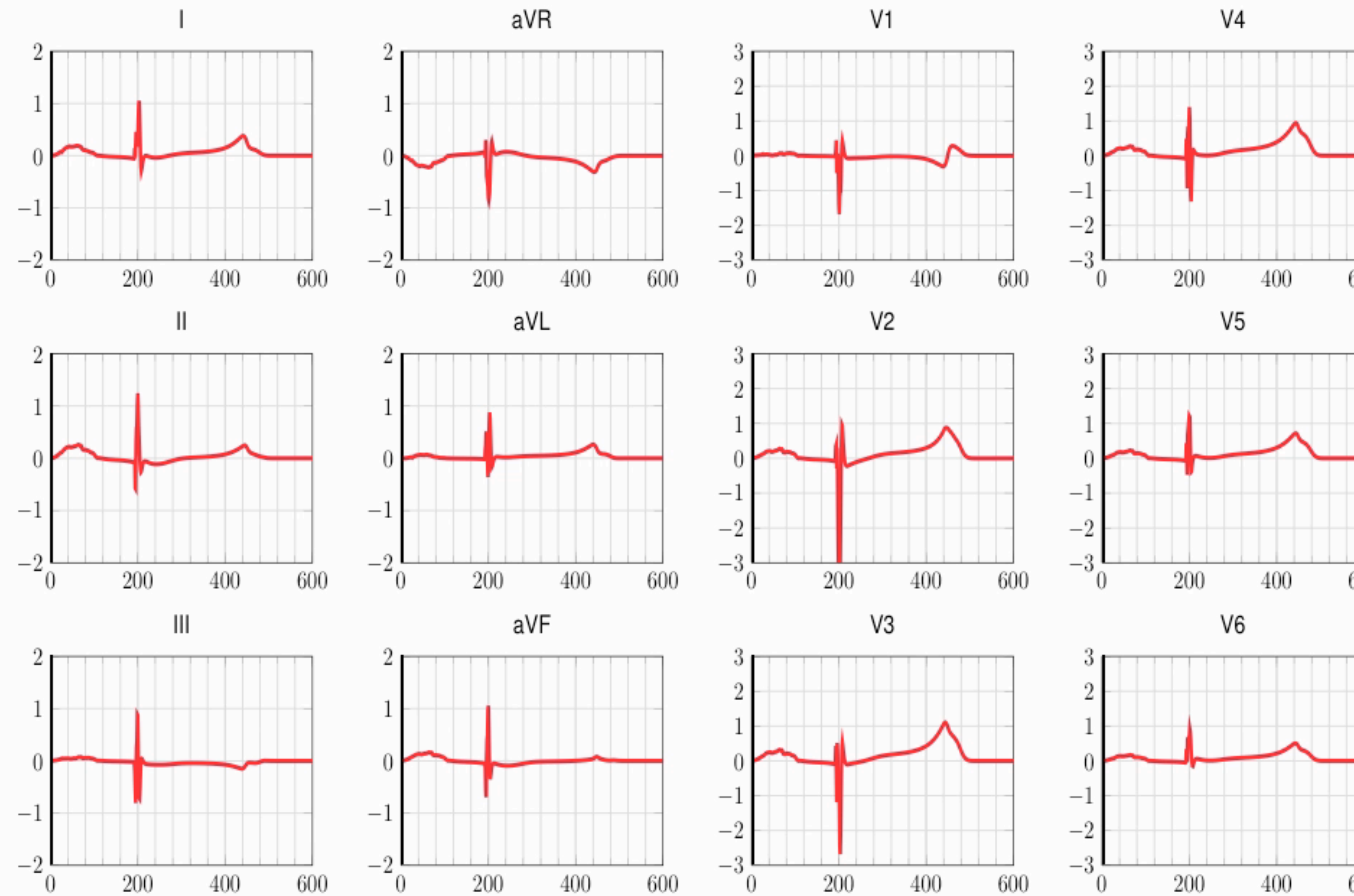
Standard 12-lead ECG



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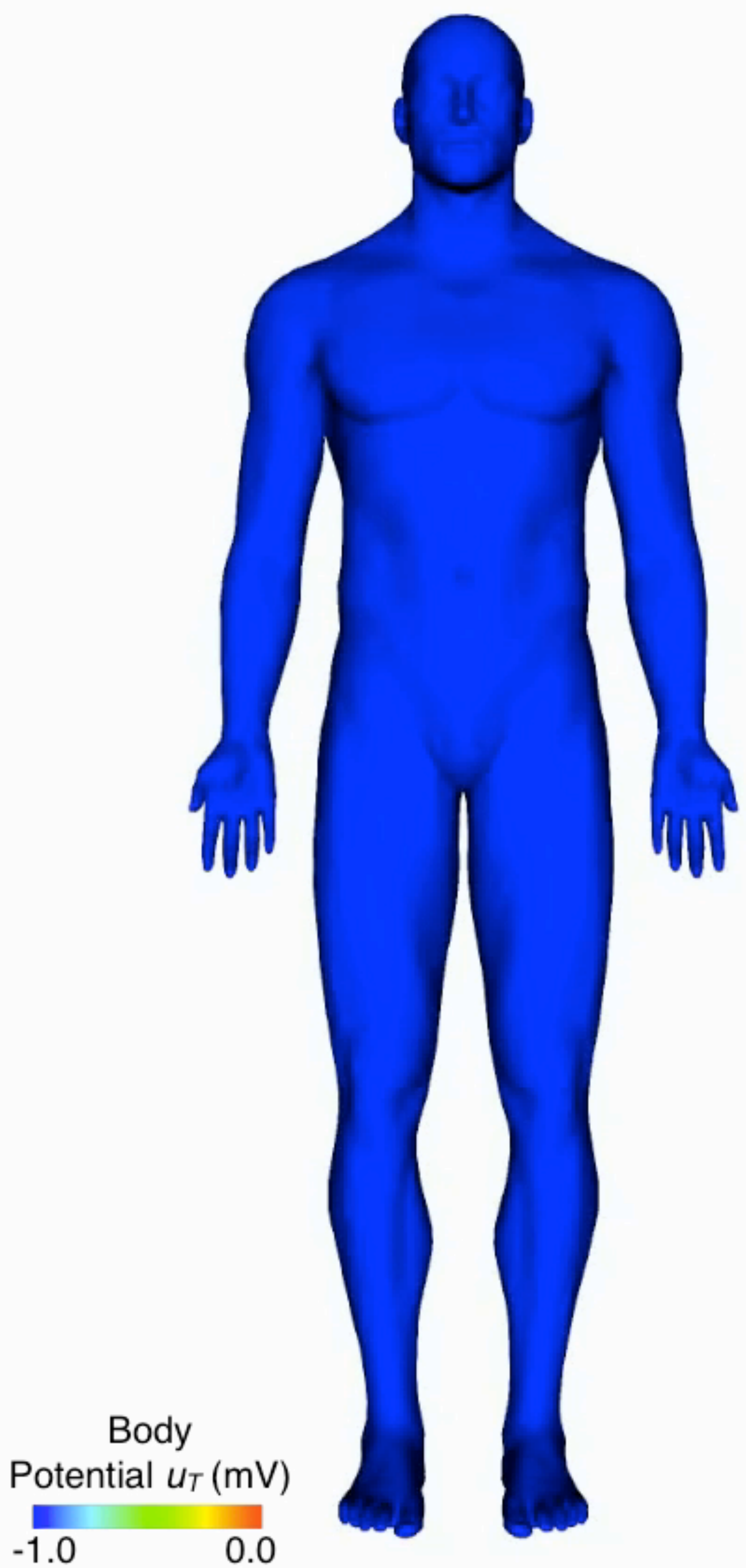

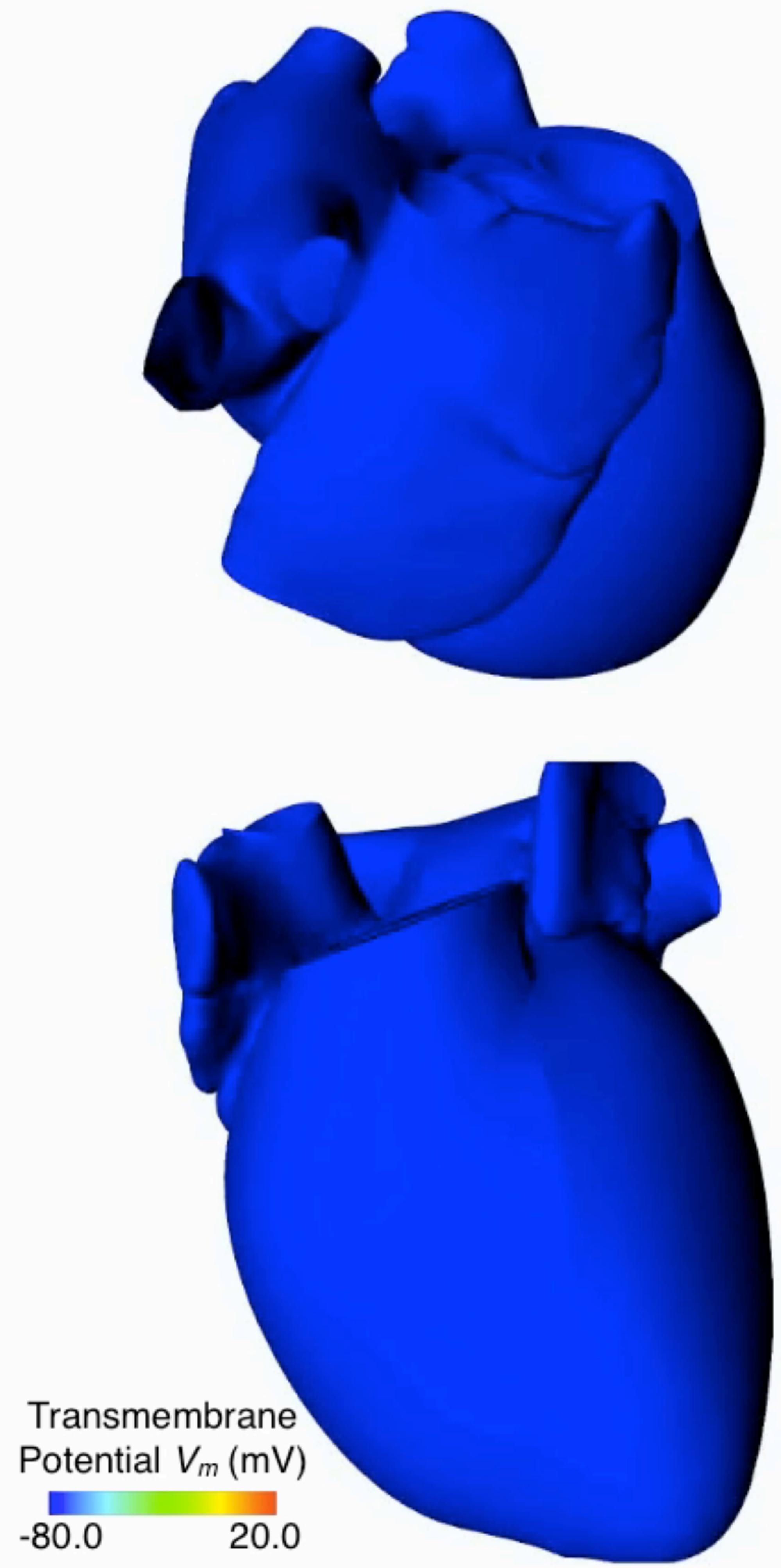

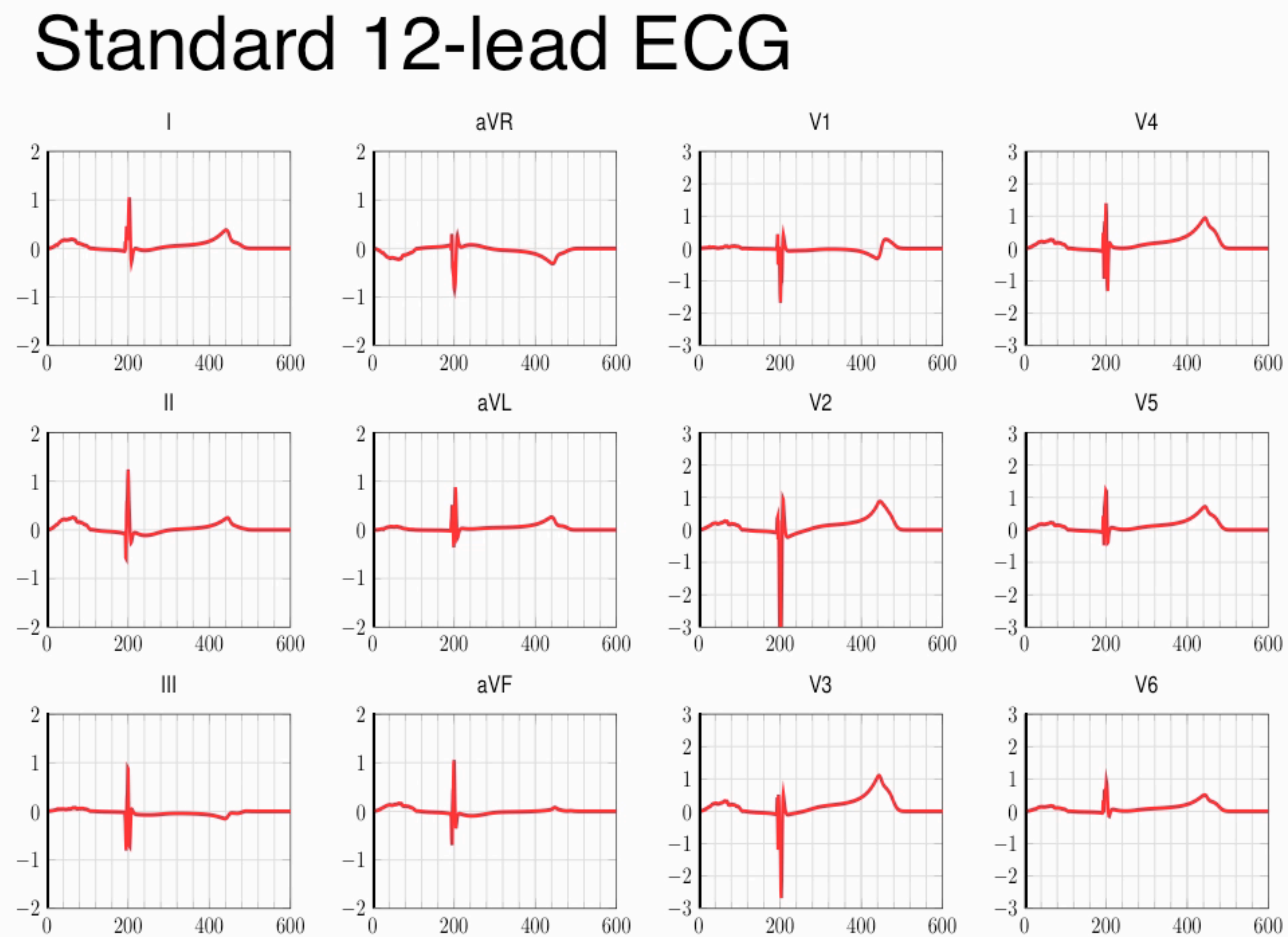

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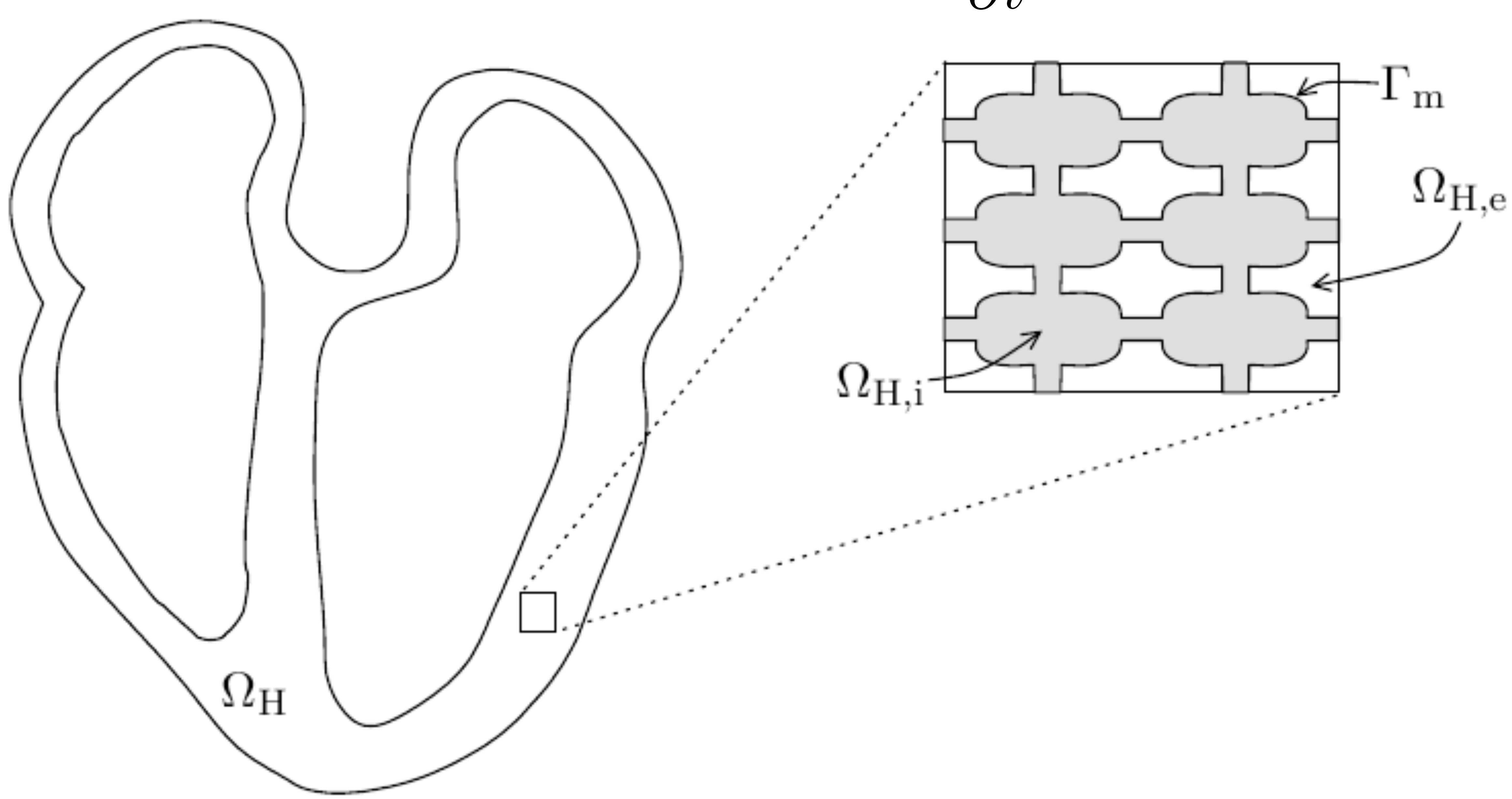
Cardiac electrophysiology model

Bidomain equations

- Bidomain equations

$$
\begin{aligned}
A_m\Big(C_m\frac{\partial v_m}{\partial t} + I_{ion}(v_m, w)\Big) - \mathrm{div}(\bar{\bar{\sigma}}_I \nabla v_m) - \mathrm{div}(\bar{\bar{\sigma}}_I \nabla u_E) &= A_m I_{app} \\
-\mathrm{div}((\bar{\bar{\sigma}}_I + \bar{\bar{\sigma}}_E)\nabla u_E) - \mathrm{div}(\bar{\bar{\sigma}}_I \nabla v_m) &= 0 \\
\frac{\partial w}{\partial t} - g(v_m, w) &= 0
\end{aligned}
$$



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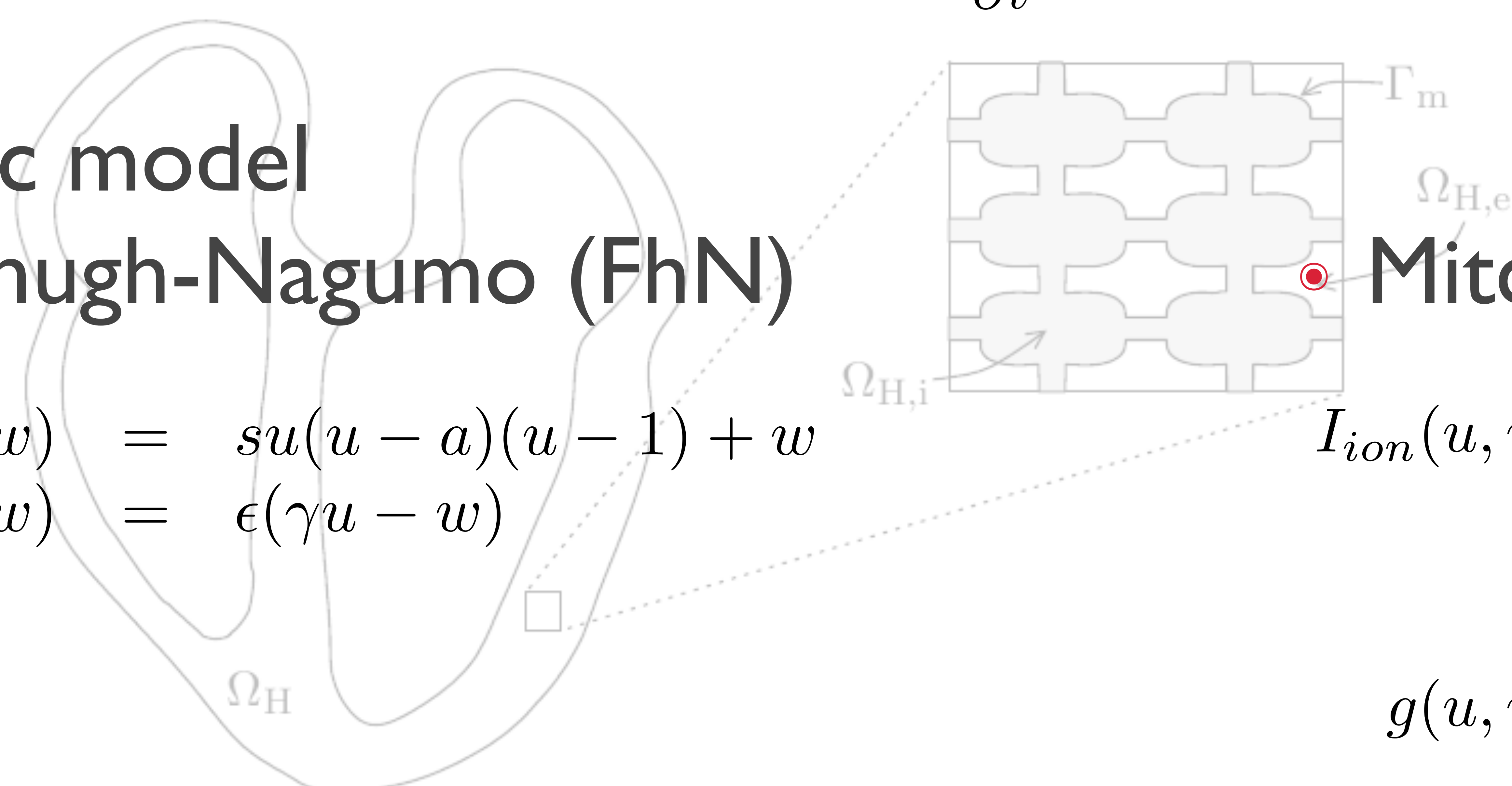


- Ionic model

- Fitzhugh-Nagumo (FhN)

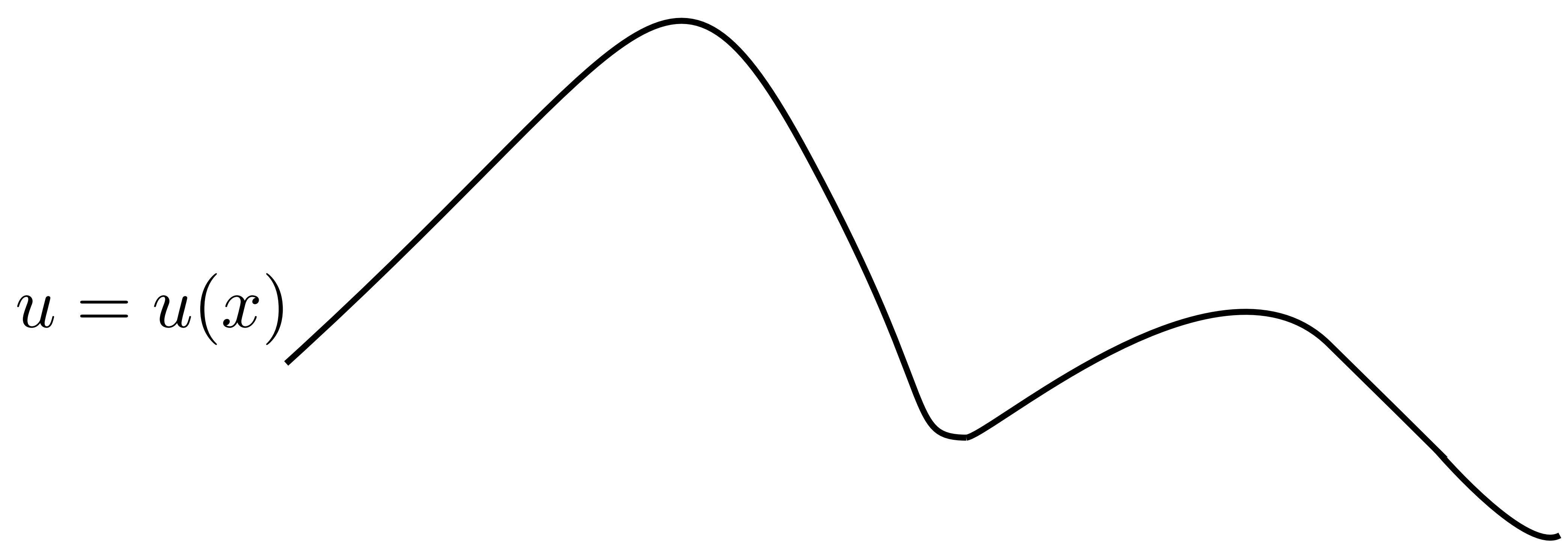
$$
\begin{aligned}
I_{ion}(u, w) &= su(u-a)(u-1) + w \\
g(u, w) &= \epsilon(\gamma u - w)
\end{aligned}
$$

- Mitchell-Schaeffer (MS)

$$
\begin{aligned}
I_{ion}(u, w) &= w\frac{u^2(1-u)}{\tau_{\text{in}}} - \frac{u}{\tau_{\text{out}}} \\
g(u, w) &= \begin{cases} \dfrac{1-w}{\tau_{\text{open}}} & u \leq u_{\text{gate}} \\ -\dfrac{w}{\tau_{\text{close}}} & u > u_{\text{gate}} \end{cases}
\end{aligned}
$$

A numerical approximation
Overview on Finite Element Method (FEM)

FEM space



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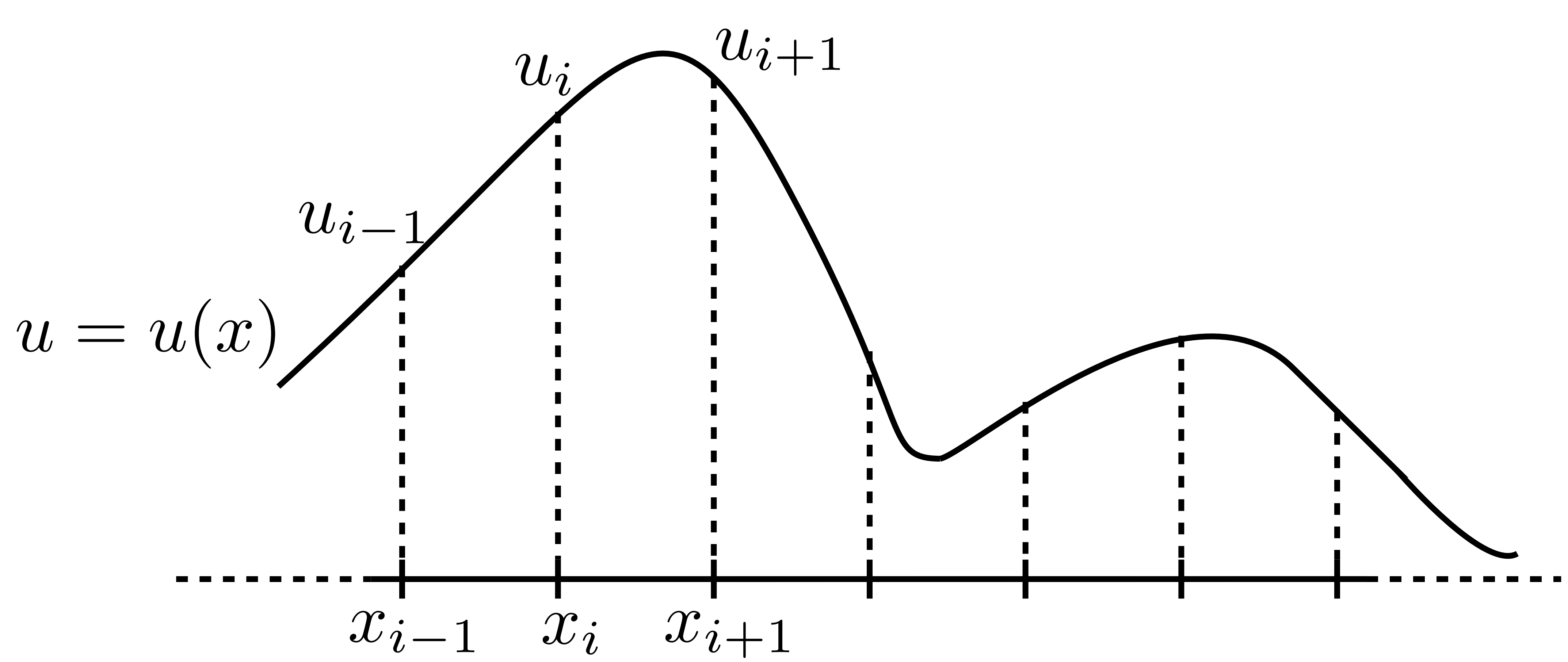

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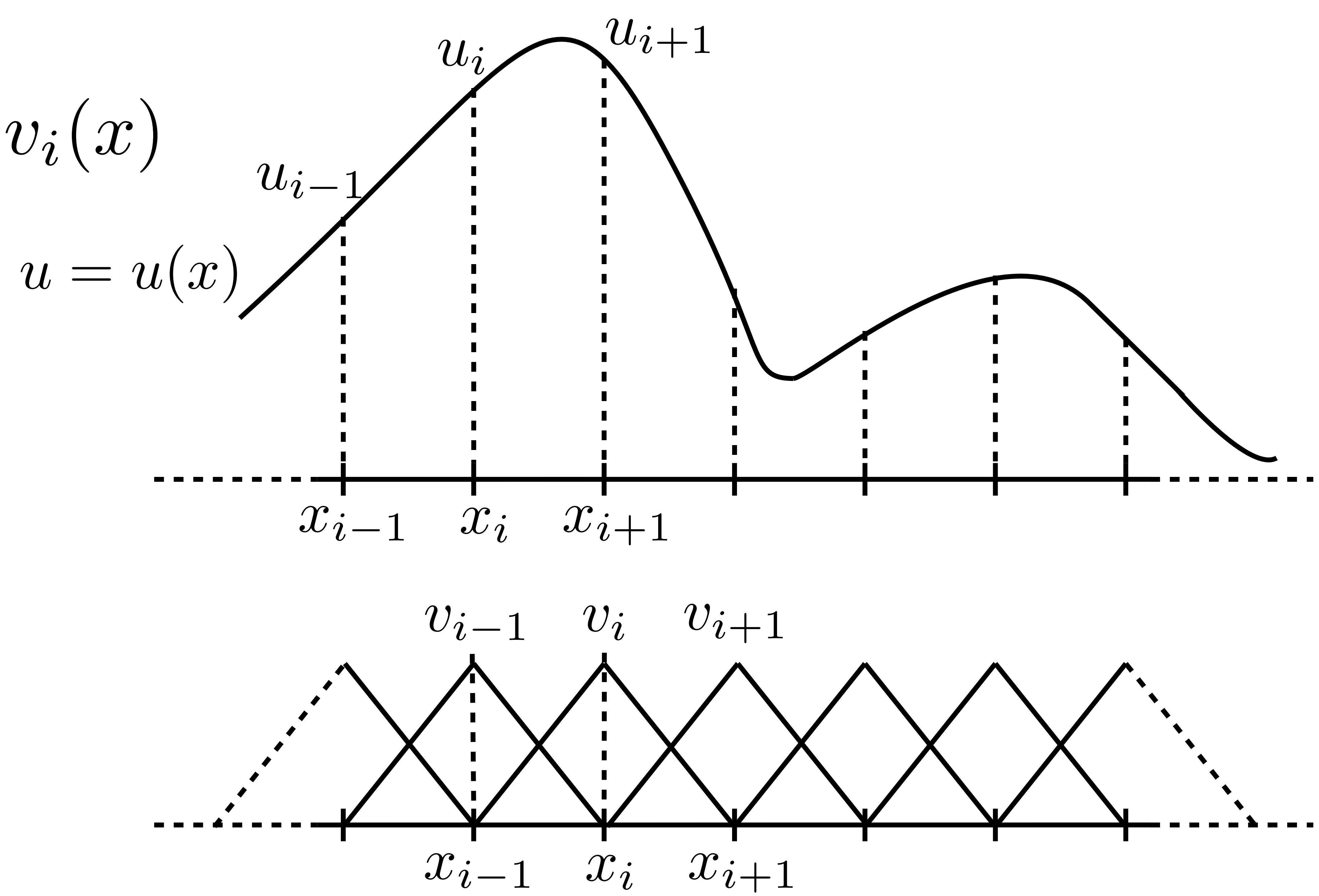
$$u(x) \simeq \sum_{i}^{\mathcal{N}} u_i v_i(x)$$



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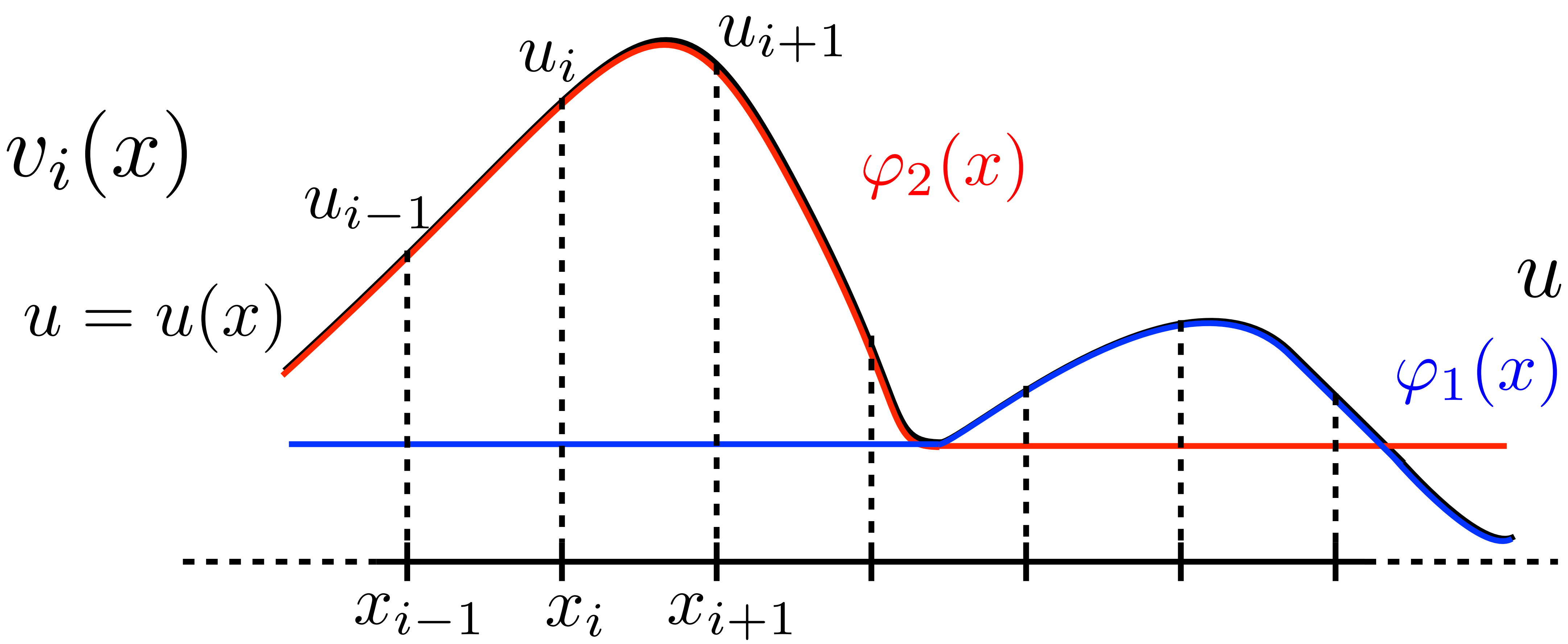


$$u(x) \simeq \tilde{u}_1 \varphi_1(x) + \tilde{u}_2 \varphi_2(x)$$

FEM versus ROM

FEM space

$$V_{\mathcal{N}} \equiv \mathrm{span}\{v_1, \dots, v_{\mathcal{N}}\} \subset V$$

$$\dim(V_{\mathcal{N}}) = \mathcal{N}$$

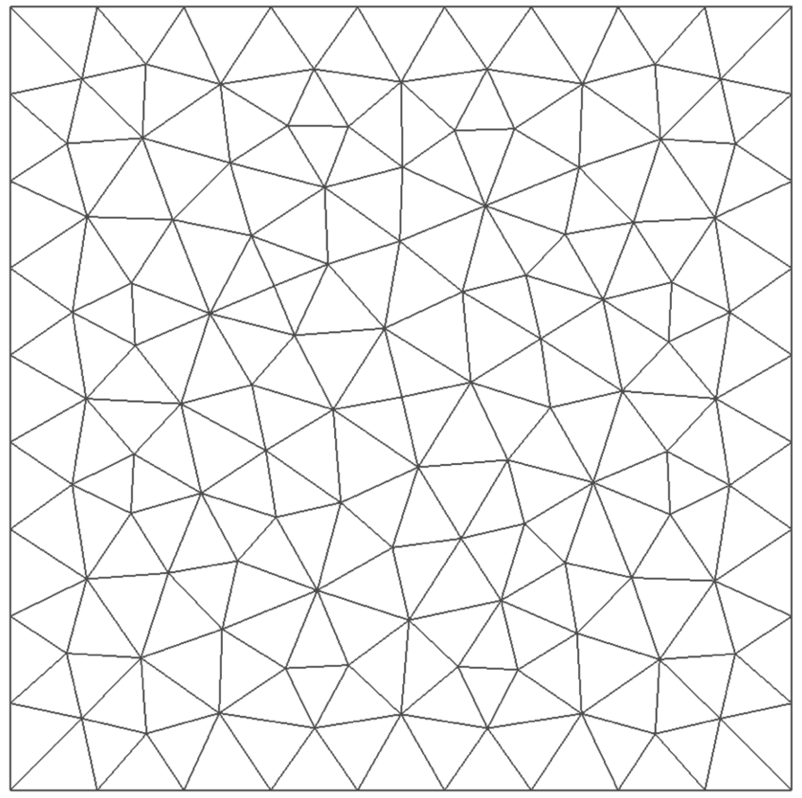
$$\mathbf{u} = \sum_{i}^{\mathcal{N}} u_i v_i(x), \quad u_i = \langle u, v_i \rangle$$

ROM space

$$V_N \equiv \mathrm{span}\{\varphi_1, \dots, \varphi_N\} \subset V_{\mathcal{N}}$$

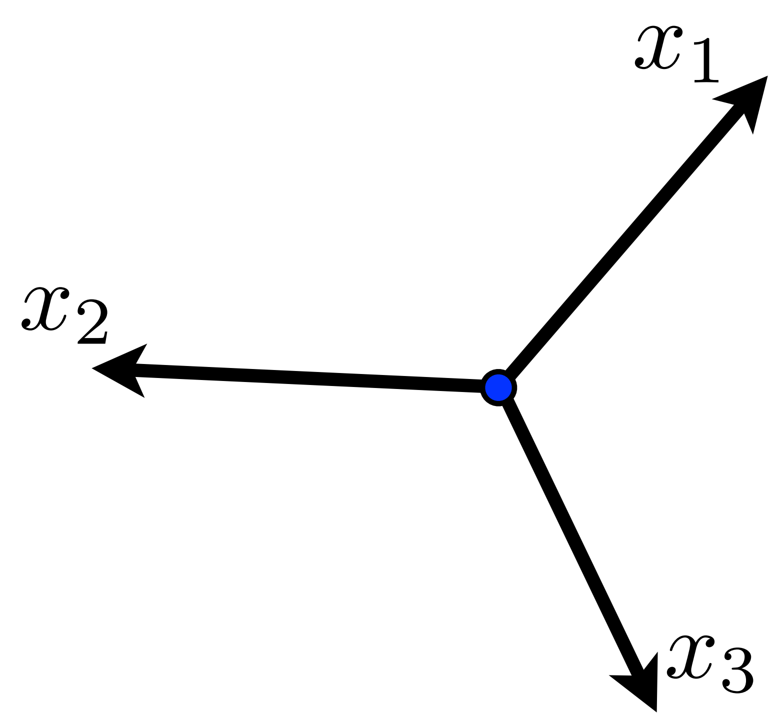
$$\dim(V_N) = N \ll \mathcal{N}$$

$$\widetilde{\mathbf{u}} = \sum_{i}^{N} \tilde{u}_i \varphi_i(x), \quad \tilde{u}_i = \langle u, \varphi_i \rangle$$



$$\Phi_N = [\varphi_1 \dots \varphi_N] \in \mathbb{R}^{\mathcal{N} \times N}$$

$$\widetilde{\mathbf{u}} = \Phi_N^T \mathbf{u}$$



Proper Orthogonal Decomposition (POD)

A "classical" approach to Reduced Order Models

- Use information given by FEM solution(s) to build a suitable basis

Proper Orthogonal Decomposition (POD)

A "classical" approach to Reduced Order Models

- Use information given by FEM solution(s) to build a suitable basis

1. Solve with FEM the problem(s)

$$\partial_t u = F(u, \partial_x^{(n)}, \pi), \ \text{in } \Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_+$$

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2. Collect snapshots of FEM solution(s) in a matrix

$$S = (\mathbf{u}^1, \dots, \mathbf{u}^p) \in \mathbb{R}^{\mathcal{N} \times p} \qquad \mathbf{u}^i = \begin{bmatrix} u(x_1, t_i) \\ \vdots \\ u(x_{\mathcal{N}}, t_i) \end{bmatrix} \in \mathbb{R}^{\mathcal{N}}, \ \forall i = 1, \dots, \mathcal{N}$$

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3. Compute Singular Value Decomposition (SVD) of the matrix

$$S = \Phi \, \Sigma \, \Psi^T, \quad \Sigma = \mathrm{diag}(\sigma_i)$$

4. Keep first eigenvectors as ROM basis functions

- Can reproduce events described by the FEM solution(s)

POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction

- Infarcted tissue: damaged area which cannot be activated

➡ to build the POD basis, we use a FEM solution with <u>NO infarction</u>

[Boulakia, Schenone, Gerbeau - 2012]

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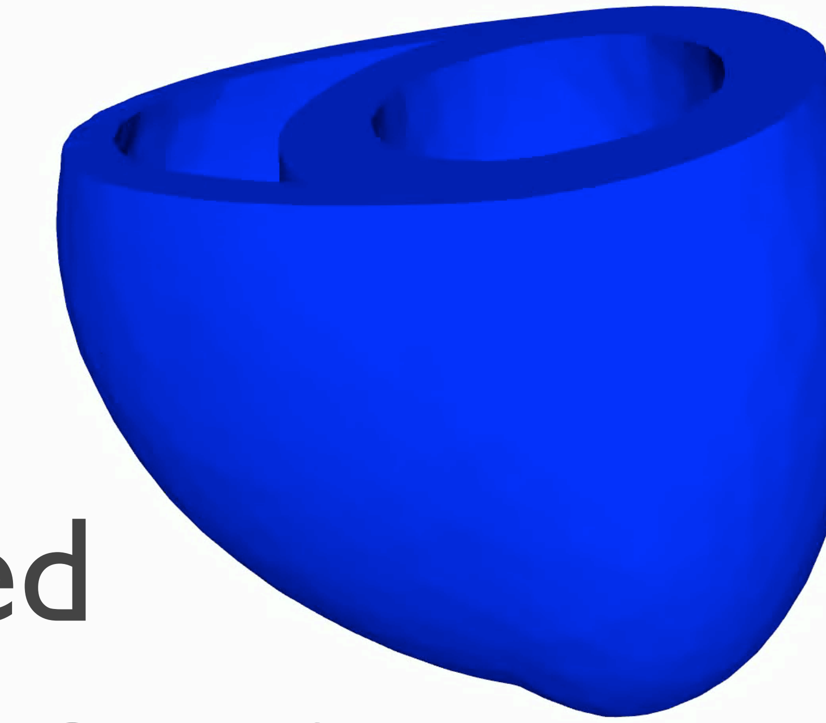

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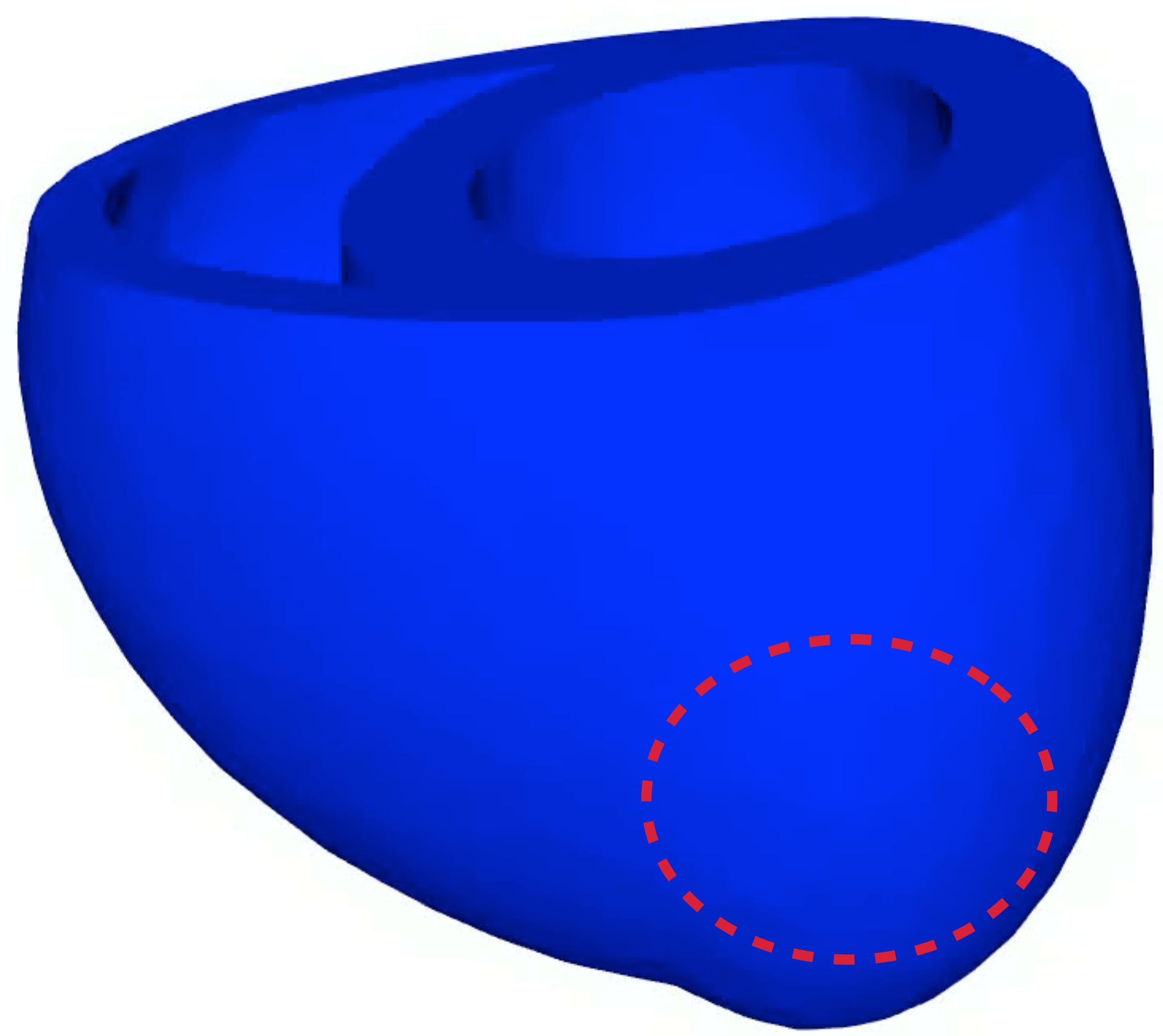
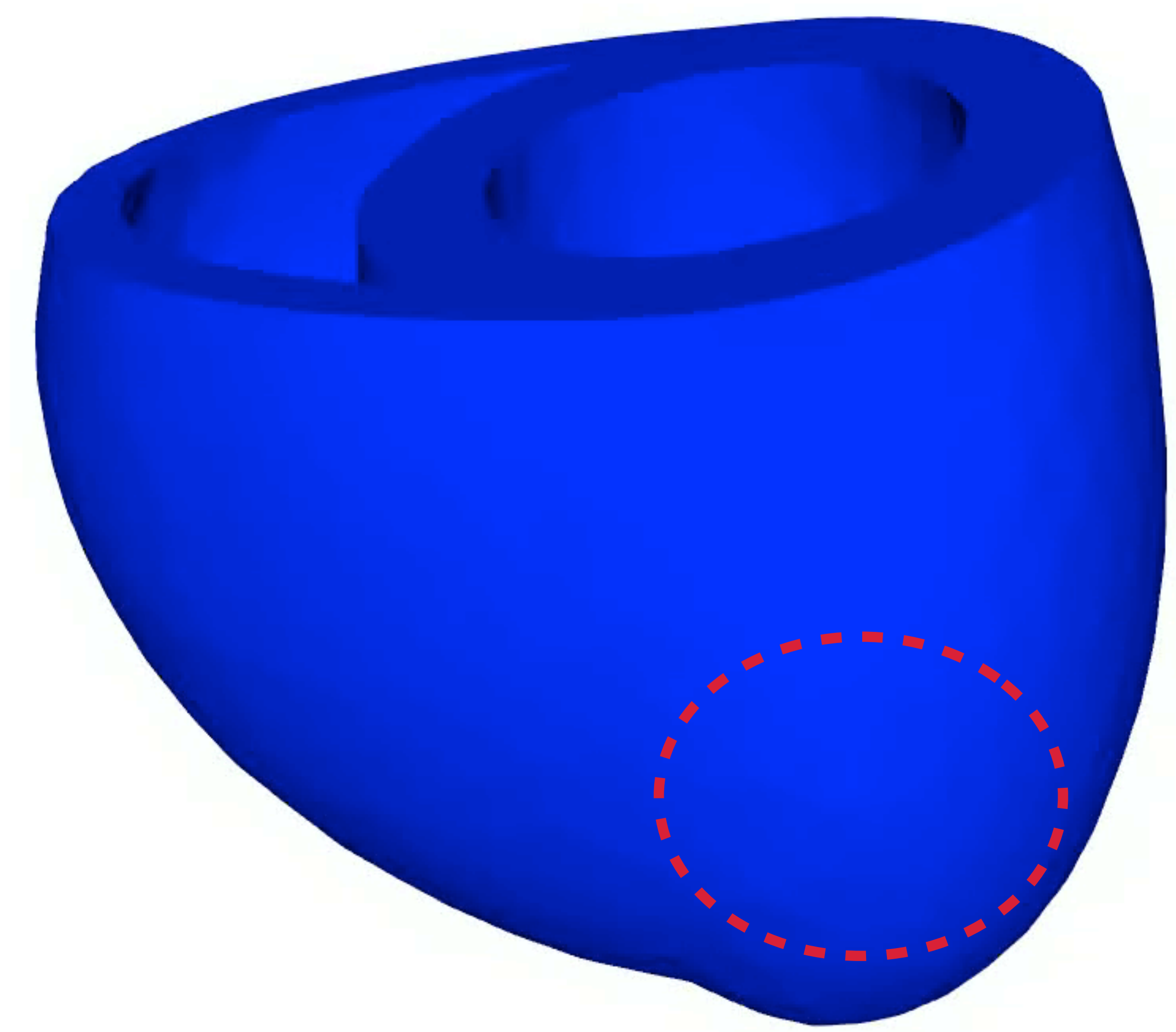


FEM (79,537 basis)



POD (100 basis)

[Boulakia, Schenone, Gerbeau - 2012]

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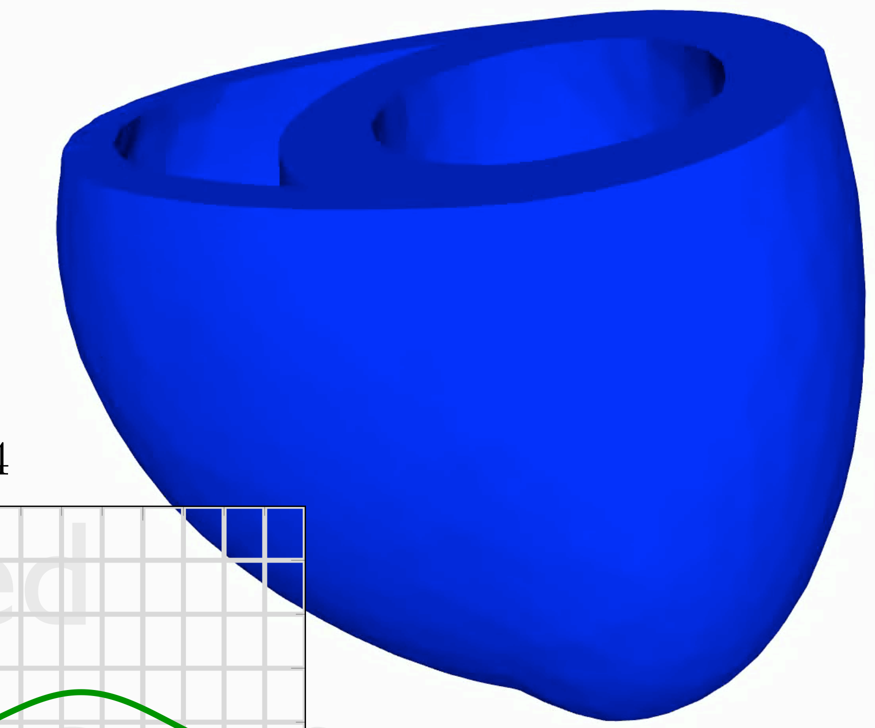

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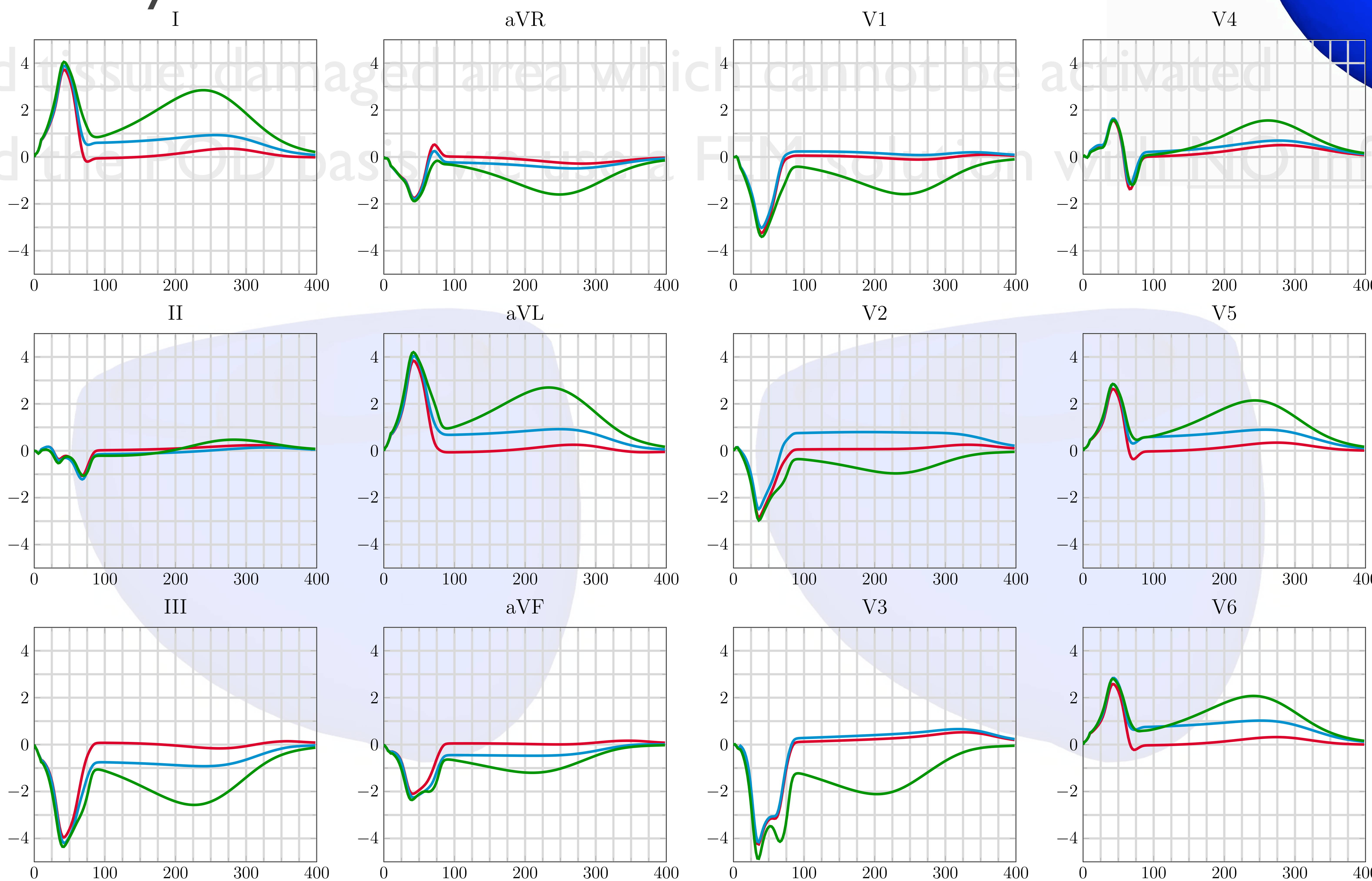

FEM (79,537 basis) POD (100 basis)

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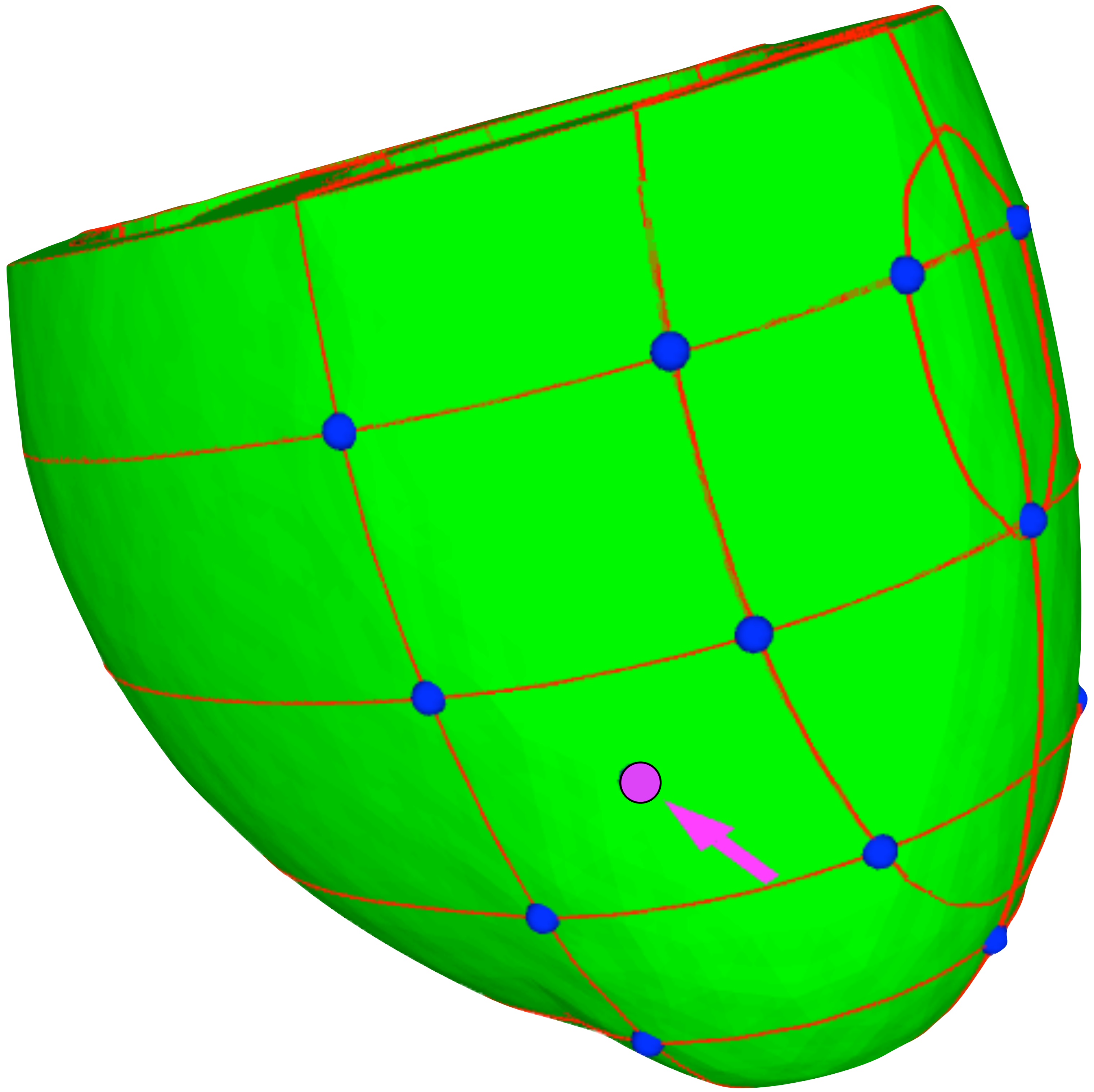
POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction

- An efficient POD to simulate an infarction in any point of the heart
  ➡ to build the POD basis, we use <u>many</u> FEM solutions <u>with infarction</u>



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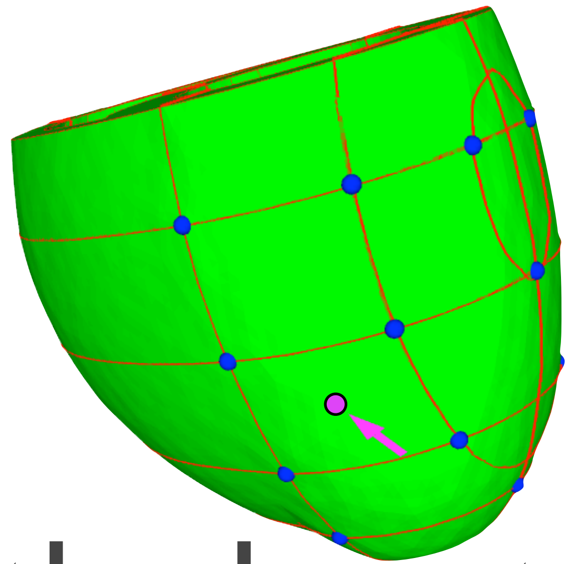


- An efficient POD to simulate an infarction in any point of the heart

➡ to build the POD basis, we use many FEM solutions with infarction

- healthy FEM solution $S_h = \left[\mathbf{u}_h^1 | \mathbf{u}_h^2 | \dots | \mathbf{u}_h^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$
- infarct 1 FEM solution $S_{I_1} = \left[\mathbf{u}_{I_1}^1 | \mathbf{u}_{I_1}^2 | \dots | \mathbf{u}_{I_1}^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$
- infarct 2 FEM solution $S_{I_2} = \left[\mathbf{u}_{I_2}^1 | \mathbf{u}_{I_2}^2 | \dots | \mathbf{u}_{I_2}^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$

$$\dots \quad \dots \quad \dots$$

- infarct m FEM solution $S_{I_m} = \left[\mathbf{u}_{I_m}^1 | \mathbf{u}_{I_m}^2 | \dots | \mathbf{u}_{I_m}^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$

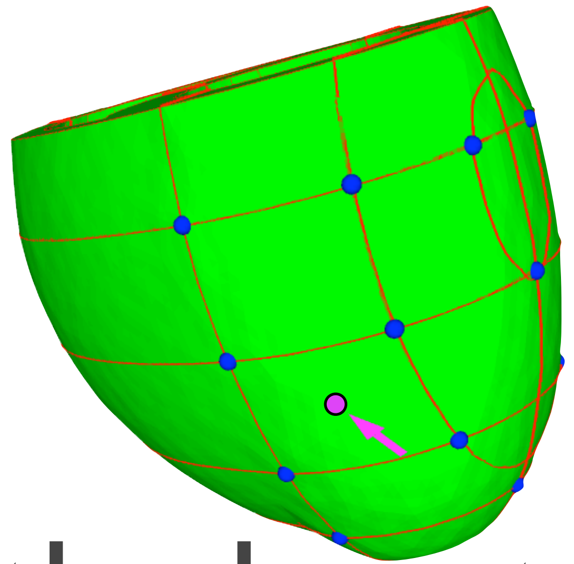
➡ compute the SVD on an enlarged matrix

$$S = \left[S_h | S_{I_1} | S_{I_2} | \dots | S_{I_m}\right] \in \mathbb{R}^{\mathcal{N} \times (m+1) N_T}$$

[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction



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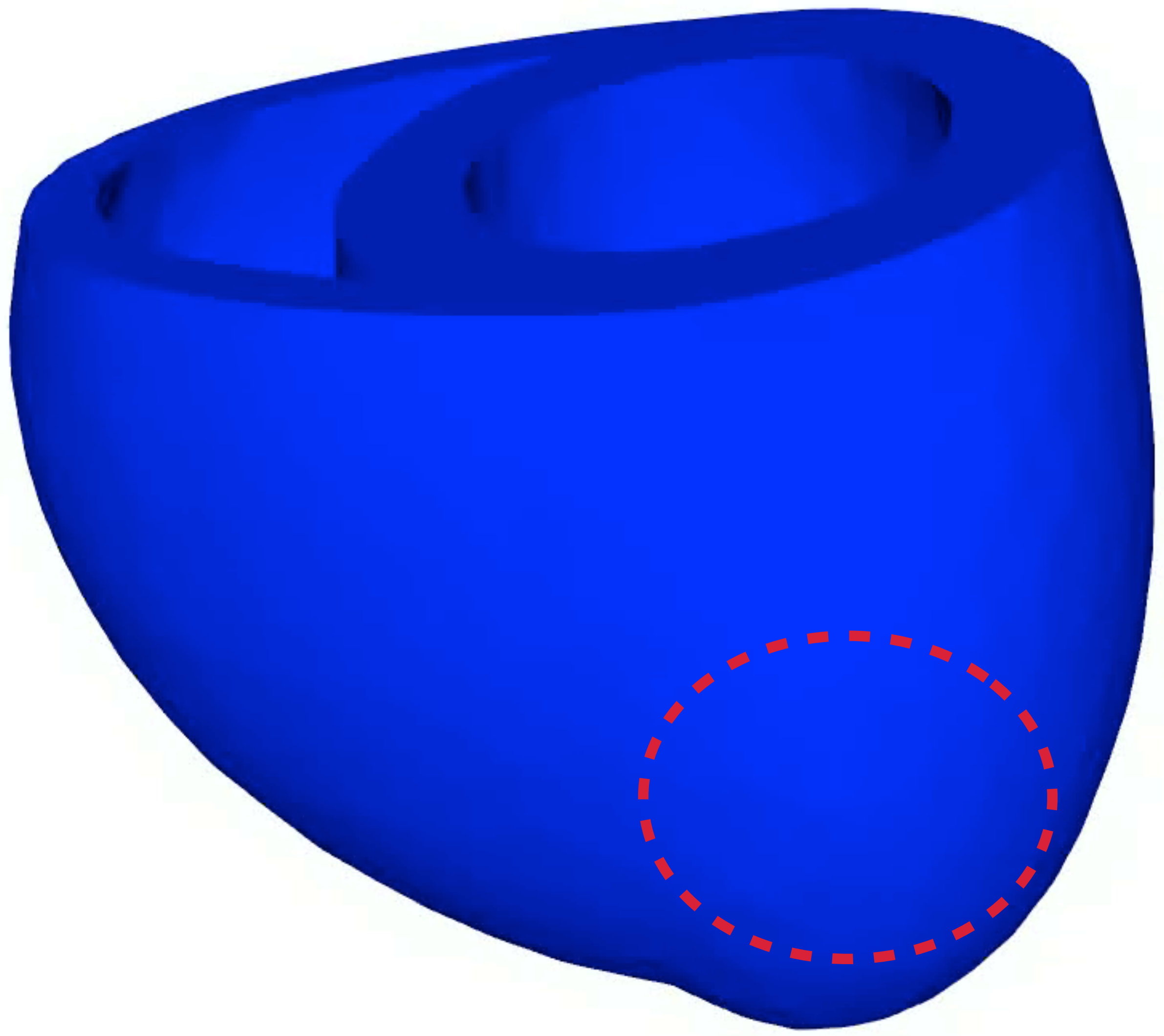

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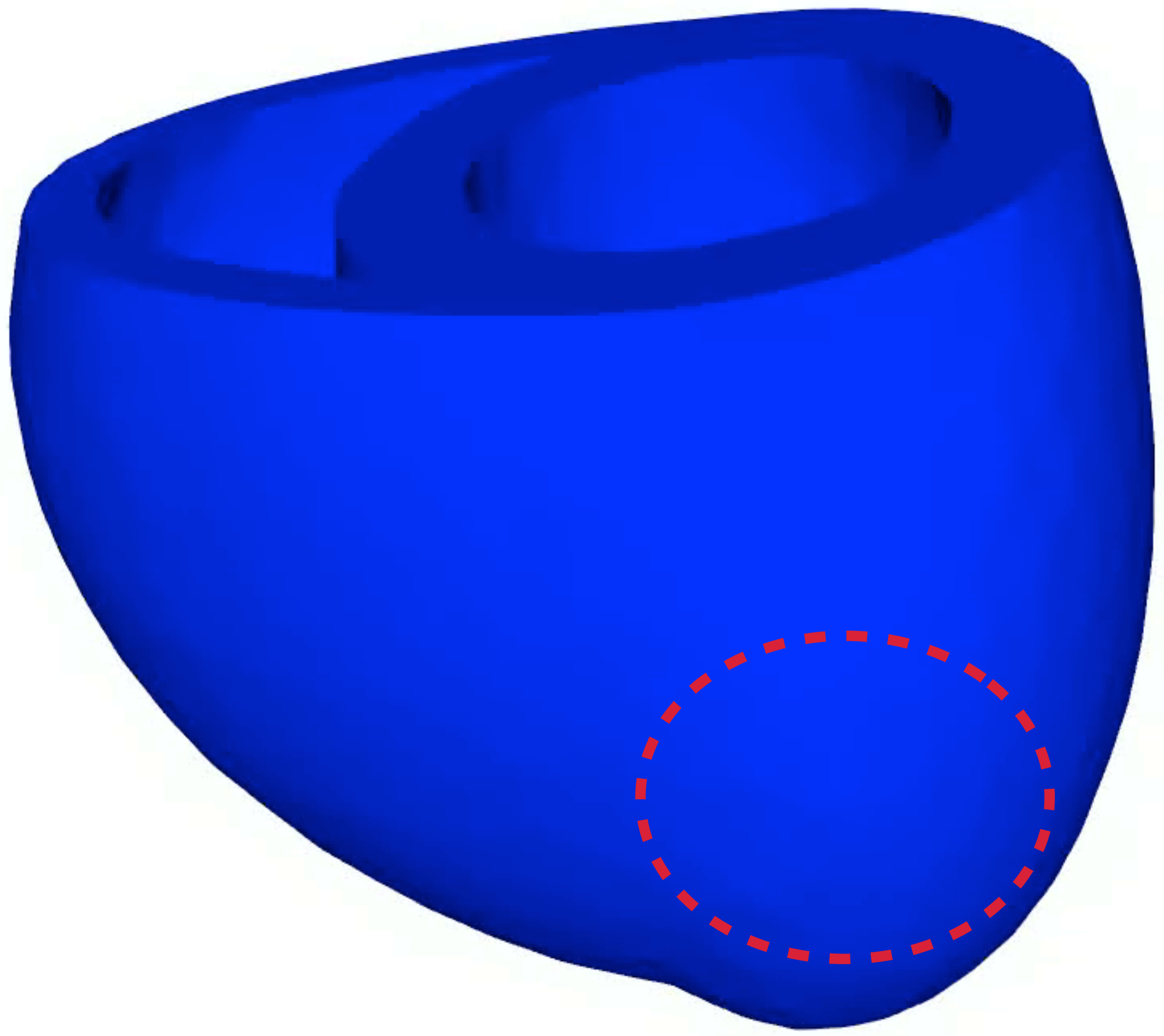

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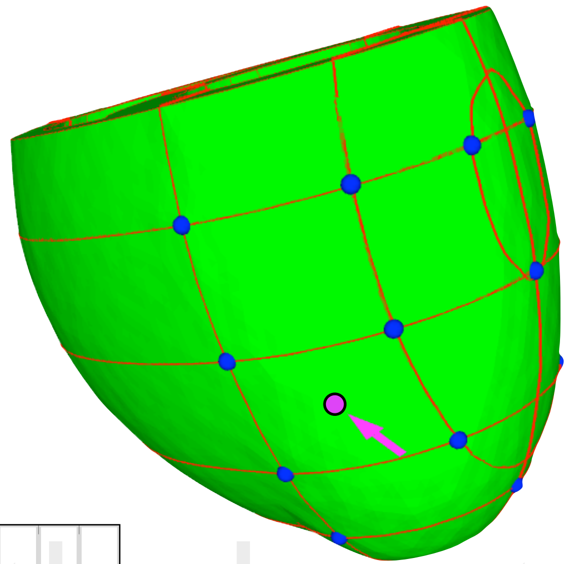

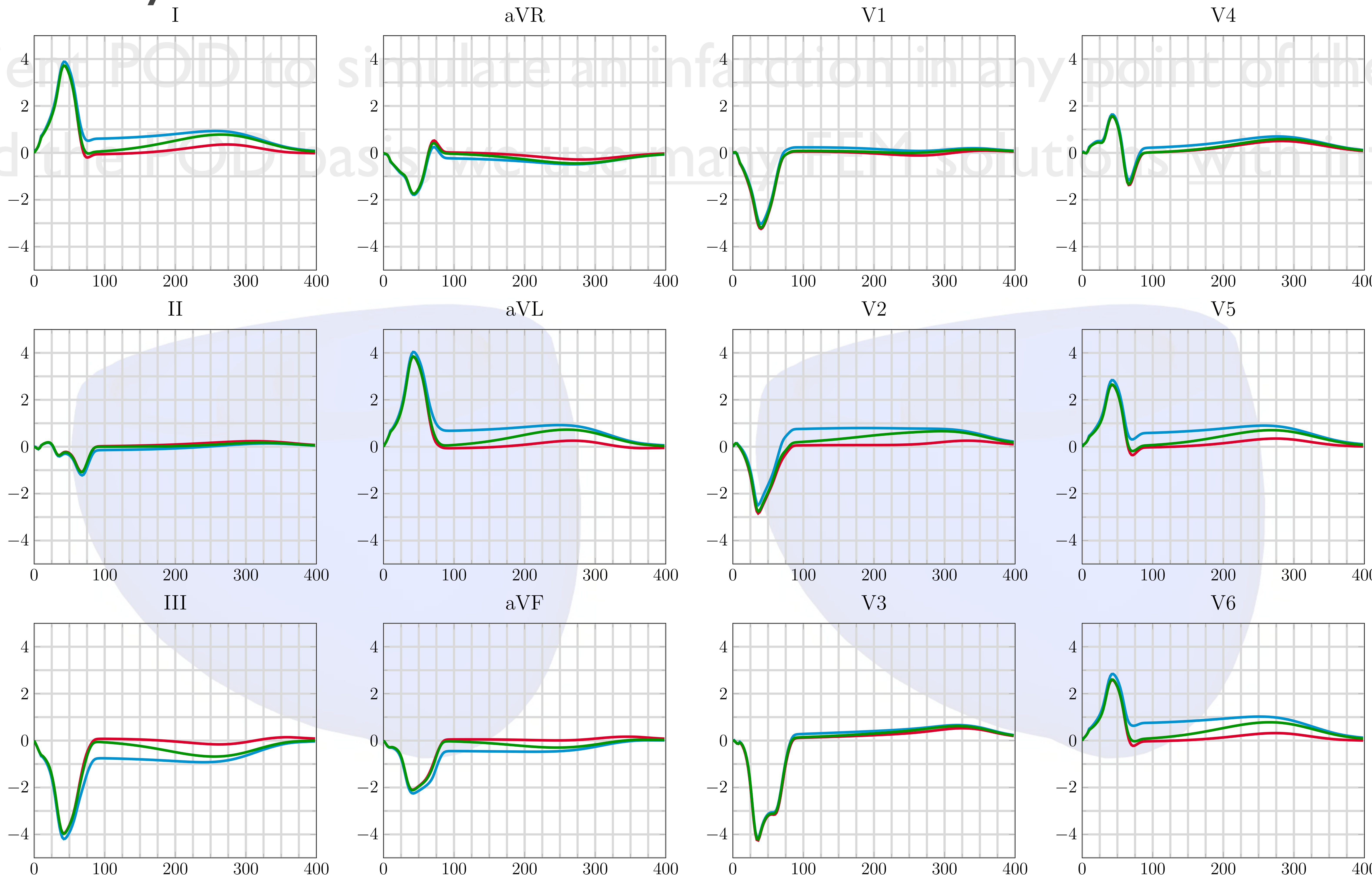

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ALP (Approximated Lax Pairs) method

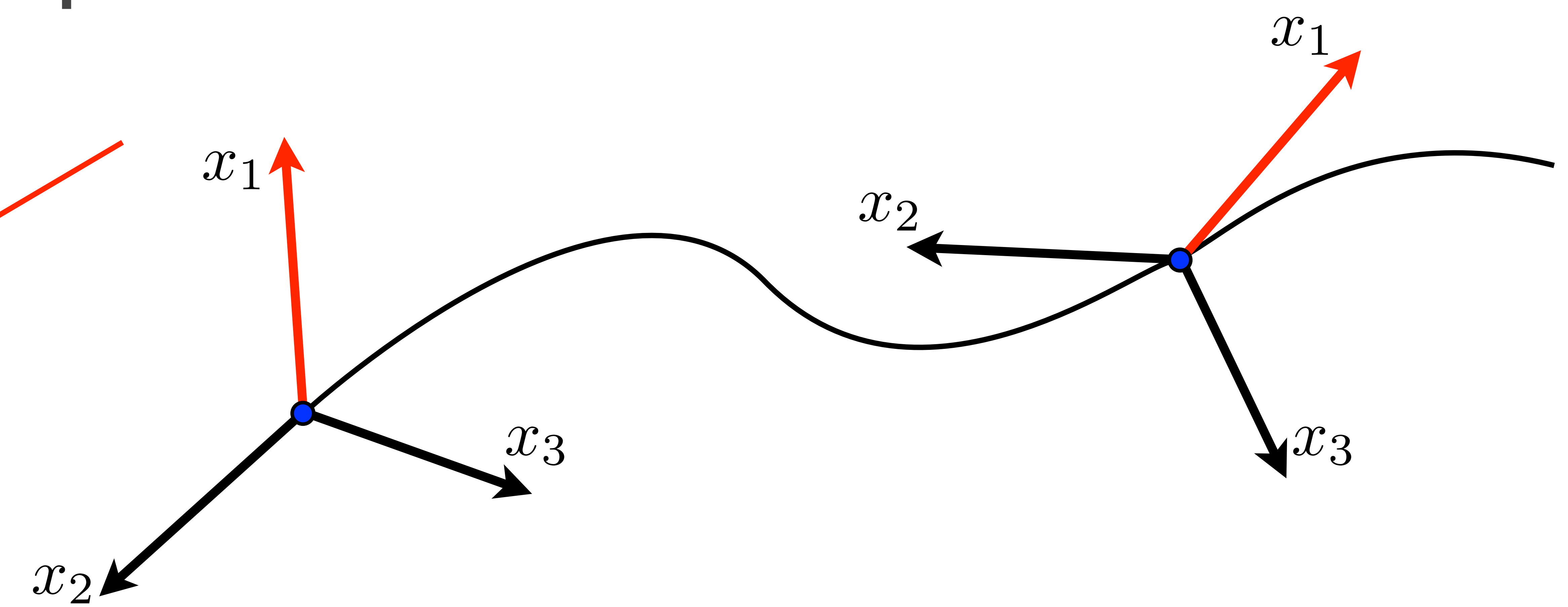
A new approach to Reduced Order Models

- Solve a generic PDE system for any set of parameters

$$\partial_t u = F(u, \partial_x^{(n)}, \pi), \ \text{in } \Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_+$$

- Define a time evolving modal expansion

$$\boxed{\hat{u} = \sum_{i=1}^{N} \beta_i(t)\varphi_i(x,t)}$$



ALP method
Summary

- Solve a generic PDE system for any set of parameters

$$\begin{cases} \partial_t u = F(u), & \Omega \times [0,T] \\ u(t=0) = u_0, & \Omega \end{cases}$$

- Method

1. Choose an initial basis
2. Define the basis evolution
3. Representation in the reduced space

ALP method

1. Choose an initial basis, 2. Define the basis evolution

1. Definition of the basis using PDE solution

$$\mathcal{L}(u)\varphi_m(t) = \lambda_m(t)\varphi_m(t)$$

Solution of $\partial_t u = F(u)$

1.1) Choose of an operator $\mathcal{L}$ $\longrightarrow$ Schrödinger

$$\mathcal{L}_\chi(u)\varphi = -\Delta\varphi - \chi u\varphi$$

1.2) Solve the eigenvalue problem $\mathcal{L}_\chi(u)\varphi_m = \lambda_m\varphi_m$

1.3) The initial basis is $(\varphi_m)_{m\geq 1}$

2. Find an operator $\mathcal{M}$ s.t. $\partial_t\varphi_m(t) = \mathcal{M}(t)\varphi_m(t)$

ALP method in electrophysiology

Discretization of bidomain equations

- Bidomain equations + FhN model

$$
\begin{aligned}
A_m C_m \partial_t v_m &= f(v_m, u_E, w) \\
\partial_t w &= g(v_m, w) \\
q(v_m, u_E) &= 0
\end{aligned}
$$

where
$$
\begin{aligned}
f(v_m, u_E, w) &= A_m s v_m (v_m - a)(1 - v_m) - A_m w + \mathrm{div}(\sigma_I \nabla v_m) + \mathrm{div}(\sigma_I \nabla u_E) + A_m I_{app} \\
g(v_m, w) &= \epsilon(\gamma v_m - w) \\
q(v_m, u_E) &= -\mathrm{div}((\sigma_I + \sigma_E)\nabla u_E) - \mathrm{div}(\sigma_I \nabla v_m)
\end{aligned}
$$

- Write the solution in the RO space

$$
\hat{v}_m = \sum_{i=1}^{N} \beta_i(t)\varphi_i(x,t) \qquad \hat{w} = \sum_{i=1}^{N} \mu_i(t)\varphi_i(x,t) \qquad \hat{u}_E = \sum_{i=1}^{N} \xi_i(t)\varphi_i(x,t)
$$

$$
f(\hat{v}_m, \hat{u}_E, \hat{w}) = \sum_{i=1}^{N} \gamma_i(t)\varphi_i(x,t) \qquad g(\hat{v}_m, \hat{w}) = \sum_{i=1}^{N} \eta_i(t)\varphi_i(x,t) \qquad q(\hat{v}_m, \hat{u}_E) = E\hat{v}_m + Q\hat{u}_E
$$

ALP method in electrophysiology

Discretization of bidomain equations

Update of ALP solution	$\dot{\beta} + M\beta - \gamma = 0$ $\dot{\mu} + M\mu - \eta = 0$ $E\beta + Q\xi = 0$
Update of eigenvalue	$\dot{\lambda}_i + \chi \sum_{m=1}^{N} T_{iim}\gamma_m = 0, \quad i = 1 \dots N$
Update of matrices/tensors	$\dot{B} = [M, B] \qquad \dot{T} = \{M, T\}^{(3)}$ $\dot{E} = [M, E] \qquad \dot{T}^{(s)} = \{M, T^{(s)}\}^{(3)}$ $\dot{Q} = [M, Q] \qquad \dot{Y} = \{M, Y\}^{(4)}$
Update of evolution operator	$M_{ij} = \dfrac{\chi}{\lambda_j - \lambda_i} \sum_{m=1}^{N} T_{ijm}\gamma_m, \quad i,j = 1 \dots N$
Compute new equation RHS	$\gamma = \gamma(\beta, \xi, \mu)$ $\eta = \eta(\beta, \mu)$

Numerical results (1)

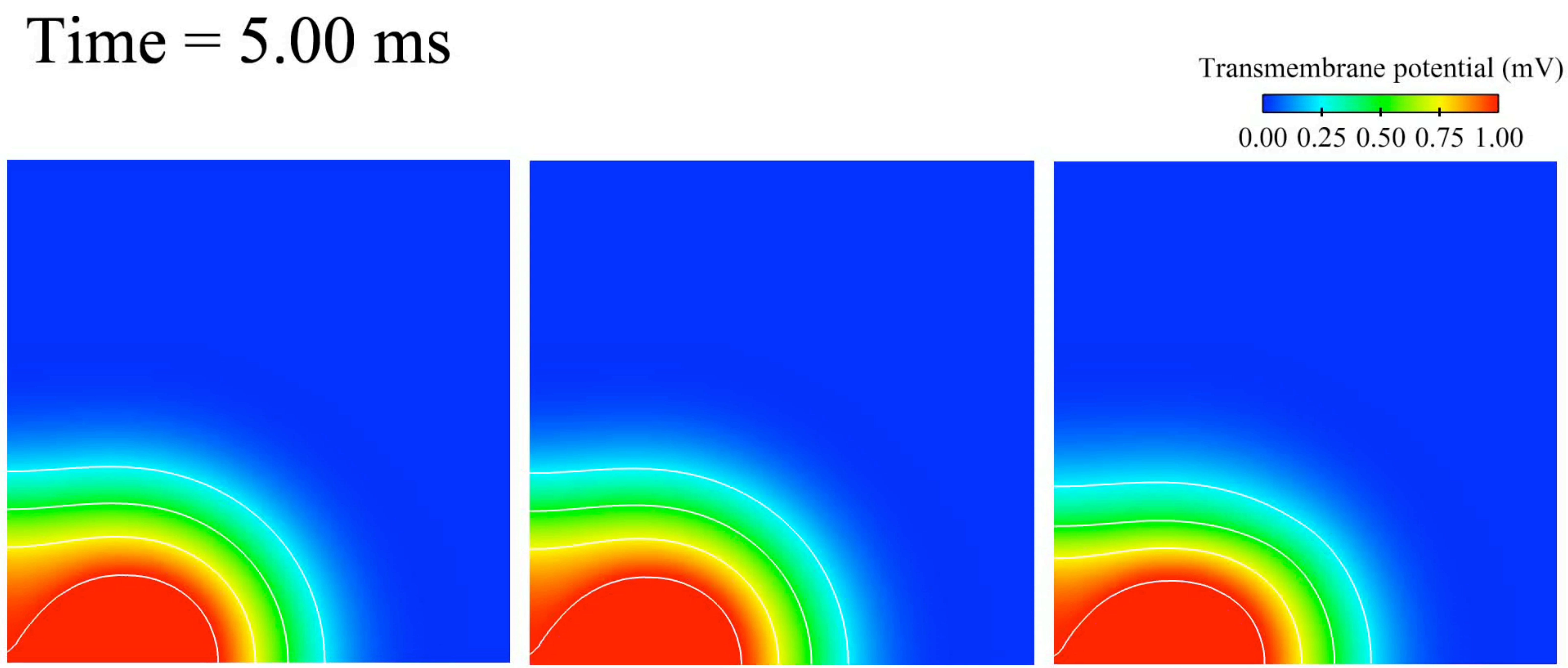
Bidomain equations in a 2D mesh

- FEM solution
  - ➡ 2D mesh with 5978 vertices
- ALP solution
  - ➡ Initial basis computed with FEM solution (at t = 5 msec)
  - ➡ Number of ROM modes N=25
- POD solution
  - ➡ POD built from a homogeneous parameters simulation
  - ➡ Number of ROM modes N=25
  - ➡ Number of FEM snapshots = 100, with sampling time 0.5 msec

[Gerbeau, Lombardi, Schenone - 2014]

Numerical results (1)

Bidomain equations in a 2D mesh



FEM (5,978 basis) ALP(25 basis) POD (25 basis)

[Gerbeau, Lombardi, Schenone - 2014]

Numerical results (2)

Bidomain equations in 2D - heterogeneous parameters

- FEM solution
  - 2D mesh with 5978 vertices
  - heterogeneous parameter s=s(x)

$$I(V_{\mathrm{m}}, w) = s(x)u(u-a)(u-1)+w$$

- ALP solution
  - Initial basis computed with FEM solution (at t = 5 msec)
  - Number of ROM modes N=25
  - heterogeneous parameter s=s(x)
- POD solution
  - POD built from a homogeneous parameters simulation
  - Number of ROM modes N=25
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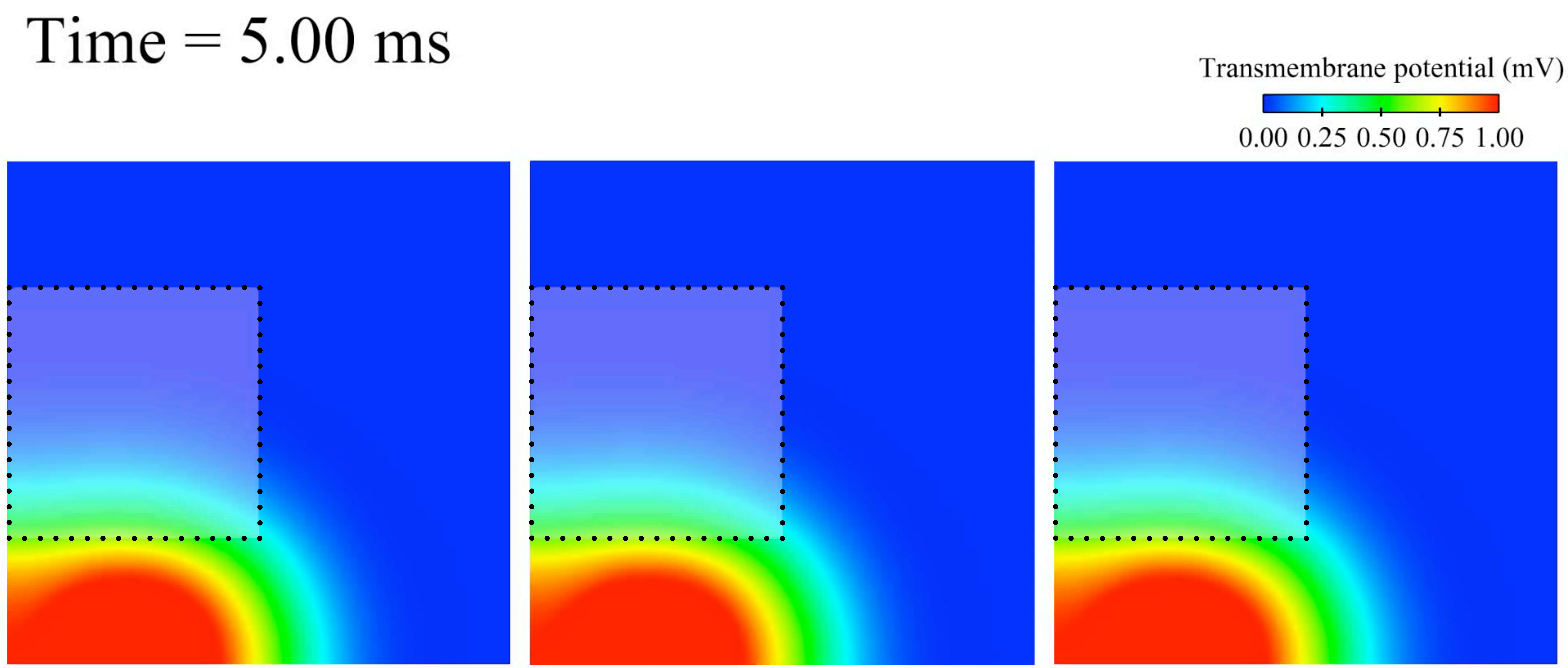
[Gerbeau, Lombardi, Schenone - 2014]

Numerical results (2)

Bidomain equations in 2D - heterogeneous parameters



FEM (5,978 basis) ALP(25 basis) POD (25 basis)

[Gerbeau, Lombardi, Schenone - 2014]

Numerical results (3)

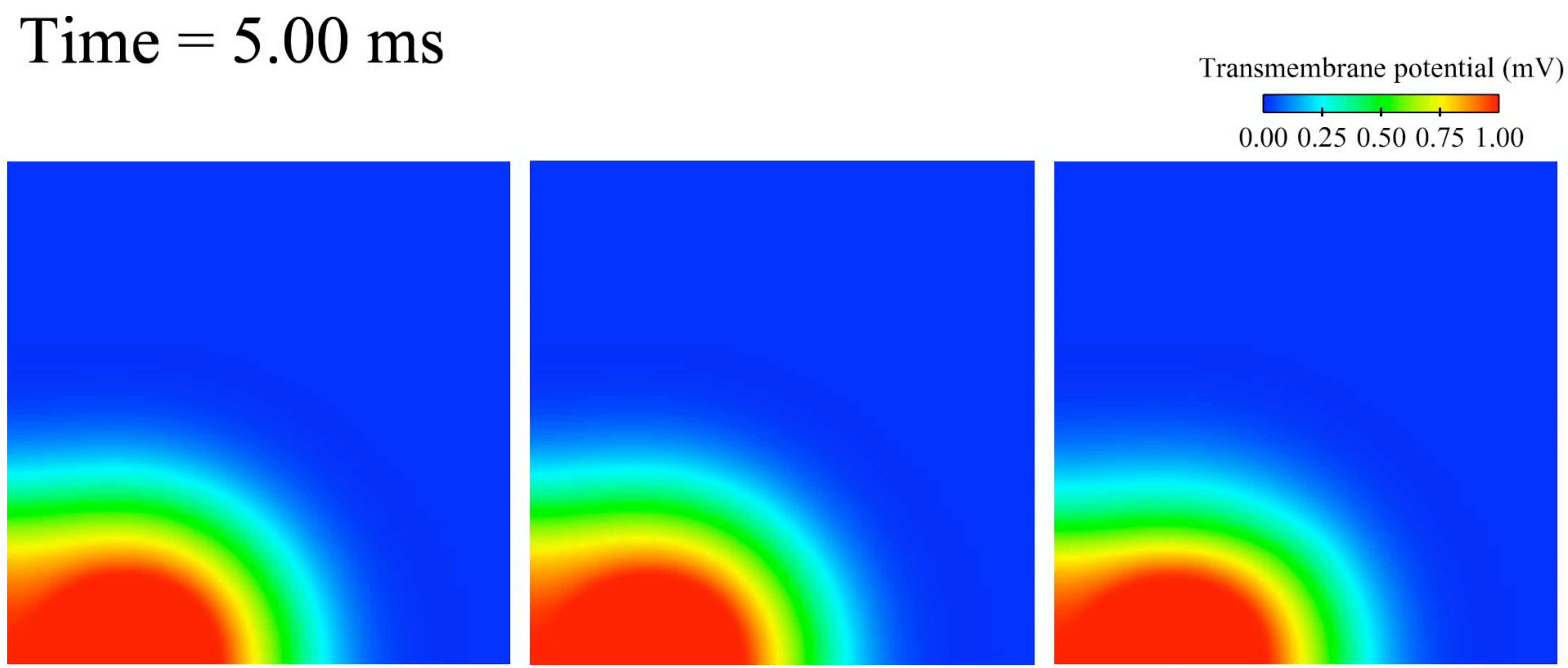
Bidomain equations in 2D - source term

- FEM solution
  - ➡ 2D mesh with 5978 vertices
  - ➡ ectopic pacemaker (source at t = 0 msec + t = 60 msec)
- ALP solution
  - ➡ Initial basis computed with FEM solution (at t = 5 msec)
  - ➡ Number of ROM modes N=25
  - ➡ ectopic pacemaker (source at t = 60 msec)
- POD solution
  - ➡ POD built from a homogeneous parameters simulation
  - ➡ Number of ROM modes N=25
  - ➡ Number of FEM snapshots = 100, with sampling time 0.5 msec

[Gerbeau, Lombardi, Schenone - 2014]

Numerical results (3)

Bidomain equations in 2D - source term



FEM (5,978 basis) ALP(25 basis) POD (25 basis)

[Gerbeau, Lombardi, Schenone - 2014]

Conclusions

- Numerical simulations make presentations nice and colorful
- Cardiac electrophysiology equations are complicated
- FEM are boring
- ROM are much more fun,

... but use them carefully!

- POD does not always work and often need many FEM solutions to give good results
- ALP does not need an *a priori* knowledge of the solution, no data-base is needed

- A lot of things are still to do ... maybe during my post-doc!

A special thank to the organizers for these two years working together!