CANARD-INDUCED LOSS OF STABILITY ACROSS A HOMOCLINIC BIFURCATION

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Multiscale dYnamiCs in neuroENdocrine AxEs
Research interests:
- theoretical and numerical study of slow-fast systems with complex oscillations,
- non-conservative transport equations for cellular dynamics
- macroscopic limits for stochastic neural networks and neural fields
Canard explosion in slow-fast systems models the transition between a state of equilibria and relaxation oscillation,

Many natural phenomena can be described by such systems,

In a category of planar slow-fast systems, some systems present bifurcations which display a dramatical change in the phase portrait for a very small change of parameter.

→ Main point: study of the stability boundaries
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Definition

Slow-fast system of dimension 2:

\[
\begin{align*}
\varepsilon \dot{x} &= f(x, y, \lambda) \\
\dot{y} &= g(x, y, \lambda),
\end{align*}
\]

Where $0 < \varepsilon \ll 1$, $\lambda \in \mathbb{R}^n$, $f, g \in C^\infty$.

$\varepsilon$ is the ratio of the time scale in which each variable involves.

They naturally appears when analysing for example the activity of GnRh neurons.
Canards in the FitzHugh-Nagumo model

The FitzHugh-Nagumo model is a simplification of the Hodgking-Huxley model which captures the action potential in the squid giant axon.

\[
\begin{align*}
x' &= x - \frac{x^3}{3} - y + I \\
y' &= \varepsilon(x + a - by).
\end{align*}
\]

Figure: Canard explosion in the FHN model for $I$ decreasing from 1.5 to 1.4.

→ Example of excitable system.
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Consider the two dimensional system

\[
\dot{x} = f(x, \alpha), \ x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}.
\]

Suppose

- \(x(\alpha)\) equilibrium point,
- \(\lambda_1(\alpha), \lambda_2(\alpha)\) complex conjugate eigenvalues of \(\text{Jac}(x(\alpha))\),
- \(\alpha\) varies \(\rightarrow\) \(\lambda_1(\alpha), \lambda_2(\alpha)\) cross the imaginary axis.

\(\leftrightarrow\) Equilibrium point changes his stability and a limit cycle appears.

**Figure:** Appearance of a stable periodic orbit via a Hopf bifurcation.

**stable focus**  
**unstable focus**
Singular Hopf bifurcation

Hopf bifurcation in slow-fast systems + some conditions (see [2]) realised ⇒ *singular Hopf bifurcation*. That is the periodic orbit which emerges from the Hopf bifurcation grows from an amplitude that is $O(\varepsilon^{1/2})$ to an amplitude that is $O(1)$ for a variation of parameter that is $O(e^{-k/\varepsilon})$ for some constant $k$. The canards explosion is an example of a singular Hopf bifurcation, see [[1],[5],[6],[8]].
Homoclinic bifurcation

**Figure:** Disappearance of a cycle across a homoclinic bifurcation.
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Idea

Goal is to compute branches of solutions of non-linear equations of the form:

$$F(X) = 0, \ F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n.$$ 

That is a under-determined system → away from singularities, the solution set is a curve. Many problems can be put in this form. In particular studies of ODEs

$$\dot{x} = F(x, \lambda).$$

- stationary problems (search for equilibria)
- Boundary value problem (BVP), including periodic orbits
Parameter continuation

Suppose we have one solution

\[ F(U_0) = 0, \quad U_0 = (u_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R} \]

Goal: Find a branch of solutions parametrized by \( \lambda \): \((u(\lambda), \lambda)\).

A first strategy is to use Newton’s method to find a solution of the augmented problem

\[
F(u_1, \lambda_1) = 0 \\
\lambda_1 = \lambda_0 + \Delta \lambda,
\]

where \( \Delta \lambda \) is given and with \( u_1^0 = u_0 + \Delta \lambda \dot{u}_0 \) as initial approximation.

**Figure:** Graphical interpretation of parameter continuation from [4].

Correction only in \( u \to \) problem at fold.
Keller’s Pseudo-Arclength Continuation

To overcome the problem at fold points of the solution branch, Keller’s Pseudo-Arclength Continuation is implemented in the software package AUTO (see [7]). Suppose we have a solution \((u_0, \lambda_0) \rightarrow\) Instead of varying \(\lambda\), we vary the arclength \(s\) and use Newton’s method to solve the augmented problem:

\[
F(u_1, \lambda_1) = 0, \\
(u_1 - u_0)^T \dot{u}_0 + (\lambda_1 - \lambda_0) \dot{\lambda}_0 - \Delta s = 0,
\]

where \(\Delta s\) is given and with \(u_1^0 = u_0 + \Delta s \dot{u}_0\), \(\lambda_1^0 = \lambda_0 + \Delta s \dot{\lambda}_0\) as initial approximation.

Figure: Graphical interpretation of pseudo-arclength continuation from [4].
Boundary Value Problem (BVP) & Orthogonal collocation

BVP

\[ u'(t) - f(u(t), \mu, \lambda) = 0; \quad t \in [0, 1] \]

with the boundary condition

\[ b(u(0), u(1), \mu, \lambda) = 0 \]

and the integral constraint

\[ \int_0^1 q(u(t), \mu, \lambda) dt = 0, \text{ Where } \lambda \text{ is the continuation parameter.} \]

Orthogonal Collocation

\[ [0, 1] = \bigcup [t_{j-1}, t_j], \quad j = 1, ..., N, \quad t_j - t_{j-1} = h. \]

- Approach \( u \) by a continuous function \( P^m \) piecewise polynomial which is polynomial of degree \( m \) on each interval \([t_{j-1}, t_j], j \in [1, N].\)
  
  \( \rightarrow \) Find solution \( p^j_h \) on degree \( m \) such that
  
  \( (p^j_h)'(z_{j,i}) = f(p^j_h(z_{j,i}), \mu, \lambda), j \in [1, N], i \in [1, m] \)

  \( \rightarrow \) \( z_{j,i} \) = roots of the \( m \)-th degree polynome of Legendre on the interval \([t_{j-1}, t_j]\)

  \( \rightarrow \) \( P^j_h \) have to satisfies the boundary and the integral condition.

- Pseudo-arclength equation becomes

\[
\int_0^1 (u(t) - u_0(t))^T \dot{u}_0(t) dt + (\mu - \mu_0)^T \dot{\mu}_0 + (\lambda - \lambda_0) \dot{\lambda}_0 - \delta s = 0.
\]
Following Periodic Solution

Search periodic solution of \( u'(t) - f(u(t), \lambda) = 0 \).

- Fix the interval periodicity by the transformation \( t \rightarrow \frac{t}{T} \).

\[
\begin{align*}
  u'(t) &= Tf(u(t), \lambda) \\
  u(0) &= u(1).
\end{align*}
\]

where \( T \) is an unknown.

- This equation don't uniquely specify \( T \) and \( u \) → We choose the solution such that:

\[
\int_0^1 u_k(t)^T u_{k-1}(t) dt = 0.
\]

That is the *integral phase condition*.

- BVP problem! → AUTO compute the solution using a orthogonal collocation scheme.
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Lienard and Hamiltonian system

The generalized polynomial Liénard equations are of the form:

\[\begin{align*}
\dot{x} &= y - f(x) \\
\dot{y} &= -\varepsilon(g(x) + \lambda),
\end{align*}\]  

(2)

Where \( f \) and \( g \) are polynomials. We propose to study perturbation of the particular case

\[\begin{align*}
\varepsilon\dot{x} &= y - f(x) \\
\dot{y} &= -f'(x)
\end{align*}\]  

(3)

Which after the rescaling \((x, y, t) \rightarrow (\varepsilon^{1/2} x, y\varepsilon, \varepsilon^{1/2} t)\) becomes

\[\begin{align*}
\dot{x} &= y - f_\varepsilon(x) \\
\dot{y} &= -f'_\varepsilon(x)
\end{align*}\]  

(4)

This system is integrable of integrand factor \(e^{-y}\).

\[H(x, y) = e^{-y}[f_\varepsilon(x) - y - 1] = h \in [-1, h_{\text{max}}].\]

and

\[y(x) = f_\varepsilon(x) - 1\]

is a solution.
Examples

To simplify the visualisation, we represent the level set of $H$ for $\varepsilon = 1$ in the examples

- $f(x) = (x + 1)x(x - 1)(x - 3/2)(x - 3)$

![Image 1](image1.png)

**Figure:** Three homoclinic loops which bound three nest. Figure from [3].

- $f(x) = x^2/2 - x^4/4$

![Image 2](image2.png)

**Figure:** A nest bounded by an heteroclinic loop. Figure from [3].
Lambert function

Reciprocal of the function $x \rightarrow xe^x$.

- Function multivalued (except at 0).
- Only two real branches: the principal branch $W_0$ and the other $W_{-1}$

Figure: The two real branches of the Lambert function.
Solution of the integrable system

\[ e^{f_\varepsilon(x)-1}[f_\varepsilon(x) - y - 1] = h e^{f_\varepsilon(x)-1}. \]

\[
\begin{align*}
\leftarrow y_+ &= f_\varepsilon(x) - 1 - W_{-1}\left(\frac{h}{\varepsilon} e^{f_\varepsilon(x)}\right), \\
y_- &= f_\varepsilon(x) - 1 - W_{0}\left(\frac{h}{\varepsilon} e^{f_\varepsilon(x)}\right).
\end{align*}
\]

**Proposition**

*Any periodic trajectory intersects transversally the critical curve in exactly two points.*

\[ y(x) = y_+(x) \text{ above the critical curve } y = f_\varepsilon(x) \text{ and } y(x) = y_-(x) \text{ below.} \]
Perturbation of a integrable system

We choose to study the system:

\[
\begin{align*}
\varepsilon \dot{x} &= y - \frac{x^2}{2} - \alpha \frac{x^3}{3} \\
\dot{y} &= -x - x^2 (\alpha - \beta) + \sqrt{\varepsilon} \mu.
\end{align*}
\]  

(5)

After rescaling this yields:

\[
\begin{align*}
\dot{x} &= y - \frac{x^2}{2} - \sqrt{\varepsilon} \alpha \frac{x^3}{3} \\
\dot{y} &= -x - \sqrt{\varepsilon} x^2 (\alpha - \beta) + \mu.
\end{align*}
\]  

(6)

Figure: Integrable system.
Numerical simulation

Numerical simulations have been done with XPPAUT (see [7]).

- For $0 < \alpha < 1$, $0 < \sqrt{\epsilon} \ll 1$ and $0 < \beta < 1$ fixed, a small canard cycle is born by varying $\mu$ across a Hopf bifurcation,
- for variation of $\mu$ of order $10^{-7}$ the cycle explodes and disappears across an homoclinic bifurcation.

Figure: (a): Small canard cycle. (b): Trajectory with same initial condition after explosion. Figure from [3].
We want to compute the value of $\mu$ for which the cycle explodes.

Figure: (a): Bifurcation diagram of system 6 in $\mu$. (b): a few limit cycles on the explosive branch (in blue) shown in panel (a), approaching the homoclinic connection. Figure from [3].
Strategy based on
- the first return map
- the derivative given by an integral of Lambert function

Consider equation

\[
\begin{align*}
    h &= e^{-\frac{y}{\varepsilon}} \left[ \frac{f(x)}{\varepsilon} - \frac{y}{\varepsilon} - 1 \right] \\
    \omega &= e^{-\frac{y}{\varepsilon}} \frac{y-f(x)}{\varepsilon} dy - e^{-\frac{y}{\varepsilon}} (-f'(x) - \delta(x)) dx \\
    &= dh - e^{-\frac{y}{\varepsilon}} \delta(x) dx.
\end{align*}
\] (7)
The following integral equation hold:

$$\int_{\gamma_{\mu,\beta,h}} \omega = \int_{\gamma_{\mu,\beta,h}} dh - \int_{\gamma_{\mu,\beta,h}} e^{-\frac{y}{\varepsilon}} \delta(x) dx,$$

(8)

**Figure:** Schema of trajectory in positive and in negative time starting from an initial condition on the vertical axis $x = 0$. 
Results

Using the parametrisation with the Lambert function we obtain the condition:

\[
\frac{L_+(h, \beta, \mu) - L_-(h, \beta, \mu)}{h} = \beta \int_{x^{-}(h)}^{x^{+}(h)} x^2 \left[ \frac{1}{W_0\left(\frac{h}{\varepsilon} e^{\varepsilon} f(x)\right)} - \frac{1}{W_{-1}\left(\frac{h}{\varepsilon} e^{\varepsilon} f(x)\right)} \right] dx
\]

\[
+ \sqrt{\varepsilon \mu} \int_{x^{-}(h)}^{x^{+}(h)} \left[ \frac{1}{W_0\left(\frac{h}{\varepsilon} e^{\varepsilon} f(x)\right)} - \frac{1}{W_{-1}\left(\frac{h}{\varepsilon} e^{\varepsilon} f(x)\right)} \right] dx
\]

\[+ O \left( (\sqrt{\varepsilon \mu}, \beta)^2 \right). \tag{9}\]

Solving the equation with MATHEMATICA for \( h = e^{-\frac{1}{6\alpha \varepsilon}} \) (level set of the homoclinic loop) we find a very good approximation of the parameter for which the loss of stability happens.
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E. J Doedel, *Lecture Notes on Numerical Analysis of Nonlinear Equation*, Department Of Computer Science, Concordia University, Montreal, Canada.


