CANARD-INDUCED LOSS OF STABILITY ACROSS A HOMOCLINIC BIFURCATION

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outline

Plan

MYCENAE

Multiscale dYnamiCs in neuroENdocrine AxEs
Research interests:

- theoretical and numerical study of slow-fast systems with complex oscillations,
- non-conservative transport equations for cellular dynamics
- macroscopic limits for stochastic neural networks and neural fields

Introduction

- Canard explosion in slow-fast systems models the transition between a state of equilibria and relaxation oscillation,
- Many natural phenomena can be described by such systems,
- In a category of planar slow-fast systems, some systems present bifurcations which display a dramatical change in the phase portrait for a very small change of parameter.

$\rightarrow$ Main point: **study of the stability boundaries**

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Definition

Slow-fast system of dimension 2:

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y, \lambda) \\ \dot{y} &= g(x, y, \lambda), \end{aligned}$$

Where $0 < \varepsilon \ll 1$, $\lambda \in \mathbb{R}^n$, f, g C^∞.
$\rightarrow \varepsilon$ is the ratio of the time scale in which each variable involves.

They naturally appears when analysing for example the activity of GnRh neurons.

Canards in the FitzHugh-Nagumo model

The FitzHugh-Nagumo model is a simplification of the Hodgking-Huxley model which captures the action potential in the squid giant axon.

$$\begin{aligned} x' &= x - \frac{x^3}{3} - y + I \\ y' &= \varepsilon(x + a - by). \end{aligned}$$

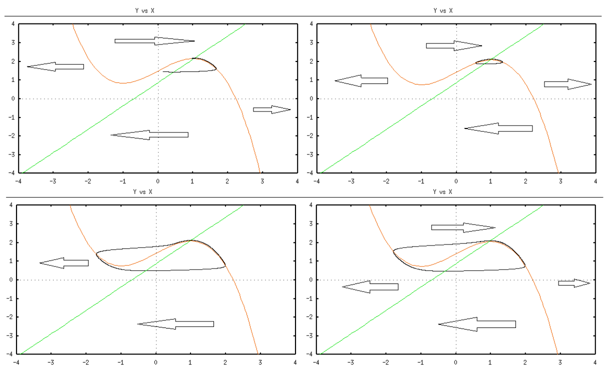

Figure: Canard explosion in the FHN model for I decreasing from 1.5 to 1.4.

→ Example of excitable system.

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Hopf Bifurcation

Consider the two dimensional system

$$\dot{x} = f(x, \alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R}.$$

Suppose

- $x(\alpha)$ equilibrium point,
- $\lambda_1(\alpha), \lambda_2(\alpha)$ complex conjugate eigenvalues of $Jac(x(\alpha))$,
- α varies $\rightarrow \lambda_1(\alpha), \lambda_2(\alpha)$ cross the imaginary axis.

$\hookrightarrow$ Equilibrium point changes his stability and a limit cycle appears.

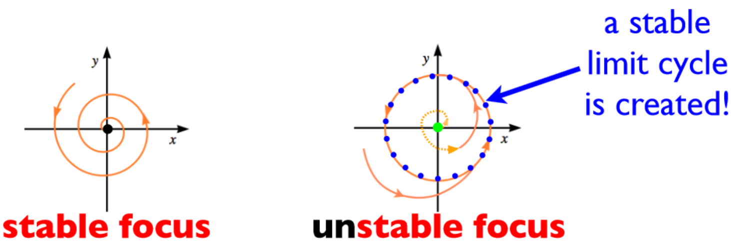

Figure: Appearance of a stable periodic orbit via a Hopf bifurcation.

Singular Hopf bifurcation

Hopf bifurcation in slow-fast systems $+$ some conditions (see [2]) realised $\Rightarrow$ *singular Hopf bifurcation*.
That is the periodic orbit which emerges from the Hopf bifurcation grows from an amplitude that is $O(\varepsilon^{1/2})$ to an amplitude that is $O(1)$ for a variation of parameter that is $O(e^{-k/\varepsilon})$ for some constant k.
The canards explosion is an example of a singular Hopf bifurcation, see [[1],[5],[6],[8]].

Homoclinic bifurcation

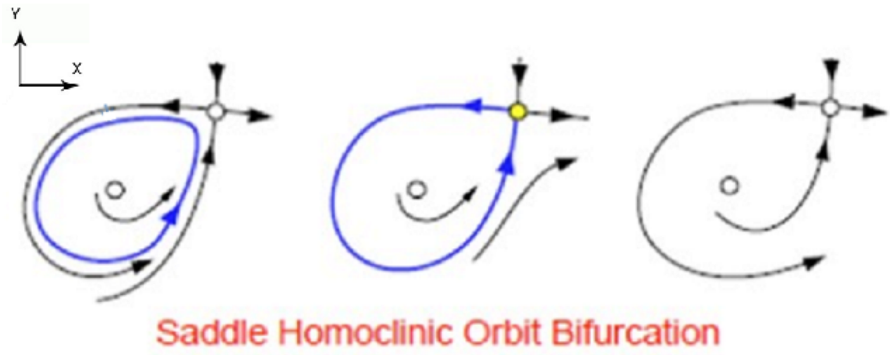

Figure: Disappearance of a cycle across a homoclinic bifurcation.

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Idea

Goal is to compute branches of solutions of non-linear equations of the form:

$$F(X) = 0, F : \mathbb{R}^{n+1} \to \mathbb{R}^n.$$

That is a under-determined system $\to$ away from singularities, the solution set is a curve. Many problems can be put in this form. In particular studies of ODEs

$$\dot{x} = F(x, \lambda).$$

- stationary problems (search for equilibria)
- Boundary value problem (BVP), including periodic orbits

Parameter continuation

Suppose we have one solution

$$F(U_0) = 0, U_0 = (u_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$$

Goal: Find a branch of solutions parametrized by λ: $(u(\lambda), \lambda)$.
A first strategy is to use Newton's method to find a solution of the augmented problem

$$\begin{aligned} F(u_1, \lambda_1) &= 0 \\ \lambda_1 &= \lambda_0 + \Delta\lambda, \end{aligned}$$

where $\Delta\lambda$ is given and with $u_1^0 = u_0 + \Delta\lambda \dot{u}_0$ as initial approximation.

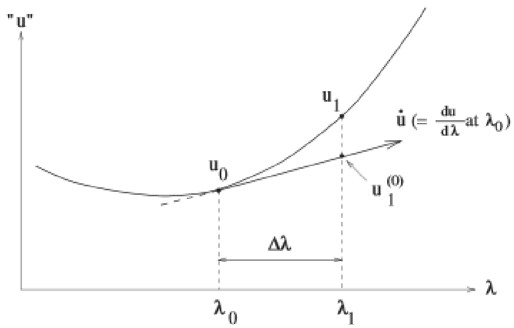

Figure: Graphical interpretation of parameter continuation from [4].

Correction only in $u \to$ problem at fold.

Keller's Pseudo-Arclength Continuation

To overcome the problem at fold points of the solution branch, Keller's Pseudo-Arclength Continuation is implemented in the software package AUTO (see [7]).
Suppose we have a solution $(u_0, \lambda_0) \rightarrow$ Instead of varying λ, we vary the arclength s and use Newton's method to solve the augmented problem:

$$\begin{aligned} F(u_1, \lambda_1) &= 0, \\ (u_1 - u_0)^T \dot{u}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s &= 0, \end{aligned} \tag{1}$$

where Δs is given and with $u_1^0 = u_0 + \Delta s \dot{u}_0$, $\lambda_1^0 = \lambda_0 + \Delta s \dot{\lambda}_0$ as initial approximation.

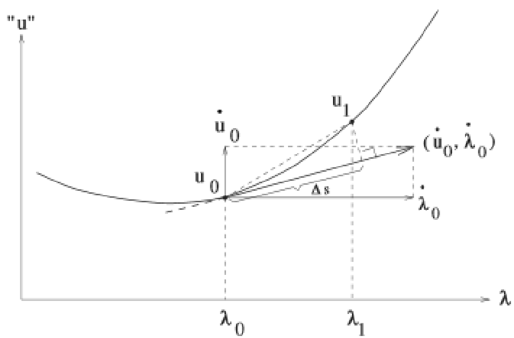

Figure: Graphical interpretation of pseudo-arclength continuation from [4].

Boundary Value Problem (BVP) & Orthogonal collocation

BVP

$\rightarrow$ solution of $u'(t) - f(u(t), \mu, \lambda) = 0; t \in [0, 1]$
with the boundary condition $b(u(0), u(1), \mu, \lambda) = 0$
and the integral constraint $\int_0^1 q(u(t), \mu, \lambda)dt = 0$, Where λ is the continuation parameter.

Orthogonal Collocation

- $[0, 1] = \cup[t_{j-1}, j_j], j = 1, ..., N, t_j - t_{j_1} = h$.
- Approach u by a continuous function P^m piecewise polynomial which is polynomial of degree m on each interval $[t_{j-1}, t_j], j \in [1, N]$.
  $\rightarrow$ Find solution p_h^j on degree m such that
  $(p_h^j)'(z_{j,i}) = f(p_h^j(z_{j,i}), \mu, \lambda), j \in [1, N], i \in [1, m]$
  $\rightarrow$ $z_{j,i}$ = roots of the mth-degree polynome of Legendre on the interval $[t_{j-1}, t_j]$
  $\rightarrow$ P_h have to satisfies the boundary and the integral condition.
- Pseudo-arclength equation becomes

$$\int_0^1 (u(t) - u_0(t))^T \dot{u}_0(t)dt + (\mu - \mu_0)^T \dot{\mu}_0 + (\lambda - \lambda_0)\dot{\lambda}_0 - \delta s = 0.$$

Following Periodic Solution

Search periodic solution of $u'(t) - f(u(t), \lambda) = 0$.

- Fix the interval periodicity by the transformation $t \to \frac{t}{T}$.
  $\hookrightarrow$
  $$\begin{array}{rcl} u'(t) & = & Tf(u(t), \lambda) \\ u(0) & = & u(1). \end{array}$$
  where T is an unknown.
- This equation don't uniquely specify T and $u \to$ We choose the solution such that:
  $$\int_0^1 u_k(t)^T u_{k-1}(t) dt = 0.$$
  That is the *integral phase condition*.
- BVP problem! $\to$ AUTO compute the solution using a orthogonal collocation scheme.

Plan

Lienard and Hamiltonian system

The generalized polynomial Liénard equations are of the form:

$$\begin{aligned} \dot{x} &= y - f(x) \\ \dot{y} &= -\varepsilon(g(x) + \lambda), \end{aligned} \tag{2}$$

Where f and g are polynomials. We propose to study perturbation of the particular case

$$\begin{aligned} \varepsilon\dot{x} &= y - f(x) \\ \dot{y} &= -f'(x) \end{aligned} \tag{3}$$

Which after the rescaling $(x, y, t) \to (\varepsilon^{1/2}x, y\varepsilon, \varepsilon^{1/2}t)$ becomes

$$\begin{aligned} \dot{x} &= y - f_\varepsilon(x) \\ \dot{y} &= -f_\varepsilon'(x) \end{aligned} \tag{4}$$

This system is integrable of integrand factor e^{-y}.

$$H(x, y) = e^{-y}[f_\varepsilon(x) - y - 1] = h \in [-1, h_{max}].$$

and

$$y(x) = f_\varepsilon(x) - 1$$

is a solution.

Examples

To simplify the visualisation, we represent the level set of H for $\varepsilon = 1$ in the examples

- $f(x) = (x+1)x(x-1)(x-3/2)(x-3)$

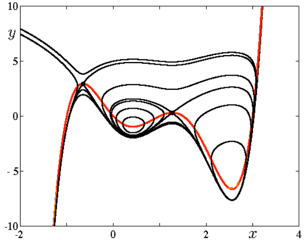

Figure: Three homoclinic loops which bound three nest. Figure from [3].

- $f(x) = x^2/2 - x^4/4$

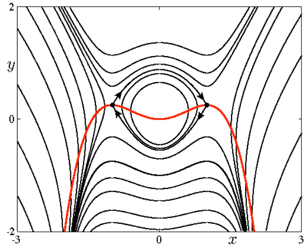

Figure: A nest bounded by an heteroclinic loop. Figure from [3].

Lambert function

Reciprocal of the function $x \to xe^x$.

- Function multivalued (except at 0).
- Only two real branches: the principal branch W_0 and the other W_{-1}

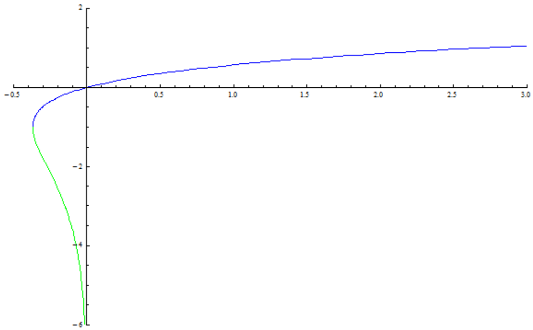

Figure: The two real branches of the Lambert function.

Solution of the integrable system

$$e^{f_\varepsilon(x)-y-1}[f_\varepsilon(x)-y-1] = he^{f_\varepsilon(x)-1}.$$

$$\hookrightarrow y_+ \quad = \quad f_\varepsilon(x) - 1 - W_{-1}\left(\tfrac{h}{e}e^{f_\varepsilon(x)}\right),$$
$$y_- \quad = \quad f_\varepsilon(x) - 1 - W_0\left(\tfrac{h}{e}e^{f_\varepsilon(x)}\right).$$

Proposition

Any periodic trajectory intersects transversally the critical curve in exactly two points.

$\rightarrow y(x) = y_+(x)$ above the critical curve $y = f_\varepsilon(x)$ and $y(x) = y_-(x)$ below.

Perturbation of a integrable system

We choose to study the system:

$$\begin{aligned} \varepsilon\dot{x} &= y - \frac{x^2}{2} - \alpha\frac{x^3}{3} \\ \dot{y} &= -x - x^2(\alpha-\beta) + \sqrt{\varepsilon}\mu. \end{aligned} \tag{5}$$

After rescaling this yields:

$$\begin{aligned} \dot{x} &= y - \frac{x^2}{2} - \sqrt{\varepsilon}\alpha\frac{x^3}{3} \\ \dot{y} &= -x - \sqrt{\varepsilon}x^2(\alpha-\beta) + \mu. \end{aligned} \tag{6}$$

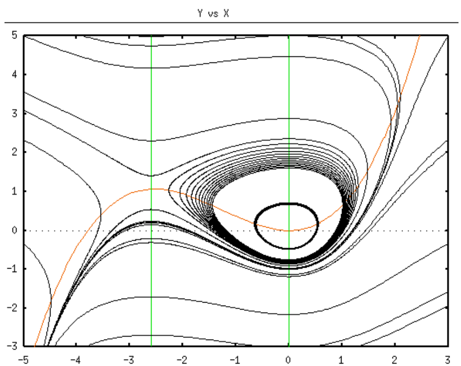

Figure: Integrable system.

Numerical simulation

Numerical simulations have been done with XPPAUT (see [7]).

- For $0 < \alpha < 1$, $0 < \sqrt{\varepsilon} \ll 1$ and $0 < \beta < 1$ fixed, a small canard cycle is born by varying μ across a Hopf bifurcation,
- for variation of μ of order 10^{-7} the cycle explodes and disappears across an homoclinic bifurcation.

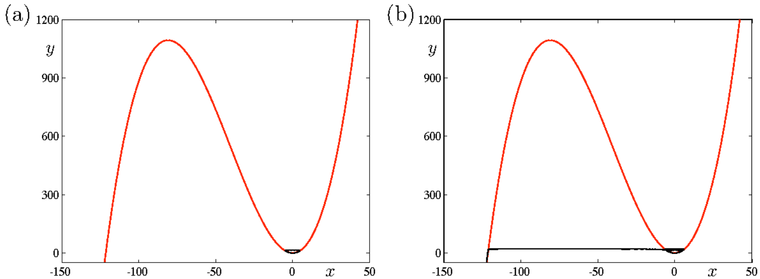

Figure: (a): Small canard cycle. (b): Trajectory with same initial condition after explosion. Figure from [3].

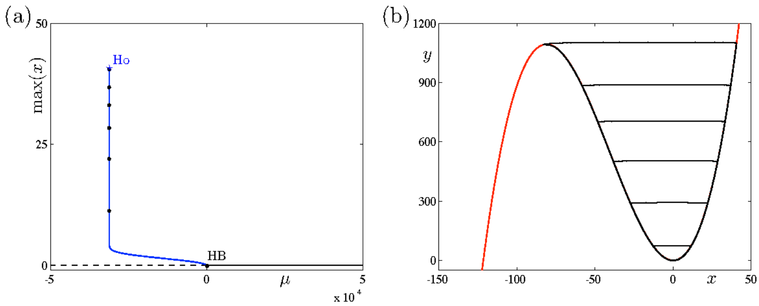

Figure: (a): Bifurcation diagram of system 6 in μ. (b): a few limit cycles on the explosive branch (in blue) shown in panel (a), approaching the homoclinic connection. Figure from [3].

$\rightarrow$ We want to compute the value of μ for which the cycle explodes.

Strategy based on

- the first return map
- the derivative given by an integral of Lambert function

Consider equation

$$\begin{aligned} h &= e^{-\frac{y}{\varepsilon}}\left[\frac{f(x)}{\varepsilon} - \frac{y}{\varepsilon} - 1\right] \\ \omega &= e^{-\frac{y}{\varepsilon}}\frac{y-f(x)}{\varepsilon}dy - e^{-\frac{-y}{\varepsilon}}(-f'(x) - \delta(x))dx \\ &= dh - e^{-\frac{y}{\varepsilon}}\delta(x)dx. \end{aligned} \tag{7}$$

The following integral equation hold:

$$\int_{\gamma_{\mu,\beta,h}} \omega = \int_{\gamma_{\mu,\beta,h}} dh - \int_{\gamma_{\mu,\beta,h}} e^{-\frac{y}{\varepsilon}} \delta(x) dx, \tag{8}$$

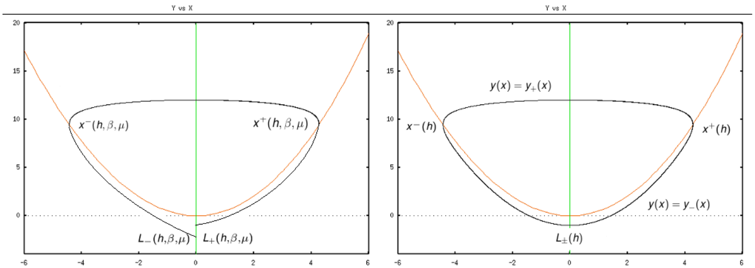

Figure: Schema of trajectory in positive and in negative time starting from an initial condition on the vertical axis $x = 0$.

Results

Using the parametrisation with the Lambert function we obtain the condition:

$$\begin{aligned}\frac{L_+(h,\beta,\mu)-L_-(h,\beta,\mu)}{h} = & \beta \int_{x^-(h)}^{x^+(h)} x^2 \left[\frac{1}{W_0(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})} - \frac{1}{W_{-1}(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})}\right] dx \\ & +\sqrt{\varepsilon}\mu \int_{x^-(h)}^{x^+(h)} \frac{1}{W_0(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})} - \frac{1}{W_{-1}(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})} dx \\ & +O\left((\sqrt{\varepsilon}\mu, \beta)^2\right).\end{aligned} \tag{9}$$

Solving the equation with MATHEMATICA for $h = e^{-\frac{1}{6\alpha^2\varepsilon}}$ (level set of the homoclinic loop) we find a very good approximation of the parameter for which the loss a stability happens.

Plan

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