An adaptive inexact Uzawa algorithm based on a posteriori estimates for the Stokes problem

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post-doc in the **POMDAPI** team

Rocquencourt, 20 January, 2015

(with some slides from M. Vohralík)

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Outline

1. A posteriori error estimate
2. Adaptive inexact method
3. Application to the Uzawa algorithm for the Stokes problem
4. Conclusion

Outline

Numerical Simulation

Physical phenomena

↓ **Physical assumptions**

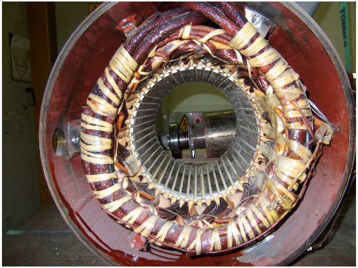
Physical modeling

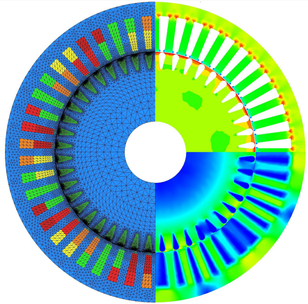
↓ **Mathematical assumptions**

Mathematical modeling (u)

↓ **Numerical methods**

Numerical results (u_h)





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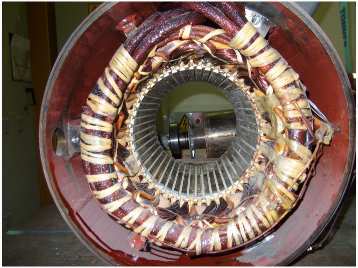
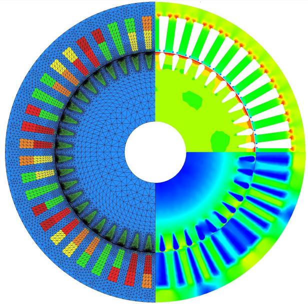
Numerical methods

Numerical errors

Numerical results (u_h)





What is an a posteriori error estimate

Error estimate

- A priori error estimate

$$\|u - u_h\| \leqslant C(u)h_{\max}^p$$

  - useful in theoretical assessment of convergence.
  - $C(u)$ is not computable in practice
- A posteriori error estimate

$$\|u - u_h\| \leqslant C\eta(u_h)$$

  - $\eta(u_h)$ is computable in practice.

A posteriori error estimate

- Error control
  - Guaranteed upper bound $\|u - u_h\|_\Omega^2 \leqslant \sum_{K\in\mathcal{T}_h} \eta_K^2$
  - Reliability : $\|u - u_h\|_\Omega^2 \leqslant C_1 \sum_{K\in\mathcal{T}_h} \eta_K^2$
- Mesh refinement
  - Local efficiency : $\eta_K \leqslant C_2\|u - u_h\|_{\omega_K}$
- Low computational cost
  - estimators can be evaluated locally
- Robustness
  - C_1 and C_2 does not depend on data, mesh, or solution.
  - C_1 and C_2 are computable.

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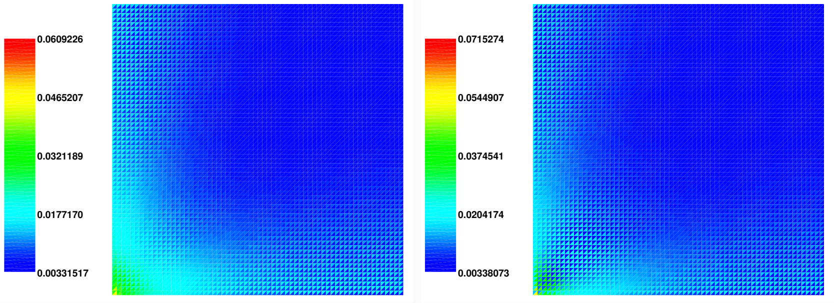
Example of an a posteriori error estimator



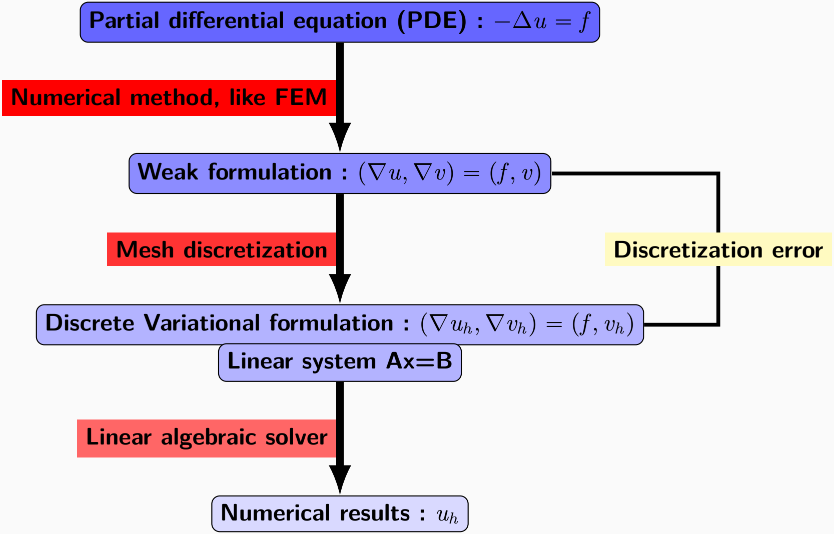
(a) Exact error distribution $\|u - u_h\|_K$ (b) Estimated error distribution η_K

Different types a posteriori estimates

- Babuška and Rheinboldt (1978), introduction
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates (equality of Prager and Synge (1947))
- Bornemann et al. (1996), Verfürth (1996) and Veeser (2002)), hierarchical estimates
- Zienkiewicz and Zhu (1987), averaging-based estimates
- Verfürth (1996, book), residual-based estimates
- Repin (1997), functional a posteriori error estimates
- Destuynder and Métivet (1999), equilibrated fluxes estimates
- Ainsworth and Oden (2000, book), equilibrated residual estimates
- Luce and Wohlmuth (2004), equilibrated fluxes estimates
- Braess and Schöberl (2008), equilibrated fluxes estimates
- some other heuristic estimates ...

Outline

Simple example of the adaptive inexact method



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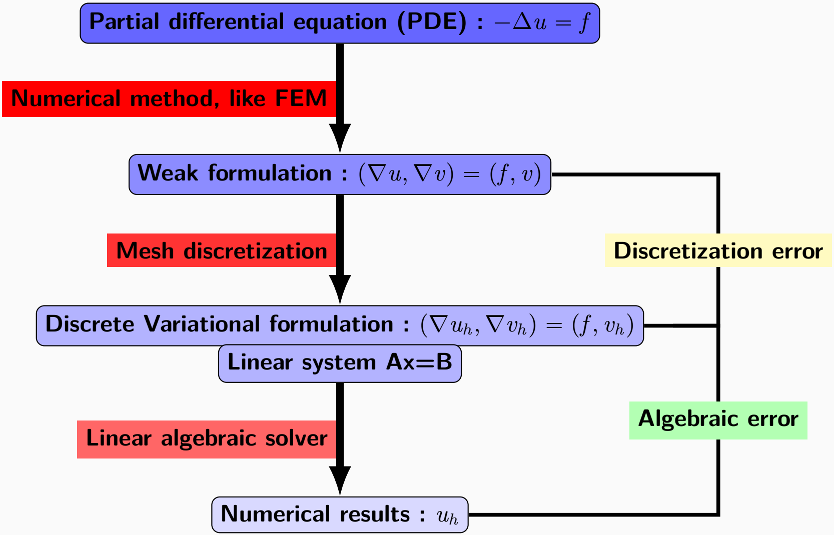

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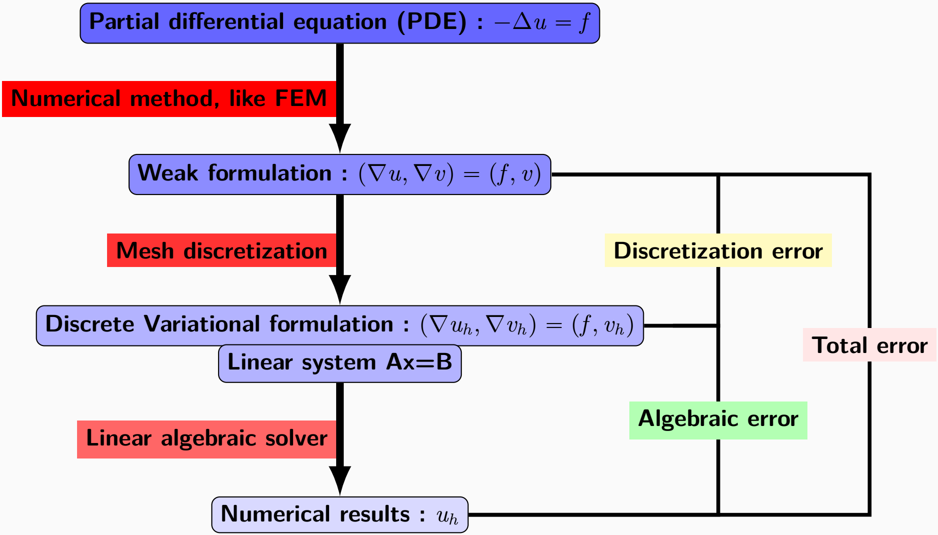
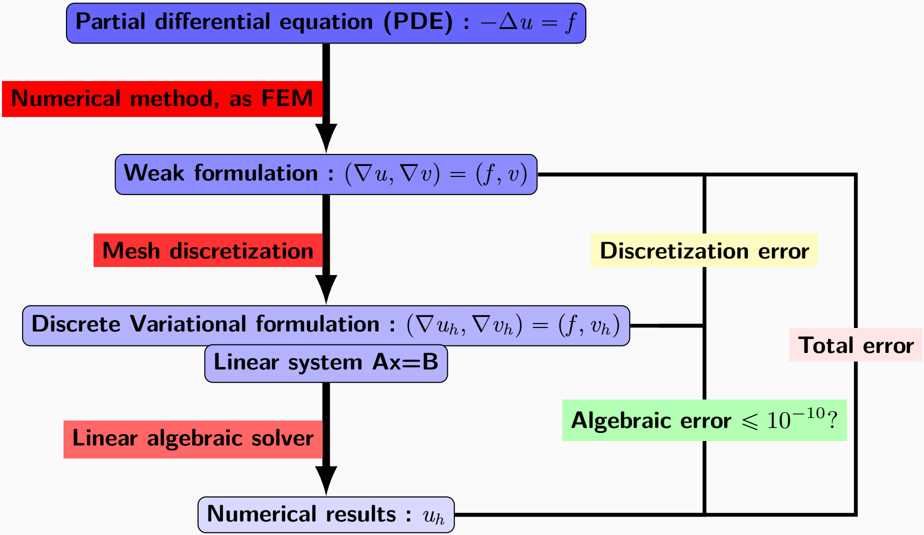


Simple example of the adaptive inexact method : exact method



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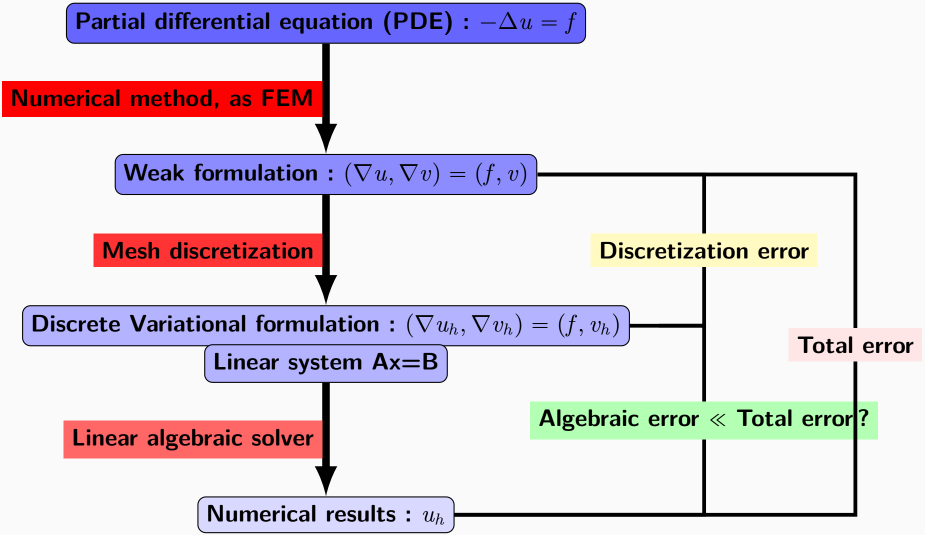

Previous results

- Becker, Johnson, and Rannacher (1995), multigrid stopping criterion
- Jiránek, Strokoš and Vohralílk (2010), finite volume method, algebraic and discretization error components
- Ern and Vohralík (2013), various discretization schemes, Newton methods for nonlinear PDEs, discretization, linearization, and algebraic error components

Conclusion

- guaranteed and robust error estimate
- distinguishes each error components
- stopping the iterations when the corresponding error no longer affects the overall error significantly
- important computational savings

Adaptive inexact method

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Adaptive stopping criterion for linear/nonlinear solver :

Theoretical development
+

Pr. Martin Vohralík
(POMDAPI, Inria)

Implementation work : Freefem++
+

Pr. Frédéric Hecht
(LJLL, UMPC)

Outline

Stokes problem

Steady linear Stokes model problem

Find a velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and a pressure $p : \Omega \to \mathbb{R}$ satisfying :

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^2$ represents the volumetric force.

Weak solution

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} (\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\ (\nabla\cdot\mathbf{u}, q) &= 0 && \forall q \in Q. \end{aligned}$$

with

$$\mathbf{V} := [H_0^1(\Omega)]^2,$$

$$Q := L_0^2(\Omega) := \{q \in L^2(\Omega); (q, 1) = 0\}.$$

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Finite element method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

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Here we obtain a corresponding system of linear algebraic equations to solve

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{U} \\ \mathrm{P} \end{pmatrix} = \begin{pmatrix} \mathrm{F} \\ 0 \end{pmatrix},$$

where $\mathbb{A}$ is a symmetric positive definite matrix and $\mathbb{B}$ has full rank.

Conforming finite element methods

- Unstabilized schemes : Taylor–Hood family, mini element, cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element, $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element
- Stabilized schemes : Brezzi–Pitkäranta method, Hughes–Franca–Balestra method, Brezzi–Douglas method

$$\begin{aligned} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\nabla\cdot \mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla\cdot \mathbf{u}_h, q_h) + s_h(\mathbf{u}_h, p_h; q_h) &= 0 && \forall q_h \in Q_h. \end{aligned}$$

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Uzawa method

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Uzawa method

1: Chose an initial approximation $p_h^0 \in Q_h$ of p, a real constant $\alpha \in (0, 2)$, and a tolerance $\varepsilon > 0$. —Initialisation

2: **For** $k = 0 \ldots + \infty$: —Uzawa iteration

a) Compute $\mathbf{u}_h^{k+1} \in \mathbf{V}_h$ such that

$$(\nabla \mathbf{u}_h^{k+1}, \nabla \mathbf{v}_h) = (\nabla \cdot \mathbf{v}_h, p_h^k) + (\mathbf{f}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.\text{—algebraic solver}$$

b) Compute $p_h^{k+1} \in Q_h$ such that

$$(p_h^{k+1}, q_h) = (p_h^k, q_h) + \alpha\, (\nabla \cdot \mathbf{u}_h^{k+1}, q_h) \quad \forall q_h \in Q_h.\text{—updating the pressure } p$$

c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation.

EndFor

Finite element method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

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Exact Uzawa method

Finite element method

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Exact Uzawa method

1: Chose an initial approximation $p_h^0 \in Q_h$ of p, a real constant $\alpha \in (0,2)$, and a tolerance $\varepsilon > 0$. —Initialisation discretization error

2: **For** $k = 0 \ldots +\infty$: —Uzawa iteration

a) Compute $\mathbf{u}_h^{k+1} \in \mathbf{V}_h$ such that

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c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation.

EndFor

Inexact Uzawa method

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2: **For** $k = 0 \ldots +\infty$: —Uzawa iteration

a) Compute $\mathbf{u}_h^{k,i} \in \mathbf{V}_h$ such that algebraic error

$$(\nabla \mathbf{u}_h^{k,i}, \nabla \mathbf{v}_h) \approx (\nabla\cdot \mathbf{v}_h, p_h^k) + (\mathbf{f}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$ —algebraic solver

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c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation. Uzawa error

EndFor

Inexact Uzawa method

Inexact method : relative work

- Vincent and Boyer (1992), mixed stabilized finite element
- Elman and Golub (1994), stopping criterion in algebraic solver
- Cheng and Zou (2003), mixed finite element methods, Lagrange multiplier methods
- Bacuta (2006), convergence analysis

Inexact Uzawa method

1: Chose an initial approximation $p_h^0 \in Q_h$ of p, a real constant $\alpha \in (0, 2)$, and a tolerance $\varepsilon > 0$. —Initialisation **discretization error**

2: **For** $k = 0 \ldots +\infty$: —Uzawa iteration

a) Compute $\mathbf{u}_h^{k,i} \in \mathbf{V}_h$ such that **algebraic error**

$$(\nabla \mathbf{u}_h^{k,i}, \nabla \mathbf{v}_h) \approx (\nabla \cdot \mathbf{v}_h, p_h^k) + (\mathbf{f}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$ —algebraic solver

b) Compute $p_h^{k+1} \in Q_h$ such that

$$(p_h^{k+1}, q_h) = (p_h^k, q_h) + \alpha\, (\nabla \cdot \mathbf{u}_h^{k,i}, q_h) \quad \forall q_h \in Q_h.$$ —updating the pressure p

c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation. **Uzawa error**

EndFor

Adaptive inexact Uzawa method

Parameters for the stopping criteria

- $\eta^{k,i}_{\text{disc}}$: discretization estimator
- $\eta^{k,i}_{\text{alg}}$: inner algebraic solver estimator
- $\eta^{k,i}_{\text{Uza}}$: Uzawa estimator

- γ_{rem}
- γ_{alg}
- γ_{Uza}

- $\eta^{k,i}_{\text{rem}}$: remainder estimator
- $\eta^{k,i}_{\text{osc}}$: data oscillation estimator

Adaptive inexact Uzawa method

1: Chose an initial approximation $p_h^0 \in Q_h$ of p, a real constant $\alpha \in (0,2)$, and a tolerance $\varepsilon > 0$. —Initialisation **discretization error** $\eta^{k,i}_{\text{disc}}$

2: **For** $k = 0 \ldots + \infty$: —Uzawa iteration

a) Compute $\mathbf{u}_h^{k,i} \in \mathbf{V}_h$ such that

algebraic error

$$\eta^{k,i}_{\text{alg}} \leqslant \gamma_{\text{alg}} \max\{\eta^{k,i}_{\text{disc}}, \eta^{k,i}_{\text{Uza}}\}$$

$$(\nabla \mathbf{u}_h^{k,i}, \nabla \mathbf{v}_h) \approx (\nabla \cdot \mathbf{v}_h, p_h^k) + (\mathbf{f}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$—algebraic solver

b) Compute $p_h^{k+1} \in Q_h$ such that

$$(p_h^{k+1}, q_h) = (p_h^k, q_h) + \alpha\,(\nabla \cdot \mathbf{u}_h^{k,i}, q_h) \quad \forall q_h \in Q_h.$$—updating the pressure p

c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation. **Uzawa error** $\eta^{k,i}_{\text{Uza}} \leqslant \gamma_{\text{Uza}} \eta^{k,i}_{\text{disc}}$

EndFor

A posteriori error estimate

Let

- k be the Uzawa step
- i be the inner algebraic iteration

with the approximations $(\mathbf{u}_h^{k,i}, p_h^k)$ and the stopping criterion. Then

Guaranteed upper bound

$$\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\| + \beta\|p - p_h^k\| \leqslant 2\left(\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{Uza}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}\right).$$

Polynomial-degree-robust local efficiency

$$\left(\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{Uza}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}\right) \leqslant C(\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\| + \beta\|p - p_h^k\|).$$

where the generic constant C only depending on the shape-regularity parameter.

A posteriori error estimate

Let

- k be the Uzawa step
- i be the inner algebraic iteration

with the approximations $(\mathbf{u}_h^{k,i}, p_h^k)$ and the stopping criterion. Then

Guaranteed upper bound

$$\|\nabla(\mathbf{u}-\mathbf{u}_h^{k,i})\| + \beta\|p-p_h^k\| \leqslant 2\left(\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{Uza}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}\right).$$

Polynomial-degree-robust local efficiency

$$\left(\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{Uza}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}\right) \leqslant C(\|\nabla(\mathbf{u}-\mathbf{u}_h^{k,i})\| + \beta\|p-p_h^k\|).$$

where the generic constant C only depending on the shape-regularity parameter.

Numerical example

Model problem

$\Omega = (0,1)^2$ with homogeneous Dirichlet boundary conditions.

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

Exact solution

$$\mathbf{u} = \begin{pmatrix} 2x^2y(x-1)^2(y-1)^2 + x^2y^2(2y-2)(x-1)^2 \\ -2xy^2(x-1)^2(y-1)^2 - x^2y^2(2x-2)(y-1)^2 \end{pmatrix} \text{ and } p = x + y - 1.$$

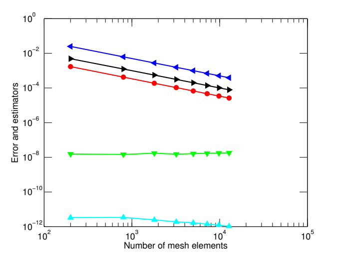
Configurations

- Taylor–Hood conforming finite element discretization of order $l = 2$, i.e.

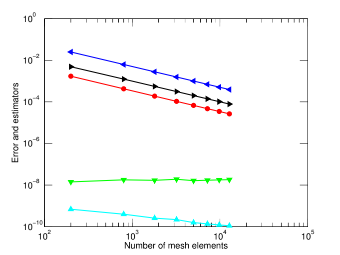
$$\mathbf{V}_h := \mathbf{V} \cap [\mathbb{P}_2(\mathcal{T}_h)]^2, Q_h := Q \cap [C^0(\overline{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_h)].$$

- $\gamma_{\text{rem}} := 1, \gamma_{\text{alg}} := 0.5, \gamma_{\text{Uza}} := 0.5.$

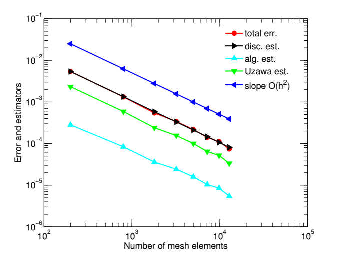
Error and estimators on uniformly refined meshes



(a) Exact



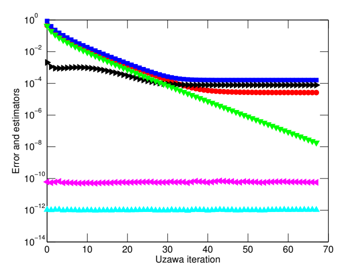
(b) Inexact



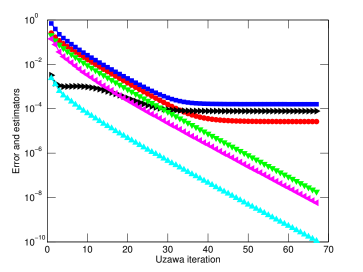
(c) Adaptive inexact

- eight levels of uniform mesh refinement
- distinguish different error components
- same order of convergence

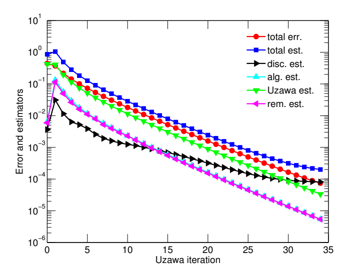
Error and estimators as a function of Uzawa iterations



(a) Exact

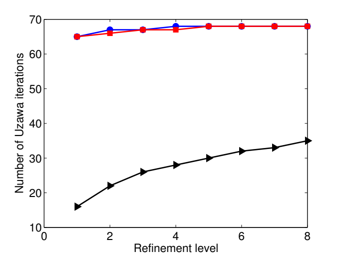


(b) Inexact

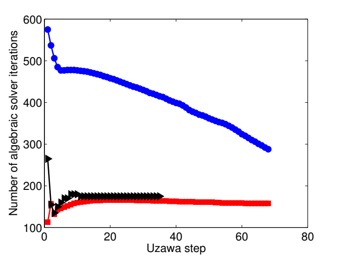


(c) Adaptive inexact

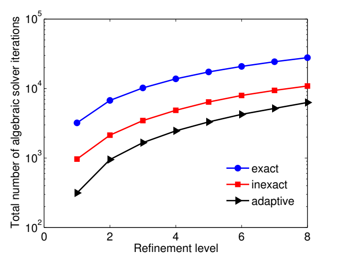
Number of linear solver and Uzawa iterations



(a) Number of Uzawa iterations per refinement level



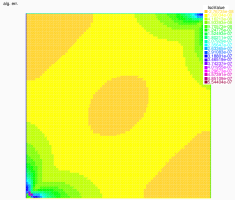
(b) Number of linear solver iterations per Uzawa step on the eighth-level mesh



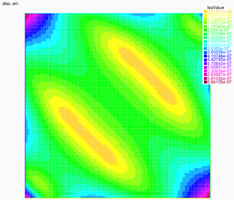
(c) Total number of linear solver iterations per refinement level

Total number of linear solver iterations at the finest mesh		
Exact	Inexact	Adaptive inexact
31499	11162	6295

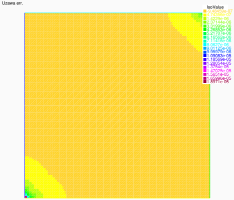
Spatial distributions of the different error components and of the corresponding estimates



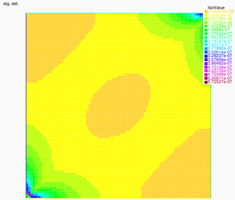
(a) Algebraic error



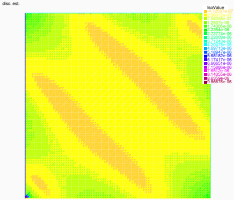
(b) Discretization error



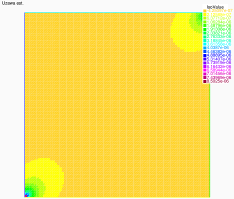
(c) Uzawa error



(d) Algebraic estimator



(e) Discretization estimator



(f) Uzawa estimator

Outline

Conclusion

- distinguishes each error components
- only a necessary number of algebraic solver iterations and Uzawa iterations
- guaranteed and robust a posteriori error estimates
- general framework for conforming finite element methods : unstabilized schemes and stabilized schemes
- implementation with Freefem++

Future directions

- convergence and optimality
- application to electromagnetic problems

Thank you for your attention !

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Conclusion and perspective

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