The mathematical finance of *Quants* and backward stochastic differential equations

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Financial derivatives

Derivative contract: agreement by which the seller will pay the buyer, in the future, a certain sum that depends on the evolution of the price of another financial asset.

Historically: started with *futures* or *forward contracts* at the Dojima Rice Exchange, Japan, 1730s.

Example: a farmer can sell 1kg of rice next year to a broker for an agreed price of 105 coins (today’s price might be 100 coins or not).
Buying option

Example: energy company, airline, meal manufacturer, international company... who is negatively affected if the market price of a certain asset (electricity, kerosene, wheat, foreign currency ...) rises above its current price. Say \( S_0 = 100 \).

A bank sells a *buying option*: allows the buyer to buy from the bank this *asset* at the agreed price \( K = 100 \), next month, whatever the market price is.

For the buyer, the net profit is \((S_T - K)^+\). Cash settlement \(\rightarrow\) the bank effectively pays the buyer \((S_T - K)^+\).
Buying option

**Figure:** Payment profile for an option to buy gas at 100 coins.
How much to charge?

Transfer of *market risk* from the company to the bank.

The bank will for sure pay a *positive* amount to the buyer. Exact amount depends on the market price in the future, but positive.

→ How much to charge for this?
A simple price model
A simple price model
A simple price model

100 (?)

50%

50%

120 (20)

80 (0)
A simple price model

100
(?)

20%

120
(20)

80%

80
(0)
A simple price model

100 (?)

80%

120 (20)

20%

80 (0)
A simple price model - solution by replication

100 (10; -40, 50)

120 (20; -40, 60)

80 (0; -40, 40)
Some comments

Cancellation of market risk: no bets, no “on average”, no risk.

Replication: option $\approx$ cash $+$ asset, in the good proportions. $\rightarrow$ fundamental price of the derivative.

1997: ”Nobel” prize in economics for Merton(, Black) and Scholes.
Some comments

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Does not depend on the probabilities!
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Does not depend on the probabilities!
... but it depends on *what is possible.*
How to figure out what to do?

\[ Y_0 = \text{initial capital: what you charge.} \]
\[ \pi_0 = \text{initial investment in the asset.} \]

\[ Y_1 = \text{trader’s wealth at payment time } T = 1. \]
Given by

\[ Y_1 = Y_0 + \pi_0 \frac{S_1 - S_0}{S_0}. \]

\[ S_0 = \text{initial price and} \]
\[ S_1 = \text{price at the end, can take values } S_1(+) \text{ and } S_1(-). \]
How to figure out what to do?

Want to adjust $Y_0$ and $\pi_0$ such that:

- $Y_1 = 20$ if $S_1 = 120$,
- $Y_1 = 0$ if $S_1 = 80$.

That means solving the 2 equations:

\[
\begin{cases}
Y_0 + \pi_0 \frac{120 - 100}{100} = 20 \\
Y_0 + \pi_0 \frac{80 - 100}{100} = 0
\end{cases}
\]

Linear equations!
2 cases: simplistic → 3-case model
3-case model
3-case model

100 (?,?,--=?,?)

120 (20)

100 (0)

80 (0)
Finer model
Finer model
Finer model

100 (?)

110

90

120 (20)

100 (0)

80 (0)
One way to solve this

4 decision parameters: $Y_0$, $\pi_0$, $\pi_1(\mathbb{+})$ and $\pi_1(\mathbb{-})$.

$Y_0$ = capital to start with.

$\pi_0$ = investment to hold in the asset at time 0.

$\pi_1$ = investment to hold in the asset at time 1.

Terminal wealth:

$$Y_2 = Y_0 + \pi_0 \frac{S_1 - S_0}{S_0} + \pi_1 \frac{S_2 - S_1}{S_1}. \quad \underbrace{Y_1}_Y$$

4 scenarios: $++$, $+-$, $-+$ and $--$.

$\leadsto$ 4 unknown, 4 equations and they are linear: solve(d)!

(Slightly tedious though.)
Another way to solve this
Another way to solve this
Another way to solve this
Another way to solve this
Finer and finer models

Discrete times $t_0, t_1 \ldots t_N (= T) \rightarrow$ continuous time $[0, T]$.

Price dynamics:
given by a (forward) stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t .$$

$B$ : a Brownian motion.
Finer and finer models

**Figure**: 4 realizations of the geometric Brownian motion model with $\mu = 0.05$ and $\sigma = 0.4$. 
How to figure out what to do?

Value of the trader's position: \((Y_0, Y_1, Y_2, \ldots) \rightarrow (Y_t)_{t \in [0, T]}\).

Investment in the asset: \((\pi_0, \pi_1, \pi_2, \ldots) \rightarrow (\pi_t)_{t \in [0, T]}\).

Dynamics of \(Y\):

\[
dY_t = \pi_t \frac{dS_t}{S_t} = [\pi_t \mu] dt + \pi_t \sigma dB_t.
\]

\(Y_0\) and \((\pi_t)_{t \in [0, T]}\) must be found such that \(Y_T\) equals the payment amount \(g(S_T)\) (with probability 1).

This is a \textit{backward stochastic differential equation} (BSDE).
General BSDEs

General form:

\[
\begin{aligned}
\begin{cases}
  dY_t &= -f(t, S_t, Y_t, Z_t)dt + Z_t dB_s \\
  Y_T &= g((S_t)_{t \in [0, T]}) .
\end{cases}
\end{aligned}
\]

Can also be written:

\[
Y_s = E\left( Y_u + \int_s^u f(Y_t) dt \mid F_t \right) ,
\]

for \( s < u \), and with \( Y_T = g((S_t)_{t \in [0, T]}) \).
Big picture in continuous time

Model (≈scenarios):
- Tree/finite number of paths
  ↑
  (forward) stochastic differential equation.

Solving the pricing and hedging problem:
- system of equations // dynamic programming principle
  ↑
  backward stochastic differential equation.
Nonlinear equations $\leadsto$ numerical methods.

Finance:
- perfect market: linear equations, closed-form solutions (Monte-Carlo evaluation);
- market imperfections $\leadsto$ nonlinearities.

General fact:
solving for $Y_t \leftrightarrow$ solving a parabolic PDE, when payment depends only on final price.
The principle for BSDE numerics

1) Time-discretization

\[ [0, T] \rightarrow \text{discrete times } 0 = t_0, t_1, \ldots, t_N = T. \]

Time step \( \Delta t = T/N. \)

\[ Y_{t_i} = E \left( Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_t) \, dt \mid \mathcal{F}_t \right) \rightarrow \]

\[ Y_{i}^N = E \left( Y_{i+1}^N + f(Y_{i+1}^N) \Delta t \mid \mathcal{F}_{t_i} \right). \]

2) Approximate the conditional expectations.
BSDE numerics

<what I did = "some stuff I tried actually worked", proud graduation: granted >

Time-discretization when the nonlinearity $f$ is superlinear: explicit Euler scheme (as above) is bad!

When the terminal condition $g$ is not bounded, the scheme can explode.
Why explosion is possible

**Figure:** Explicit Euler scheme for an ODE, for various initial conditions and step sizes, with nonlinear driver \( f(y) = -y^3 \) (similar nonlinearity as FitzHugh–Nagumo or Allen–Cahn PDEs).
The idea behind BSDE numerics

When the terminal condition $g$ is not bounded, the explicit scheme can explode.

Remedies: implicit scheme
... or truncate $g$ into a function $g^N$. (Truncation so that the effect vanishes as $N$ goes to $+\infty$.)
... or truncate the driver into an at-most-linear one.

[Joint works with Lukasz Szpruch and Gonçalo dos Reis, University of Edinburgh.]

</what I did>
Conclusions

Pricing and hedging: a central area in mathematical finance.

Starts with simple models and solutions \( \rightsquigarrow \) requires more advanced techniques as the models are refined.
(Rk: those models are not meant to be predictive. Rather: the decision problem should be able to be solved on them.)

BSDEs = how the replication argument writes when the model is given by a forward SDE.
More generally: fundamental connections with (path-dependent) PDEs.
That’s all folks …

THANK YOU

FOR YOUR

ATTENTION