The mathematical finance of *Quants* and backward stochastic differential equations

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INRIA-PRO Junior Seminar
17th February 2015

Financial derivatives

Derivative contract : agreement by which the seller will pay the buyer, in the future, a certain sum that depends on the evolution of the price of another financial asset.

Historically : started with *futures* or *forward contracts* at the Dojima Rice Exchange, Japan, 1730s.

Example : a farmer can sell 1kg of rice next year to a broker for an agreed price of 105 coins (today's price might be 100 coins or not).

Buying option

Example : energy compagy, airline, meal manufacturer, international company... who is negatively affected if the market price of a certain asset (electricity, kerosene, wheat, foreign currency ...) rises above its current price.
Say $S_0 = 100$.

A bank sells a *buying option* : allows the buyer to buy from the bank this *asset* at the agreed price $K = 100$, next month, whatever the market price is.

For the buyer, the net profit is $(S_T - K)^+$.
Cash settlement $\longrightarrow$ the bank effectively pays the buyer $(S_T - K)^+$.

Buying option

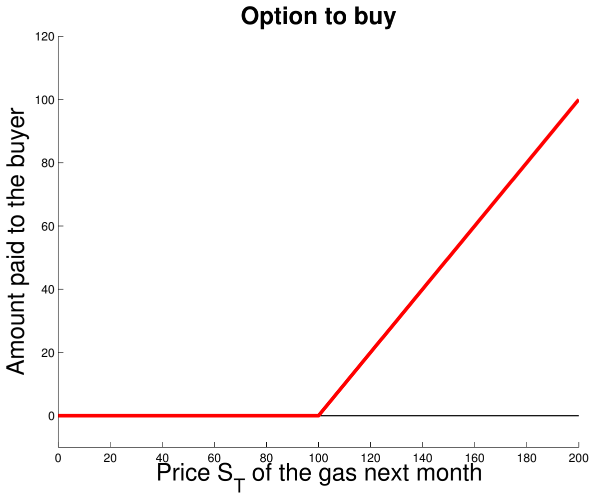


FIGURE: Payment profile for an option to buy gas at 100 coins.

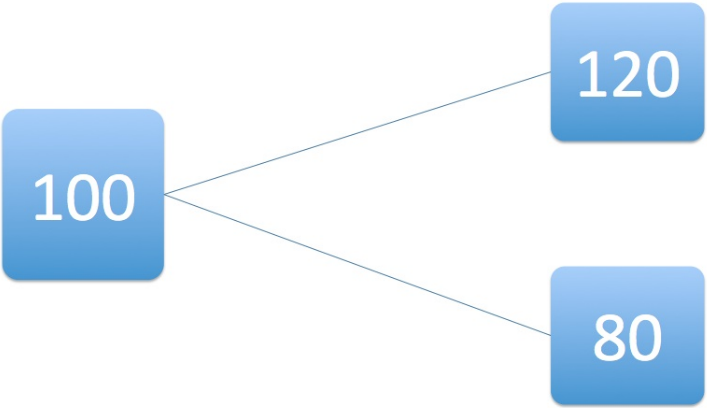
How much to charge?

Transfer of *market risk* from the company to the bank.

The bank will for sure pay a *positive* amount to the buyer. Exact amount depends on the market price in the future, but positive.

$\rightarrow$ How much to charge for this?

A simple price model



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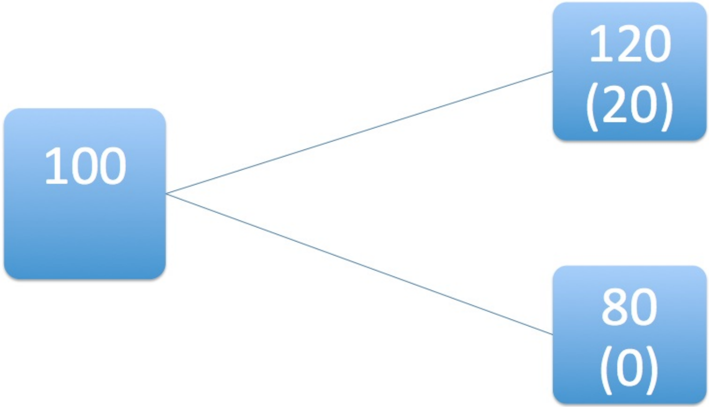

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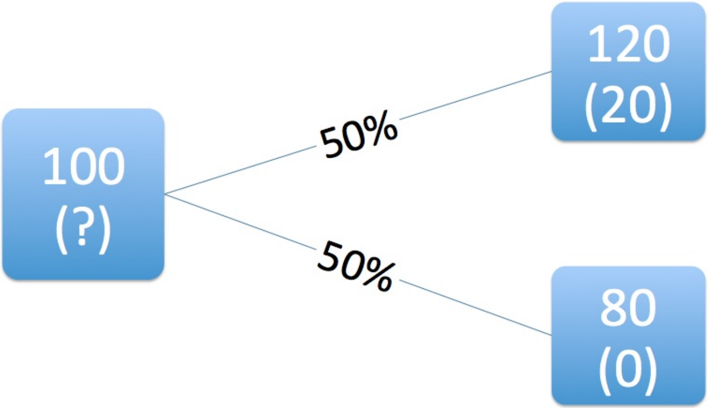

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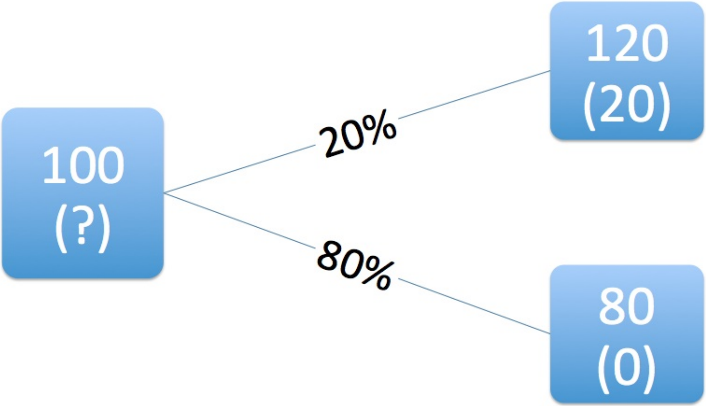

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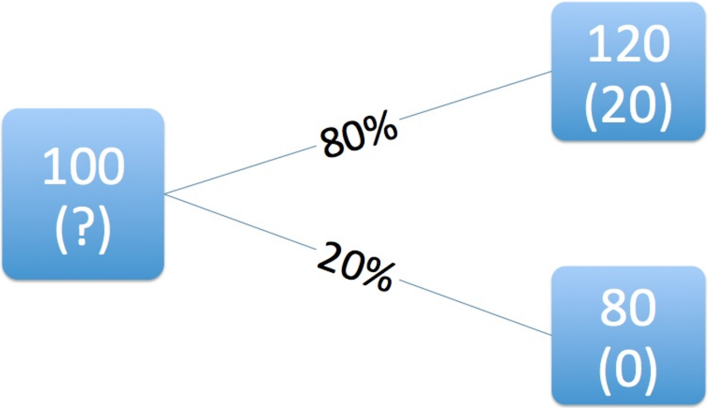
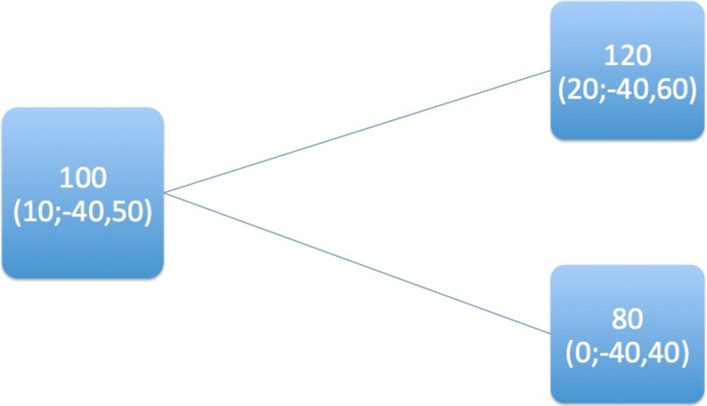


A simple price model - solution by replication



Some comments

Cancellation of market risk : no bets, no “on average”, no risk.

Replication : option $\approx$ cash $+$ asset, in the good proportions.
$\rightsquigarrow$ fundamental price of the derivative.

1973 : opening of the Chicago Board Options Exchange and seminal paper of Black and Scholes.
1997 : "Nobel" prize in economics for Merton(, Black) and Scholes.

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Does not depend on the probabilities !
... but it depends on *what is possible*.

How to figure out what to do ?

Y_0 = initial capital : what you charge.
π_0 = initial investment in the asset.

Y_1 = trader's wealth at payment time $T = 1$.
Given by

$$Y_1 = Y_0 + \pi_0 \frac{S_1 - S_0}{S_0} .$$

S_0 = initial price and
S_1 = price at the end, can take values $S_1(+)$ and $S_1(-)$.

How to figure out what to do?

Want to adjust Y_0 and π_0 such that :

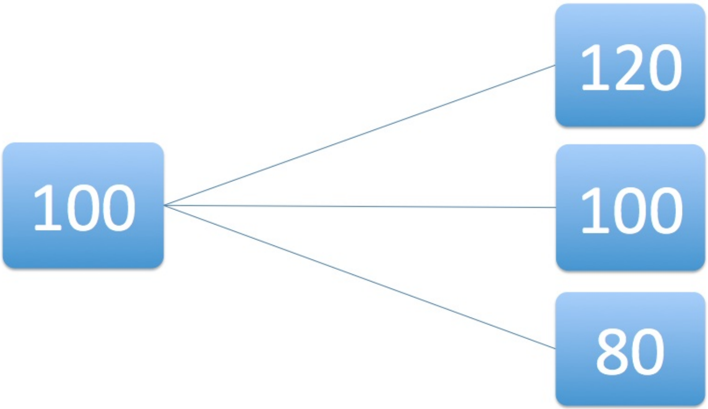
- $Y_1 = 20$ if $S_1 = 120$,
- $Y_1 = 0$ if $S_1 = 80$.

That means solving the 2 equations

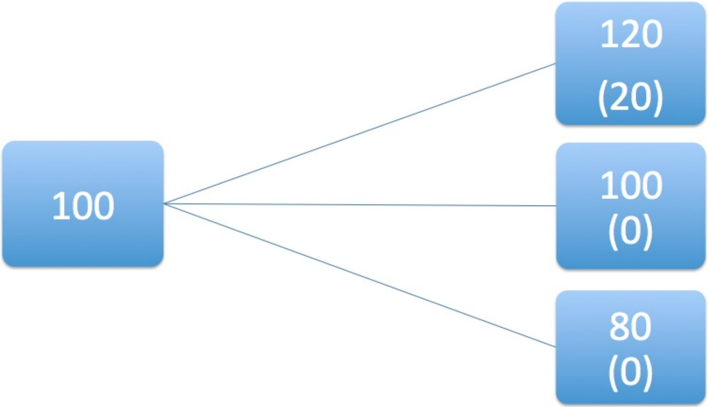
$$\begin{cases} Y_0 + \pi_0 \dfrac{120 - 100}{100} = 20 \\ \\ Y_0 + \pi_0 \dfrac{80 - 100}{100} = 0 \end{cases}$$

Linear equations!

2 cases : simplistic → 3-case model



3-case model



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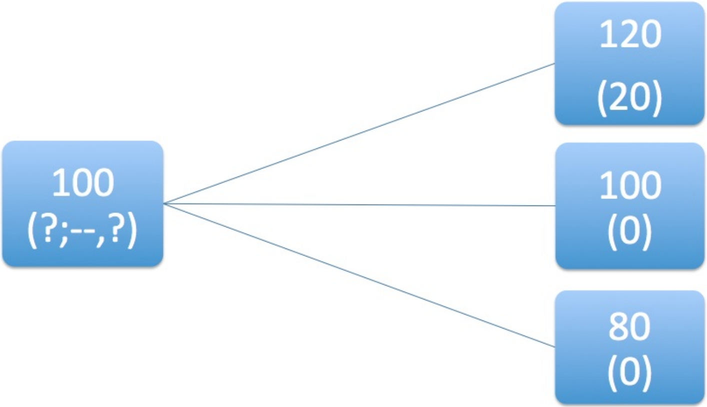
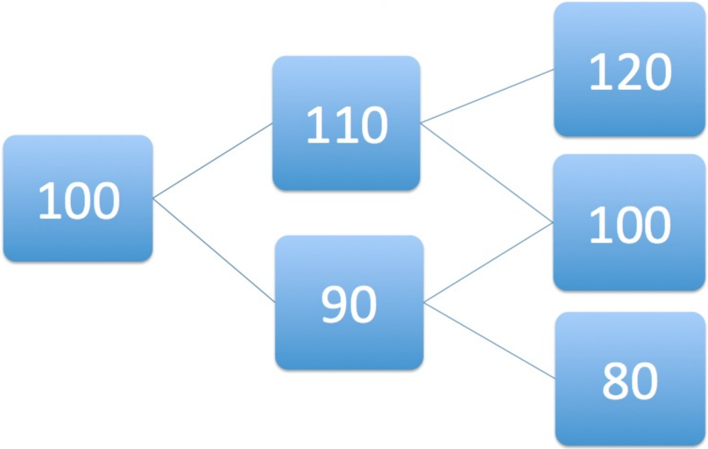


Finer model



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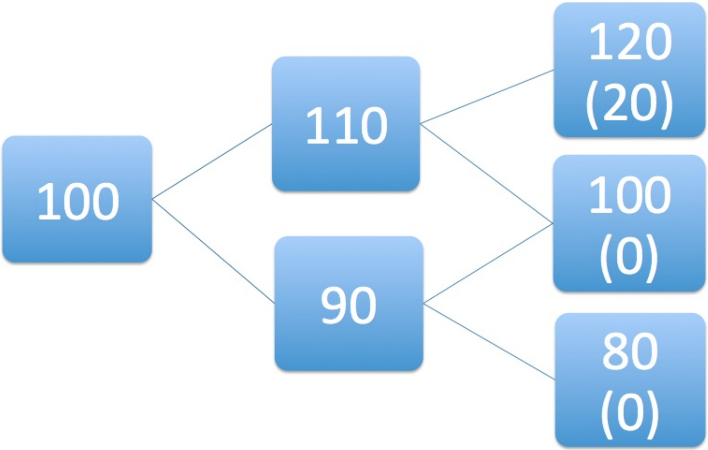

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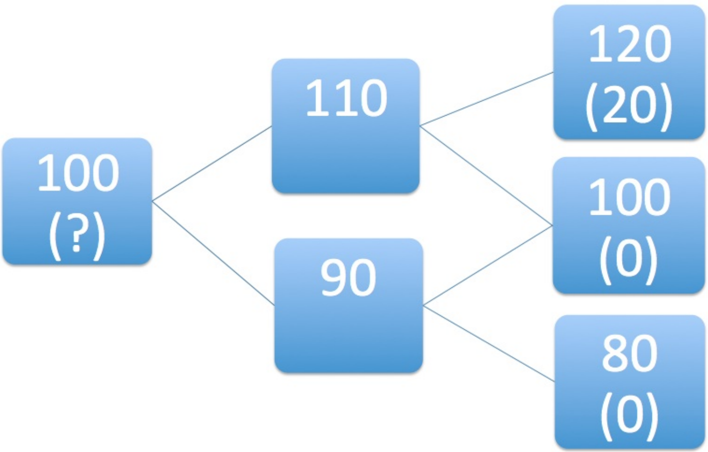


One way to solve this

4 decision parameters : Y_0, π_0, $\pi_1(+)$ and $\pi_1(-)$.
Y_0 = capital to start with.
π_0 = investment to hold in the asset at time 0.
π_1 = investment to hold in the asset at time 1.

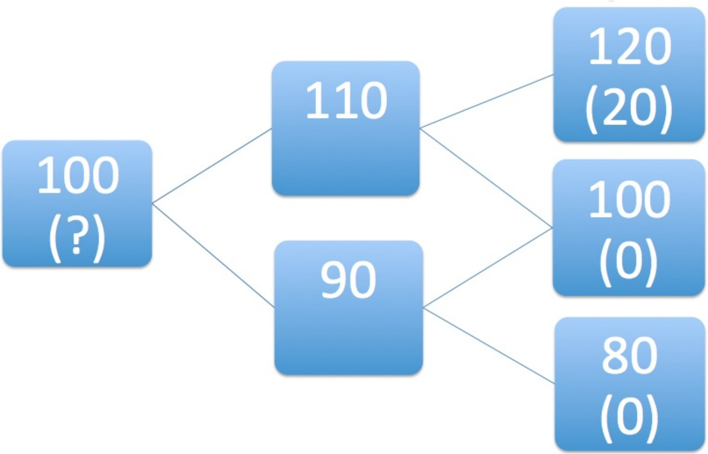
Terminal wealth :

$$Y_2 = \underbrace{Y_0 + \pi_0 \frac{S_1 - S_0}{S_0}}_{Y_1} + \pi_1 \frac{S_2 - S_1}{S_1} .$$

4 scenarios : $++$, $+-$, $-+$ and $--$.

$\rightsquigarrow$ 4 unknown, 4 equations and they are linear : solve(d) !
(Slightly tedious though.)

Another way to solve this



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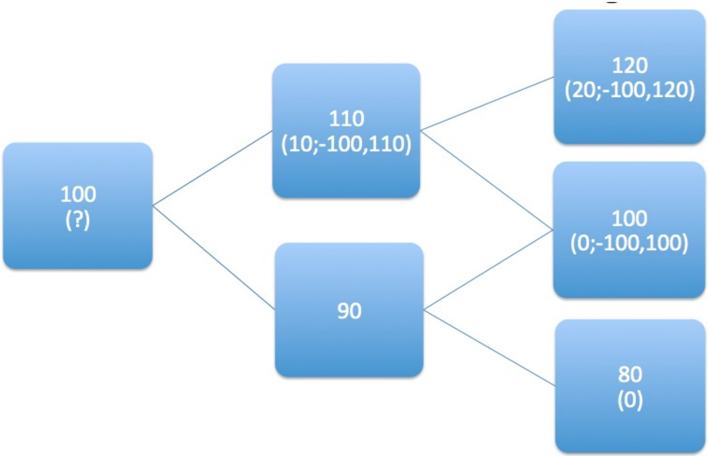

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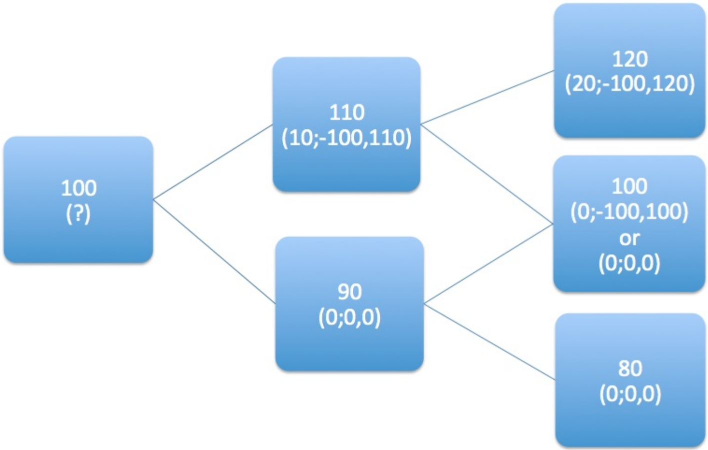

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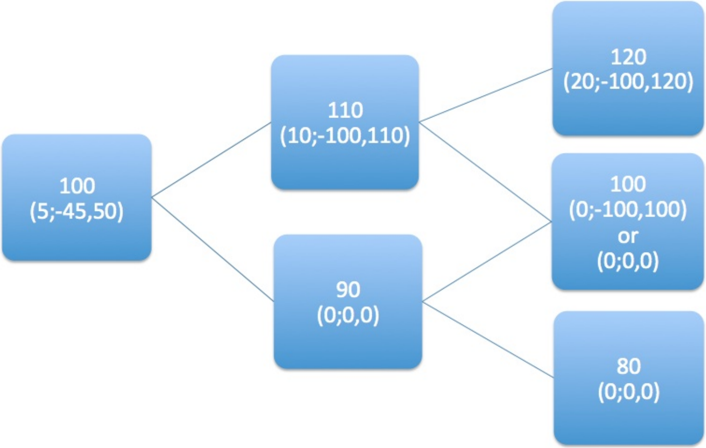

Finer and finer models

Discrete times $t_0, t_1 \dots t_N (= T) \to$ continuous time $[0, T]$.

Price dynamics :
given by a (forward) stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t \ .$$

B : a Brownian motion.

Finer and finer models

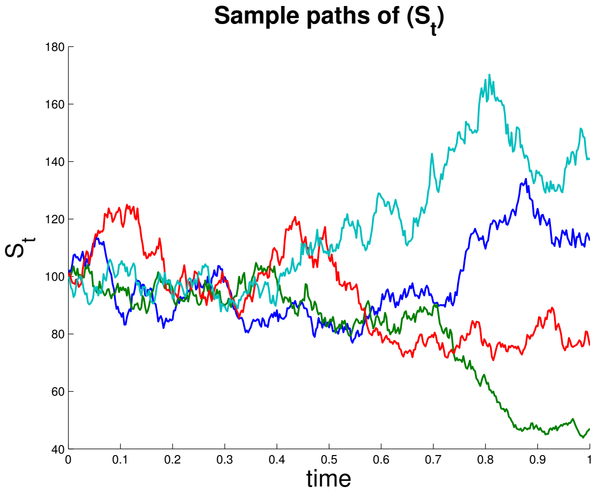

FIGURE: 4 realizations of the geometric Brownian motion model with $\mu = 0.05$ and $\sigma = 0.4$.

How to figure out what to do ?

Value of the trader's position : $(Y_0, Y_1, Y_2, \ldots) \to (Y_t)_{t\in[0,T]}$.

Investment in the asset : $(\pi_0, \pi_1, \pi_2, \ldots) \to (\pi_t)_{t\in[0,T]}$.

Dynamics of Y :

$$\begin{aligned} dY_t &= \pi_t \frac{dS_t}{S_t} \\ &= \left[\pi_t \mu\right] dt + \pi_t \sigma dB_t \ . \end{aligned}$$

Y_0 and $(\pi_t)_{t\in[0,T]}$ must be found such that Y_T equals the payment amount $g(S_T)$ (with probability 1).

This is a *backward stochastic differential equation* (BSDE).

General BSDEs

General form :

$$\begin{cases} dY_t = -f(t, S_t, Y_t, Z_t)dt + Z_t dB_s \\ Y_T = g\left((S_t)_{t\in[0,T]}\right) \end{cases} .$$

Can also be written :

$$Y_s = E\left(Y_u + \int_s^u f(Y_t)dt \middle| \mathcal{F}_t \right) ,$$

for $s < u$, and with $Y_T = g\left((S_t)_{t\in[0,T]}\right)$.

Big picture in continuous time

Model (=scenarios) :

- Tree/finite number of paths

  $\updownarrow$
- (forward) stochastic differential equation.

Solving the pricing and hedging problem :

- system of equations // dynamic programming principle

  $\updownarrow$
- backward stochastic differential equation.

Nonlinear equations $\rightsquigarrow$ numerical methods.

Finance :

- perfect market : linear equations, closed-form solutions (Monte-Carlo evaluation) ;
- market imperfections $\rightsquigarrow$ nonlinearities.

General fact :
solving for $Y_t \leftrightarrow$ solving a parabolic PDE,
when payment depends only on final price.

The principle for BSDE numerics

1)Time-discretization
$[0, T] \longrightarrow$ discrete times $0 = t_0, t_1 \ \ldots \ t_N = T$.
Time step $\Delta t = T/N$.
$Y_{t_i} = E\left(Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_t)dt \middle| \mathcal{F}_t \right) \rightarrow$

$$Y_i^N = E\left(Y_{i+1}^N + f(Y_{i+1}^N)\Delta t \middle| \mathcal{F}_{t_i} \right) .$$

2)Approximate the conditional expectations.

BSDE numerics

`<what I did ="some stuff I tried actually worked", #proud #graduation:granted>`

Time-discretization when the nonlinearity f is superlinear : explicit Euler scheme (as above) is bad !

When the terminal condition g is not bounded, the scheme can explode.

Why explosion is possible

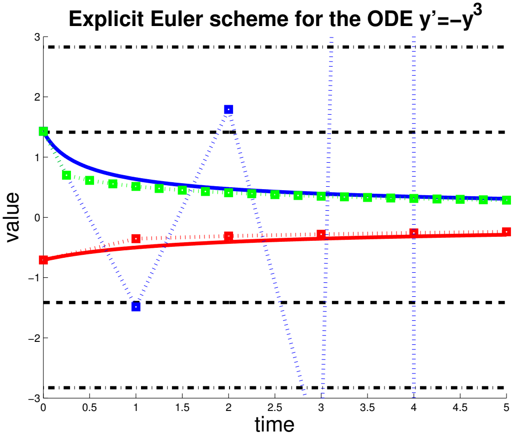

FIGURE: Explicit Euler scheme for an ODE, for various initial conditions and step sizes, with nonlinear driver $f(y) = -y^3$ (similar nonlinearity as FitzHugh–Nagumo or Allen–Cahn PDEs).

The idea behind BSDE numerics

When the terminal condition g is not bounded, the explicit scheme can explode.

Remedies : implicit scheme
... or truncate g into a function g^N. (Truncation so that the effect vanishes as N goes to $+\infty$.)
... or truncate the driver into an at-most-linear one.

[Joint works with Lukasz Szpruch and Gonçalo dos Reis, University of Edinburgh.]

`</what I did>`

Conclusions

Pricing and hedging : a central area in mathematical finance.

Starts with simple models and solutions $\rightsquigarrow$ requires more advanced techniques as the models are refined.
(Rk : those models are not meant to be predictive. Rather : the decision problem should be able to be solved on them.)

BSDEs $=$ how the replication argument writes when the model is given by a forward SDE.
More generally : fundamental connections with (path-dependent) PDEs.

That's all folks ...

THANK YOU

FOR YOUR

ATTENTION