Multi-Marginal Optimal Transportation: Numerics and Applications

Luca Nenna

I.N.R.I.A. (MoKaPlan)

Junior Seminar, 17 March 2015

Overview

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MoKaPlan

Ceremade (**Paris-Dauphine**):

– **Guillaume Carlier (PR)**

– **Gabriel Peyré (DR CNRS)**

– Francois-Xavier Vialard (MCF)

– Bernhard Schmitzer (Post Doc)

– Jingwei Liang (PhD)

– Lénaic Chizat (PhD)

– Quentin Denoyelle (PhD)

– Maxime Laborde (PhD)

– Roméo Hatchi (PhD)

– Dario Prandi (Post Doc)

INRIA :

– **Jean-David Benamou (DR INRIA)**

– Vincent Duval (Ingenieur Corps des Mines, Ďetache)

– Simon Legrand (Research engineer ADT Mokabajour)

– Luca Nenna (PhD)

Research Collaborations :

– Yves Achdou (PR U. Paris Diderot)

– Yann Brenier (DR CNRS, CMLS X)

– Quentin Merigot (CR CNRS, Ceremade)

– Jean-Marie Mirebeau (CR CNRS, Ceremade)

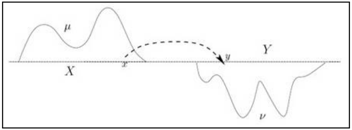
– Roman Andreev (Post Doc, U. Paris Diderot)

Applications

- Economy
- Finance
- Astrophysics
- Image Processing
- Machine Learning
- Optics (the reflector problem)
- Meteorology and Fluid models (semi-geostrophic equations)
- Density Functional Theory
- and so on $\cdots$

The Monge Problem

Once upon a time (namely 1781), Gaspard Monge...



- Two distribution μ and ν on $\mathbb{R}^d$ (for simplicity $d = 1$) with same total mass($\int \mu(x)dx = \int \nu(y)dy$)
- Find the transport map $T(x)$ such that:
    - T preserves mass ($\nu(T(x))T(x)' dx = \mu(x)dx$)
    - T minimizes the cost $\int c(x, T(x))\mu(x)dx$
- The standard cost function $c(x, y) = \frac{|x-y|^p}{p}$
    - $p = 1$ $c(x, y) = \frac{|x-y|}{2}$ (the problem introduced by Monge)
    - $p = 2 \Rightarrow$ Brenier's Theorem
- NO MASS SPLITTING

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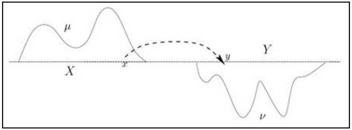

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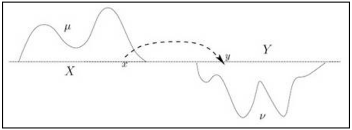

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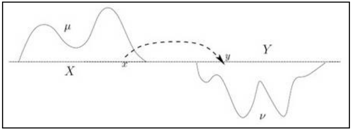

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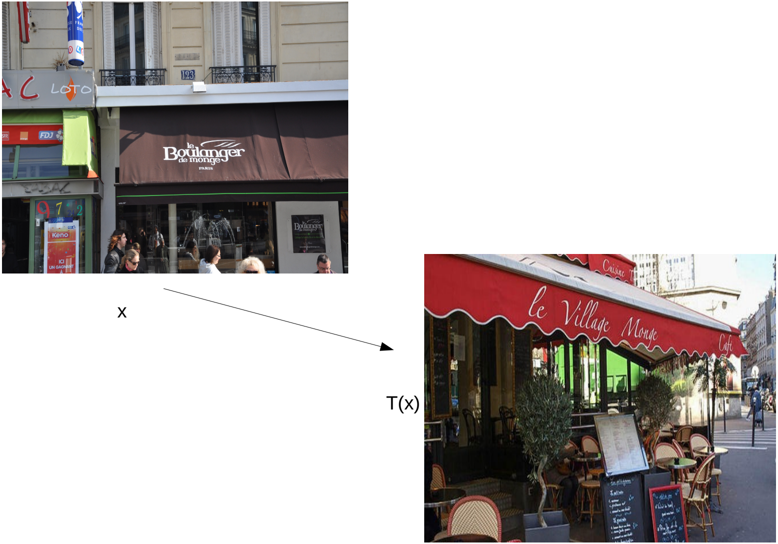
x
T(x)
le Boulanger de monge
PARIS
LOTO
Keno
le Village Monge
Café

The Kantorovich (relaxed) problem

In 1942, Kantorovich (Nobel prize in 1975) proposed a relaxed formulation of the Monge problem which allows mass splitting. Find a joint distribution $\gamma(x,y)$ such that

- $\gamma(x,y)$ has marginals equals to μ and ν:
  - $\int \gamma(x,y)dy = \mu(x)$
  - $\int \gamma(x,y)dx = \nu(y)$
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The Brenier's theorem

If $T(x)$ is a transport map then it induces a transport plan $\gamma_T(x,y) = \mu(x)\delta(y - T(x))$.

Kant pb $\Leftrightarrow$ Monge pb ?

If the optimal plan has the form $\gamma_T^\star$ (which means that no splitting of mass occurs and $\gamma_T^\star$ is concentrated) then T is an optimal transport map.

Theorem [Brenier '91] for $p = 2$

There exists a unique map of the form $T = \nabla u$ with u convex that transports μ to ν, this map is also the optimal transport between μ to ν for the quadratic cost ($p = 2$)

Thus, for $p = 2$ we have $\gamma_T^\star(x,y) = \mu(x)\delta(y - \nabla u(x))$

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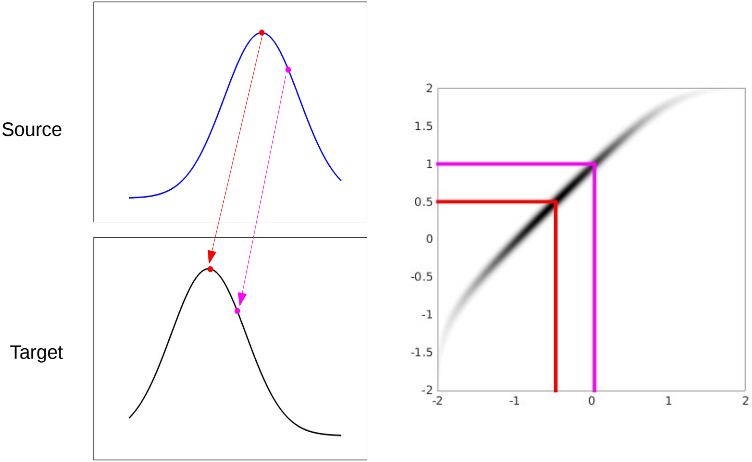
Source, Target and Transport Plan



Transport Map between ellipses and McCann's Interpolant

The Multi-Marginals (Monge) Problem [Gangbo-Święch,'98]

- N distribution μ_i ($i = 1, \cdots, N$) on $\mathbb{R}^d$ (for simplicity $d = 1$)
- Find the transport maps $T_i(x)$ such that:
  - T_i preserve mass $(\mu_i(T_i(x))T_i(x)' dx = \mu_1(x)dx$ and $T_1(x) = x)$
  - T_i minimize the cost

$$\int c(T_1(x), T_2(x), \cdots, T_N(x))\mu_1(x)dx \tag{1}$$

- The standard cost function

$$c(T_1(x), T_2(x), \cdots, T_N(x)) = \int \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{|T_i(x) - T_j(x)|^2}{2} \mu_1(x)dx. \tag{2}$$

Example, $N = 3$

$$\text{(MP)}\ c(x, T_2(x), T_3(x)) = \frac{|x - T_2(x)|^2}{2} + \frac{|T_2(x) - T_3(x)|^2}{2} + \frac{|x - T_3(x)|^2}{2}.$$

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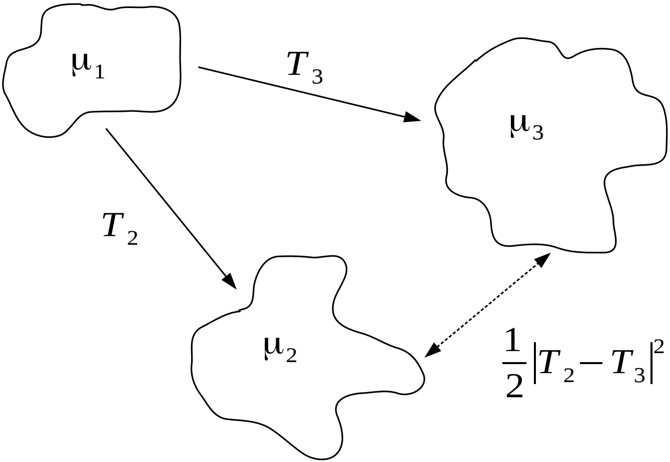
μ_1
T_3
μ_3
T_2
μ_2
$\frac{1}{2}|T_2 - T_3|^2$

The Multi-Marginals (Kantorovich) Problem [Gangbo-Święch,'98]

Find a joint distribution $\gamma(x_1, \cdots, x_N)$ such that

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$$\int \gamma(x_1, \cdots, x_i, \cdots, x_N) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_N = \mu_i(x_i) \tag{3}$$

- $\gamma(x_1, \cdots, x_N)$ minimizes the cost $\int c(x_1, \cdots, x_N)\gamma(x_1, \cdots, x_N) dx_1 \cdots dx_N$.

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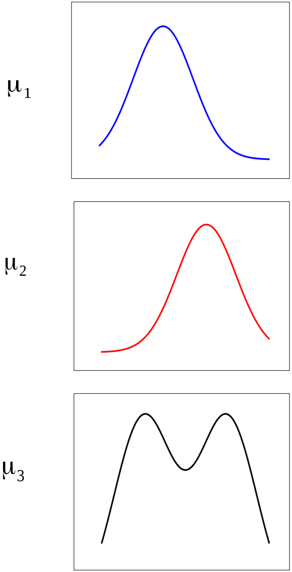
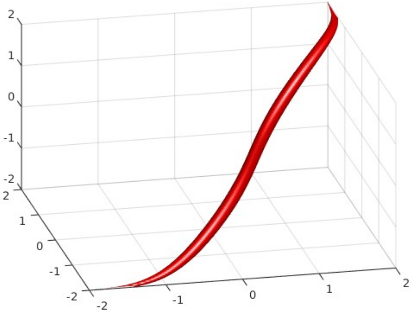
Numerical Results-1D-$N = 3$





Numerical Results-1D-Projection of $\gamma^\star$-$N=3$

$$\gamma_{\mu_i \to \mu_j} = \int \gamma^\star(x_1, \cdots, x_N) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_{j-1} dx_{j+1} \cdots dx_N$$

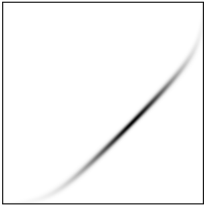


Figure : $\gamma_{\mu_1 \to \mu_2}$

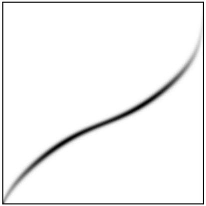


Figure : $\gamma_{\mu_1 \to \mu_3}$

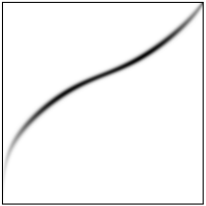


Figure : $\gamma_{\mu_2 \to \mu_3}$

Transport Maps

Transport Maps

The long (and hard) way from the DFT to Optimal Transportation

The Density Functional Theory describes the behaviour of an atom (or a molecule). After (a lot of) computations [Buttazzo,De Pascale,Gori-Giorgi '12;Cotar, Friesecke, Klüppelberg '13], we obtain the following problem: Find $\gamma(x,y)$ such that

- $\gamma(x,y)$ has marginals equals to ρ and ρ (electrons are indistinguishable so $\mu = \nu = \rho$)
- $\gamma(x,y)$ minimizes the cost $\int c(x,y)\gamma(x,y)dxdy$.

The marginals ρ are the electrons (in this case we have 2 electrons) and the cost function is the electron-electron repulsion (namely the Coulomb cost)

$$c(x,y) = \frac{1}{|x-y|}.$$

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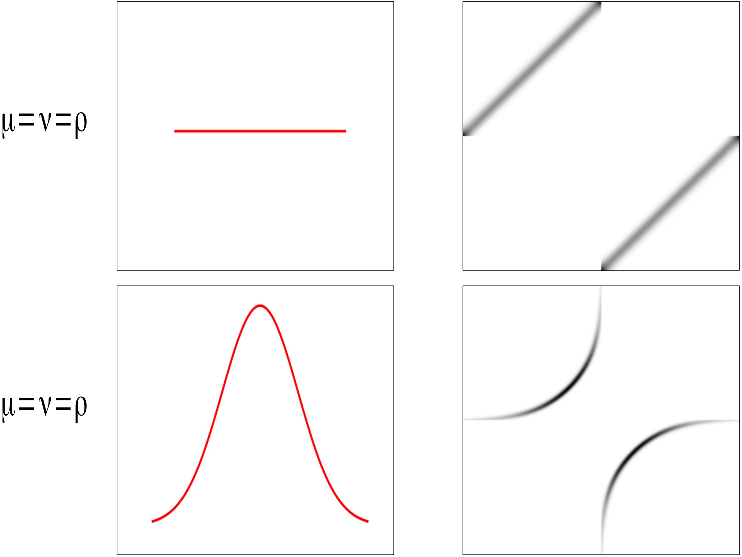
Numerical results for $N = 2$ in 1D



$$c(x,y,z) = \frac{1}{|x-y|} + \frac{1}{|y-z|} + \frac{1}{|z-x|}$$

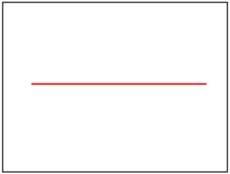


Figure : $\rho = \chi_{[0,1]}(x)$

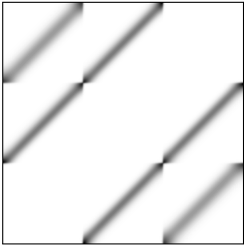


Figure : $\gamma_{\rho_1 \to \rho_3}$

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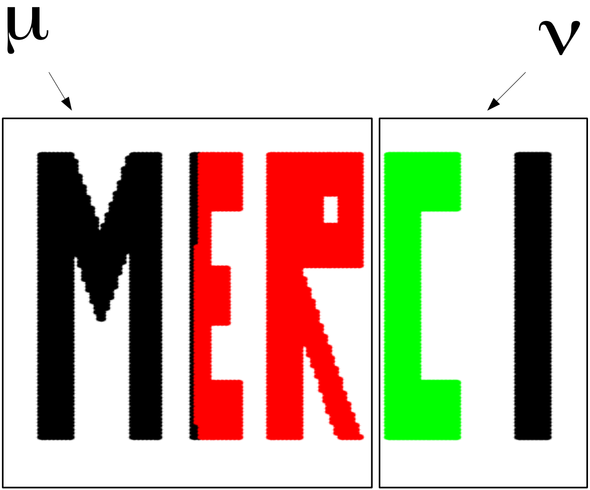
- **JD Benamou, G. Carlier, M. Cuturi, L. Nenna and G. Peyré, Iterative Bregman Projections for Regularized Transportations Problems, to appear in SIAM J. Sci. Comp. , 2015.**


μ
ν
MERCI