Sensitivity analysis for nonlinear hyperbolic equations

21/6/2016 - Junior Seminar

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Where am I from?

Advisors: Régis Duvigneau (INRIA Sophia Antipolis),
Christophe Chalons (LMV UVSQ).

University: Université Paris Saclay -
Université de Versailles Saint-Quentin-en-Yvelines.

Lab: Laboratoire de Mathématiques de Versailles

- Analysis and PDEs
- Probability and Statistics
- Algebra
- Cryptography

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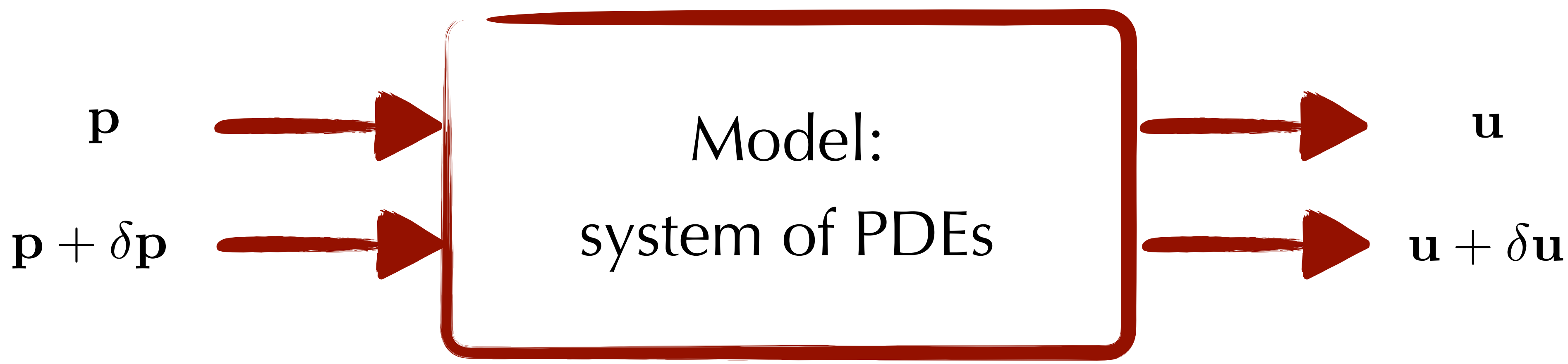
Sensitivity Analysis: the study of how **variations in the output** of a model can be attributed to different sources of **uncertainty in the model input**.



Therefore, we want to study the derivative of $\boldsymbol{u}$ with respect to $\boldsymbol{p}$:

$$\mathbf{u_p} = \frac{\partial \mathbf{u}}{\partial \mathbf{p}}$$

Applications

- Propagation of uncertainty or error: sensitivity can be used to study how uncertainty in a measurement of a parameter can affect the solution.

- Estimate of close solutions: using a first order Taylor expansion it is possible to estimate solution for different parameters values.

$$\mathbf{u}(\mathbf{p} + \delta\mathbf{p}) \simeq \mathbf{u}(\mathbf{p}) + \delta\mathbf{p}\mathbf{u}_{\mathbf{p}}(\mathbf{p})$$

- Optimisation: sensitivity can be useful to solve problems such as

$$\min_{\mathbf{p}\in\mathcal{P}} J(\mathbf{u}(\mathbf{p}))$$

for which it is necessary to compute the gradient of the cost functional:

$$\nabla_{\mathbf{p}} J = \frac{\partial J}{\partial \mathbf{u}} \mathbf{u}_{\mathbf{p}}$$

State equations

We will consider **hyperbolic equations**:

$$\begin{cases} \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 & x \in \mathbb{R},\ t > 0 \\ \mathbf{u}(x,0) = \mathbf{g}(x;\mathbf{p}) & x \in \mathbb{R}. \end{cases}$$

Hyperbolic equations are also known as **conservation laws**:

- ***u*** is the **conserved variable**
- *f(**u**)* is the **flux function**
- ***g**(x;**p**)* is the **initial condition**

Sensitivity equations

Under hypothesis of **regularity**, on can differentiate the state equations with respect to the parameter:

$$\begin{cases} \partial_{\mathbf{p}}(\partial_t \mathbf{u}) + \partial_{\mathbf{p}}(\partial_x f(\mathbf{u})) = 0 & x \in \mathbb{R},\ t > 0 \\ \partial_{\mathbf{p}} \mathbf{u}(x,0) = \partial_{\mathbf{p}} \mathbf{g}(x;\mathbf{p}) & x \in \mathbb{R}. \end{cases}$$

Exchanging the derivatives in space and time with the ones with respect to the parameter one has:

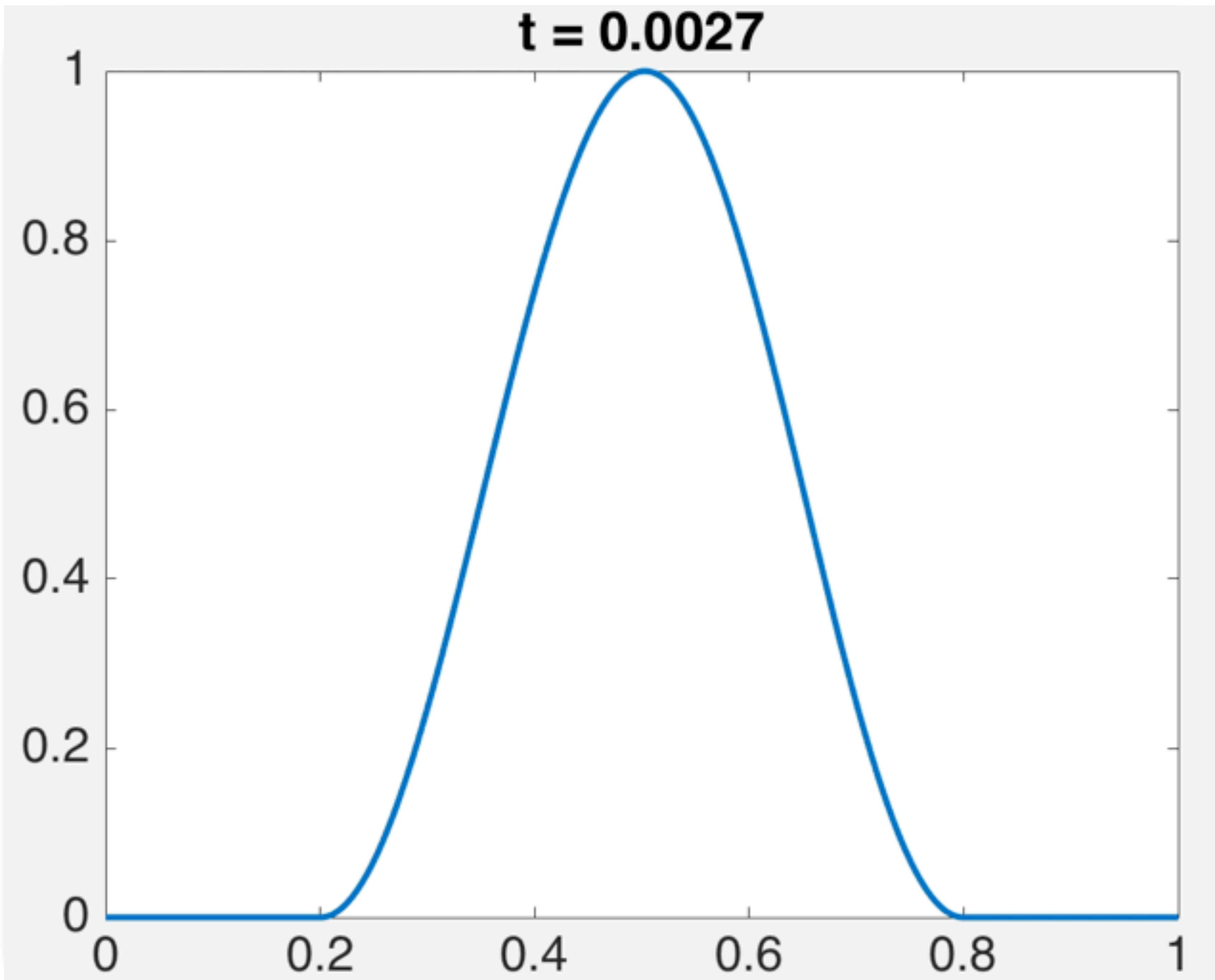
$$\begin{cases} \partial_t \mathbf{u}_{\mathbf{p}} + \partial_x \left(f'(\mathbf{u}) \mathbf{u}_{\mathbf{p}} \right) = 0 & x \in \mathbb{R},\ t > 0 \\ \mathbf{u}_{\mathbf{p}}(x,0) = \mathbf{g}_{\mathbf{p}}(x;\mathbf{p}) & x \in \mathbb{R}. \end{cases}$$

Example: the Burger's equation

$$f(u) = \frac{u^2}{2}$$

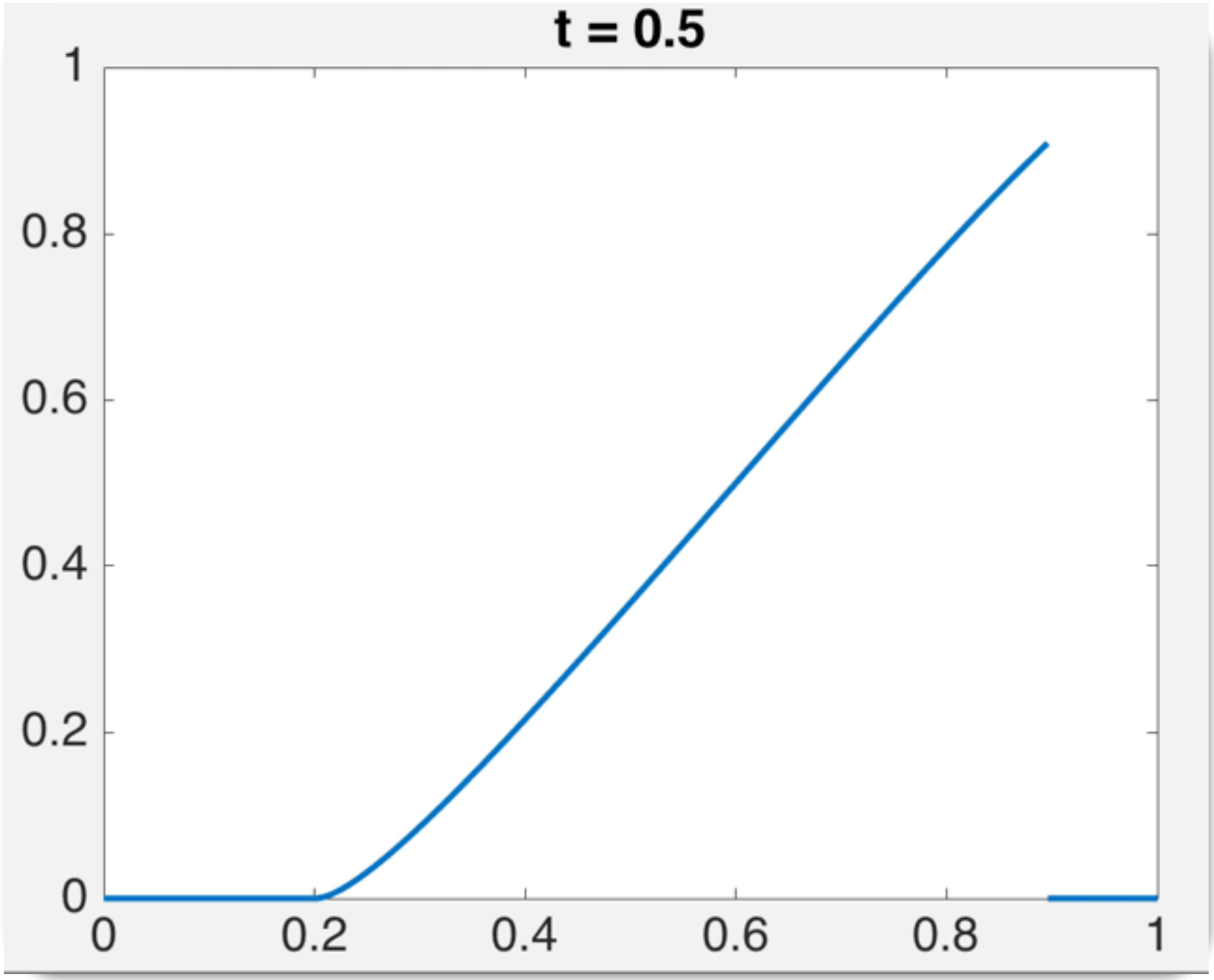
p

$$g(x; A, x_c, L) = \begin{cases} A\sin^2\left(\frac{\pi}{L}(x - x_c) + \frac{\pi}{2}\right) & x \in (x_c - \frac{L}{2}, x_c + \frac{L}{2}) \\ 0 & \text{otherwise.} \end{cases}$$



Example: the Burger's equation

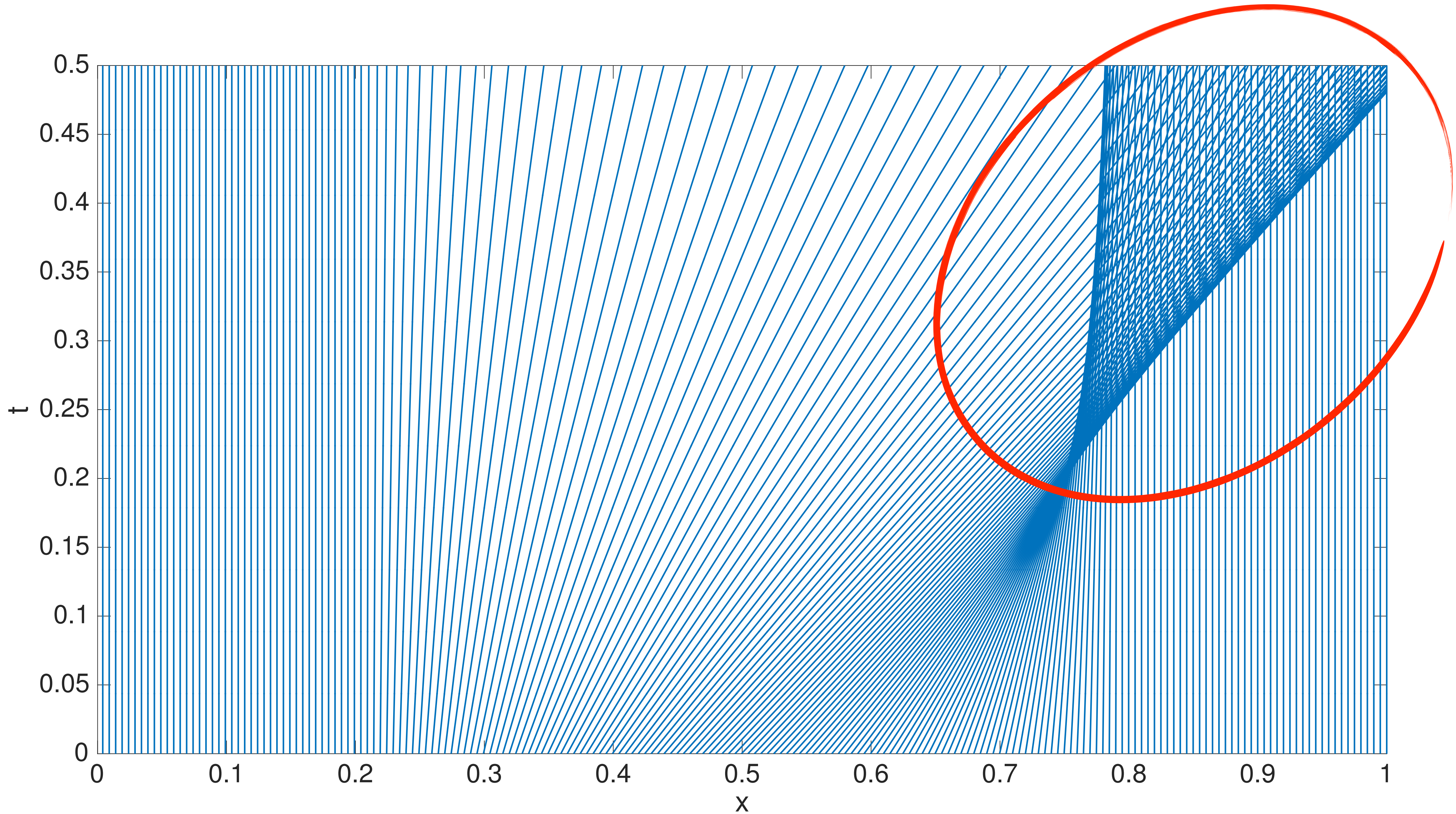
$$f(u) = \frac{u^2}{2} \qquad g(x; \underbrace{A, x_c, L}_{p}) = \begin{cases} A\sin^2(\frac{\pi}{L}(x - x_c) + \frac{\pi}{2}) & x \in (x_c - \frac{L}{2}, x_c + \frac{L}{2}) \\ 0 & \text{otherwise.} \end{cases}$$



Solution of the state equations

The Burger's equation can be rewritten as: $\partial_t u + u \, \partial_x u = 0$

speed at which the initial condition is transported



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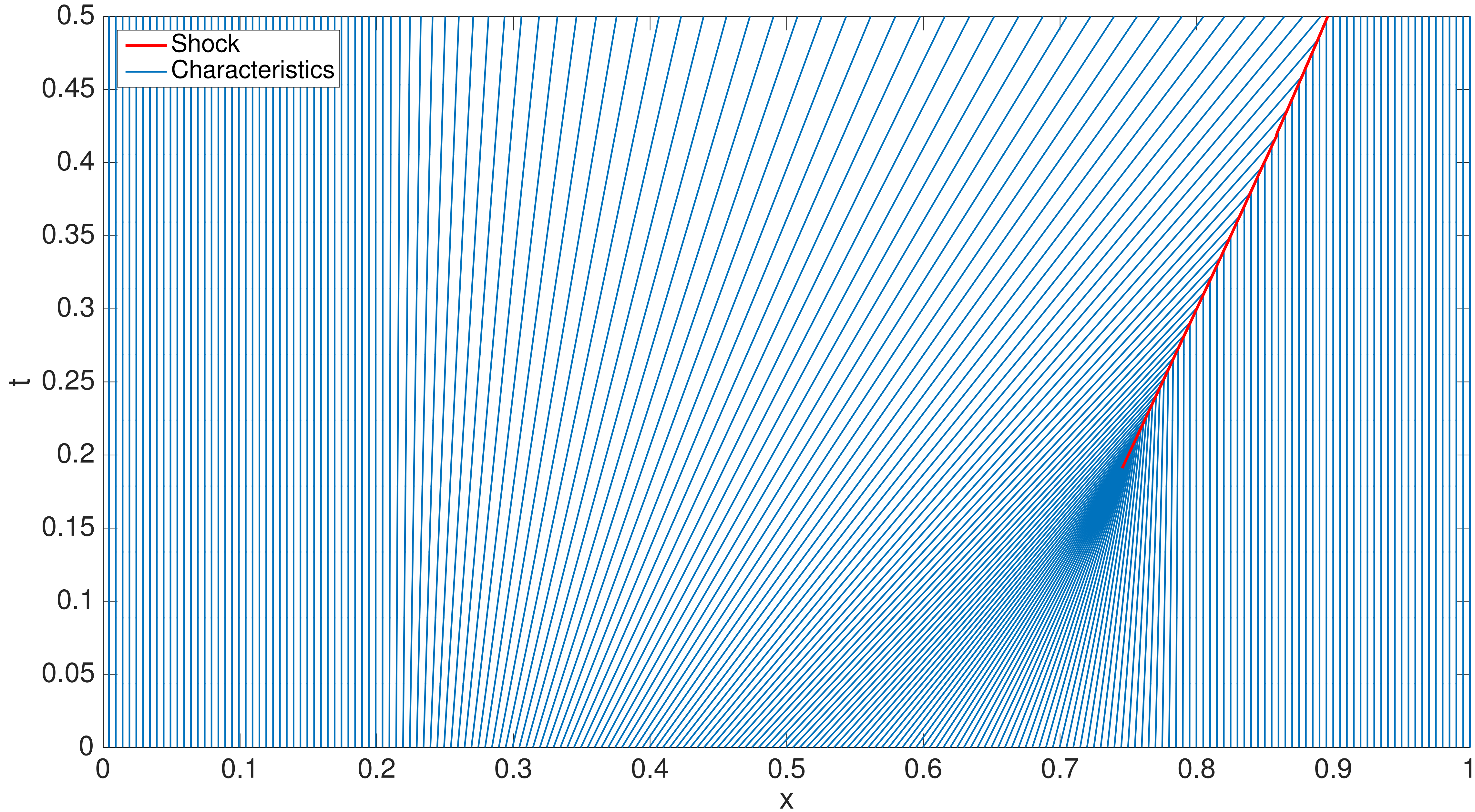
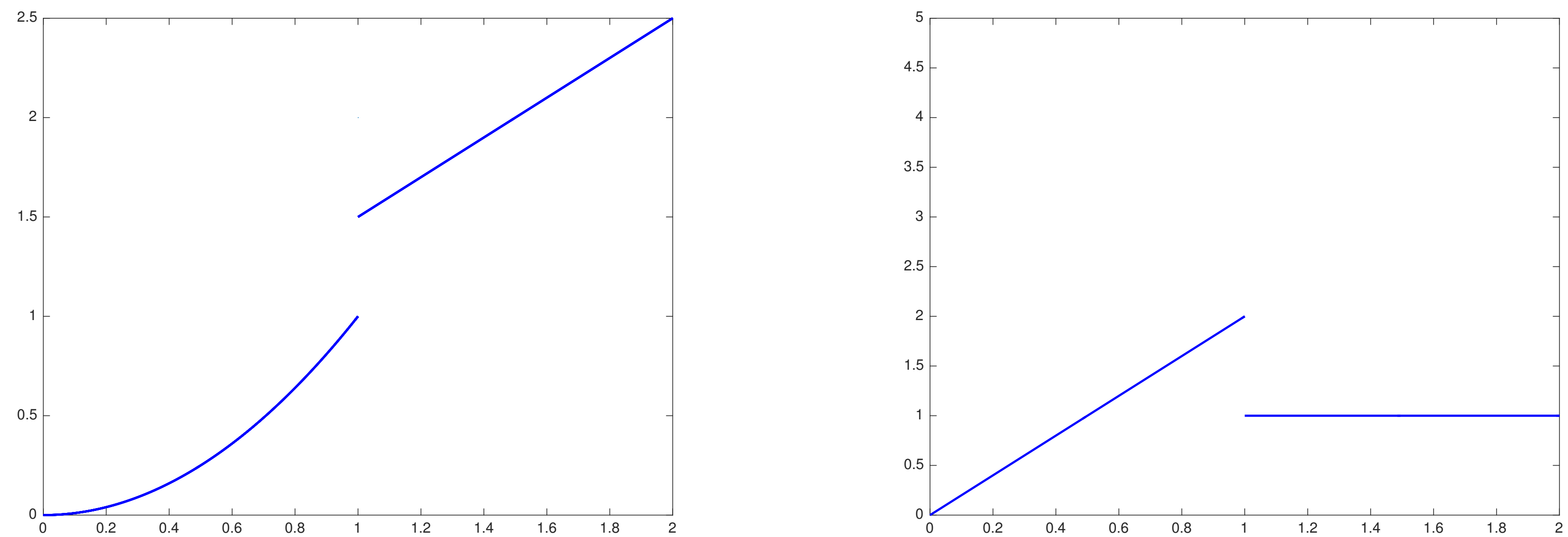
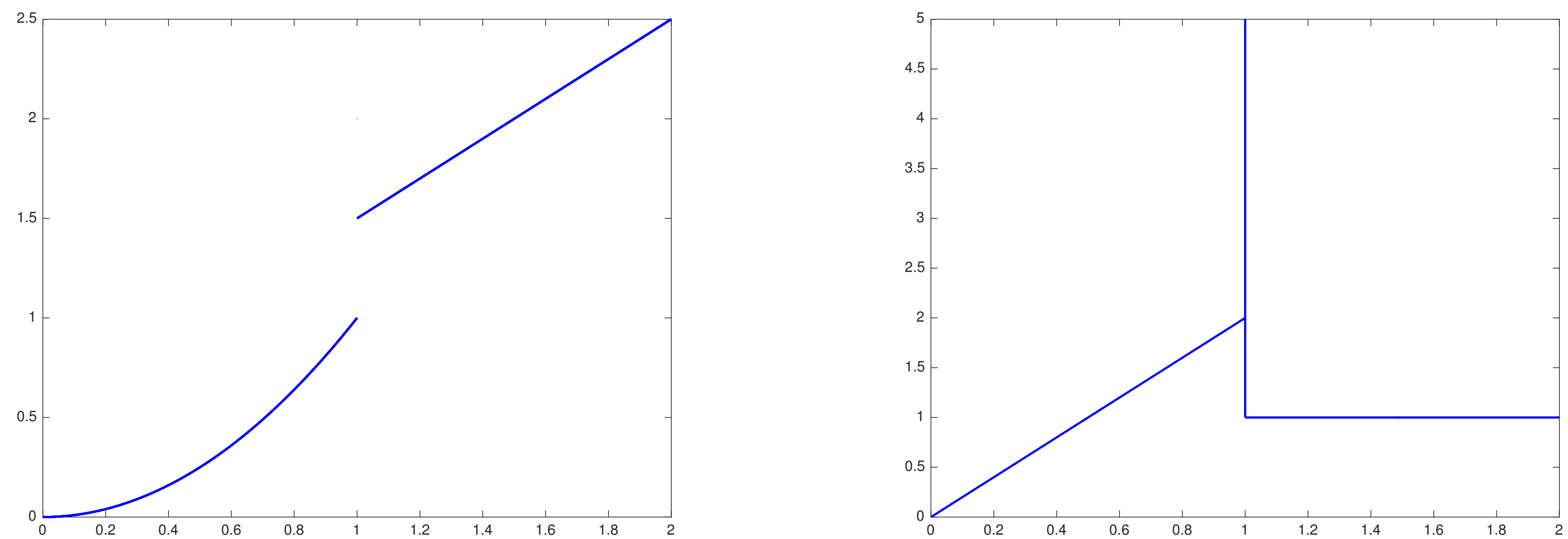


- **Classical** derivative: it is defined everywhere but in the discontinuity

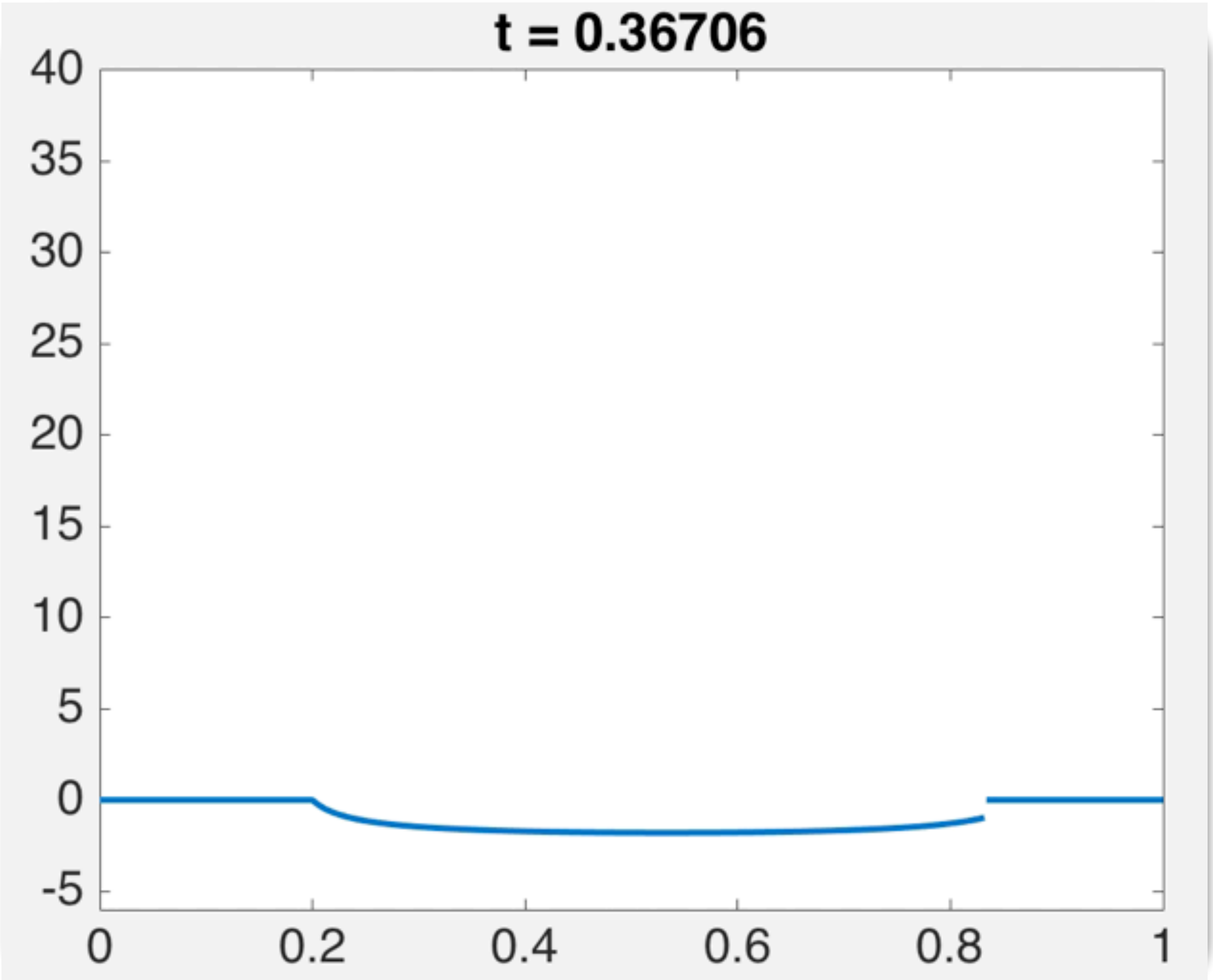


- **Weak** derivative: it is defined also in the discontinuity, where it is a **Dirac's distribution**



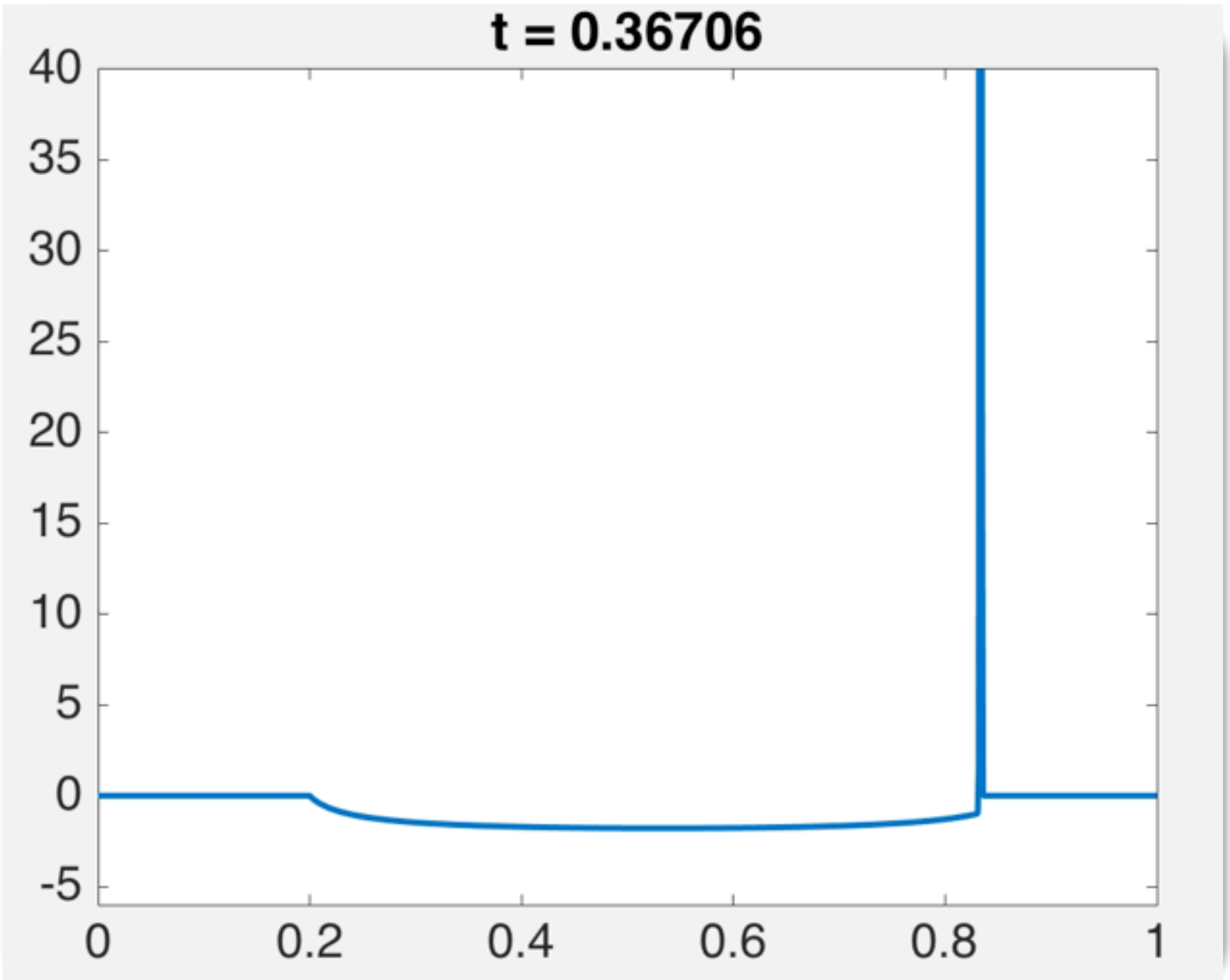
Example: the Burger's equation

$$f(u) = \frac{u^2}{2} \qquad g(x; A, x_c, L) = \begin{cases} A\sin^2\left(\frac{\pi}{L}(x - \overset{p}{x_c}) + \frac{\pi}{2}\right) & x \in (x_c - \frac{L}{2}, x_c + \frac{L}{2}) \\ 0 & \text{otherwise.} \end{cases}$$



Example: the Burger's equation

$$f(u) = \frac{u^2}{2} \qquad g(x; A, x_c, L) = \begin{cases} A\sin^2\left(\frac{\pi}{L}(x - \overset{p}{x_c}) + \frac{\pi}{2}\right) & x \in (x_c - \frac{L}{2}, x_c + \frac{L}{2}) \\ 0 & \text{otherwise.} \end{cases}$$



Choice of the derivative

Weak derivative:

- no correction to numerical schemes needed

Classical derivative:

- it does not corrupt the solution in the regular zones
- it is possible to estimate close solutions

Rankine-Hugoniot conditions

Across the shock, the **state** is governed by the Rankine-Hugoniot conditions:

$$\begin{cases} \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 & x \in \mathbb{R},\ t > 0 \\ \mathbf{u}(x,0) = \mathbf{g}(x;\mathbf{p}) & x \in \mathbb{R}. \end{cases}$$

$$f(\mathbf{u}^+) - f(\mathbf{u}^-) = \sigma(\mathbf{u}^+ - \mathbf{u}^-)$$

If we wrote the same conditions for the **sensitivity**, we would have:

$$\begin{cases} \partial_t \mathbf{u_p} + \partial_x \left(f'(\mathbf{u})\mathbf{u_p}\right) = 0 & x \in \mathbb{R},\ t > 0 \\ \mathbf{u_p}(x,0) = \mathbf{g_p}(x;\mathbf{p}) & x \in \mathbb{R}. \end{cases}$$

$$f'(\mathbf{u}^+)\mathbf{u_p}^+ - f'(\mathbf{u}^-)\mathbf{u_p}^- = \sigma(\mathbf{u_p}^+ - \mathbf{u_p}^-)$$

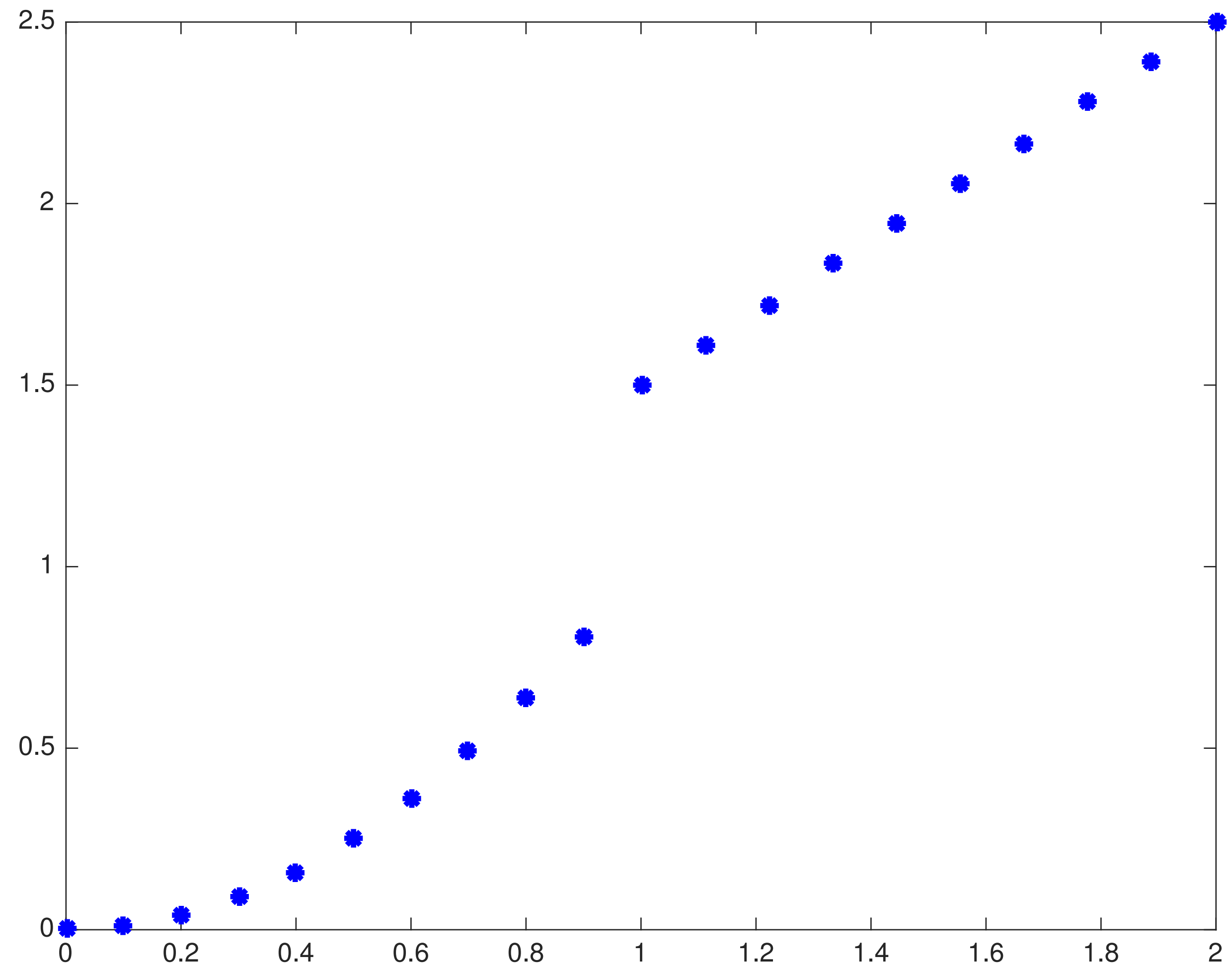
However, differentiating with respect to ***p*** the conditions for the state we obtain:

$$f'(\mathbf{u}^+)\mathbf{u_p}^+ - f'(\mathbf{u}^-)\mathbf{u_p}^- = \sigma(\mathbf{u_p}^+ - \mathbf{u_p}^-) + \partial_{\mathbf{p}}\sigma(\mathbf{u}^+ - \mathbf{u}^-)$$

Idea: add to the sensitivity equation a **source term** that balances it out.

$$\partial_t \mathbf{u_p} + \partial_x \left(f'(\mathbf{u})\mathbf{u_p}\right) = \mathbf{s}(\mathbf{u}^+, \mathbf{u}^-) \qquad x \in \mathbb{R},\ t > 0$$

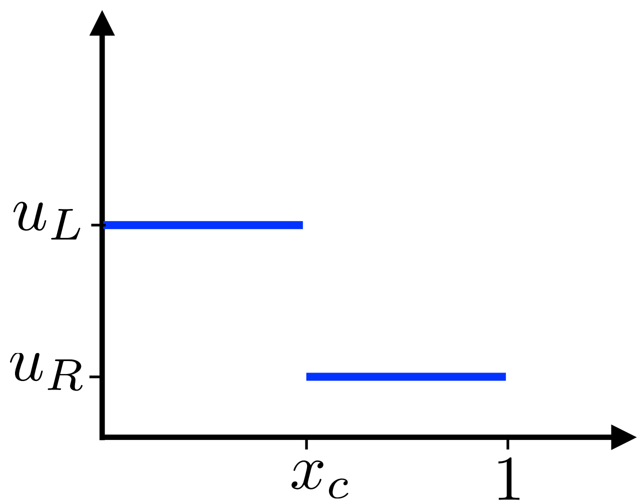
The term $\partial_{\mathbf{p}}\sigma(\mathbf{u}^+ - \mathbf{u}^-)$ is zero in the regular zones, however this is not true if we consider a **discretisation** of the equations.

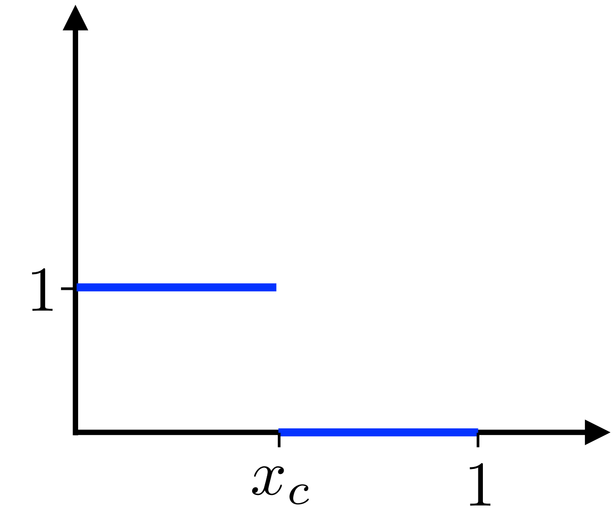


It is necessary to define a **shock detector**.

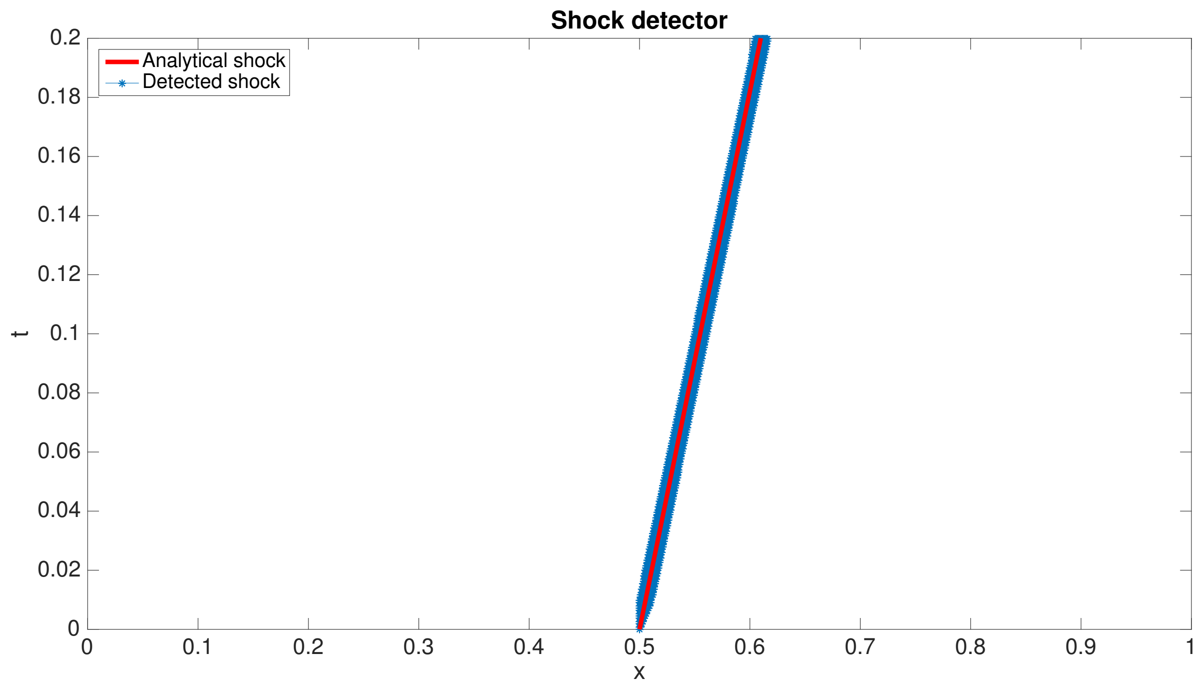
The **Riemann problem** for the Burger's equation:

$g(x; u_R, u_L, x_c)$ $\qquad$ $g_{u_L}(x; u_R, u_L, x_c)$

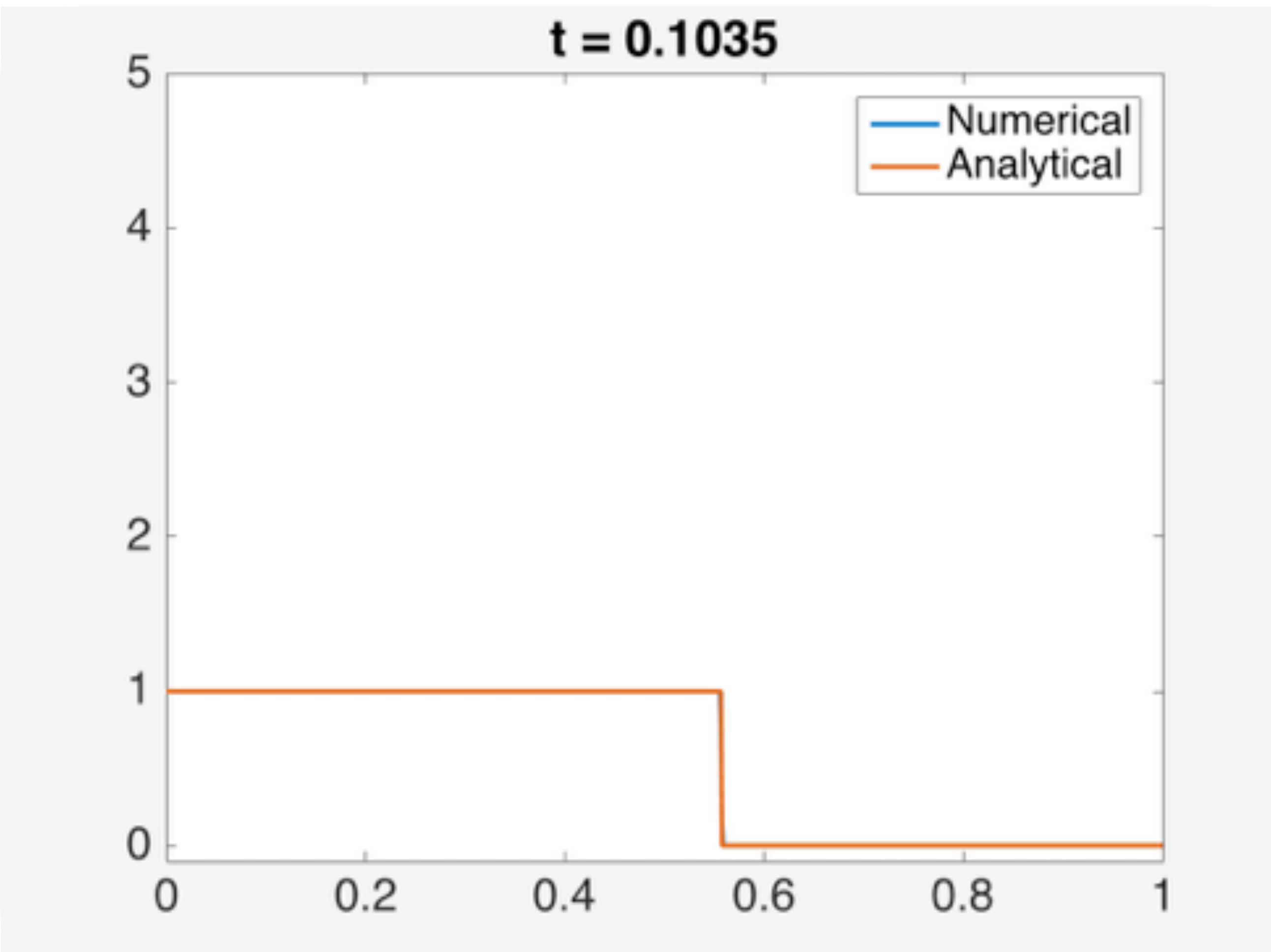




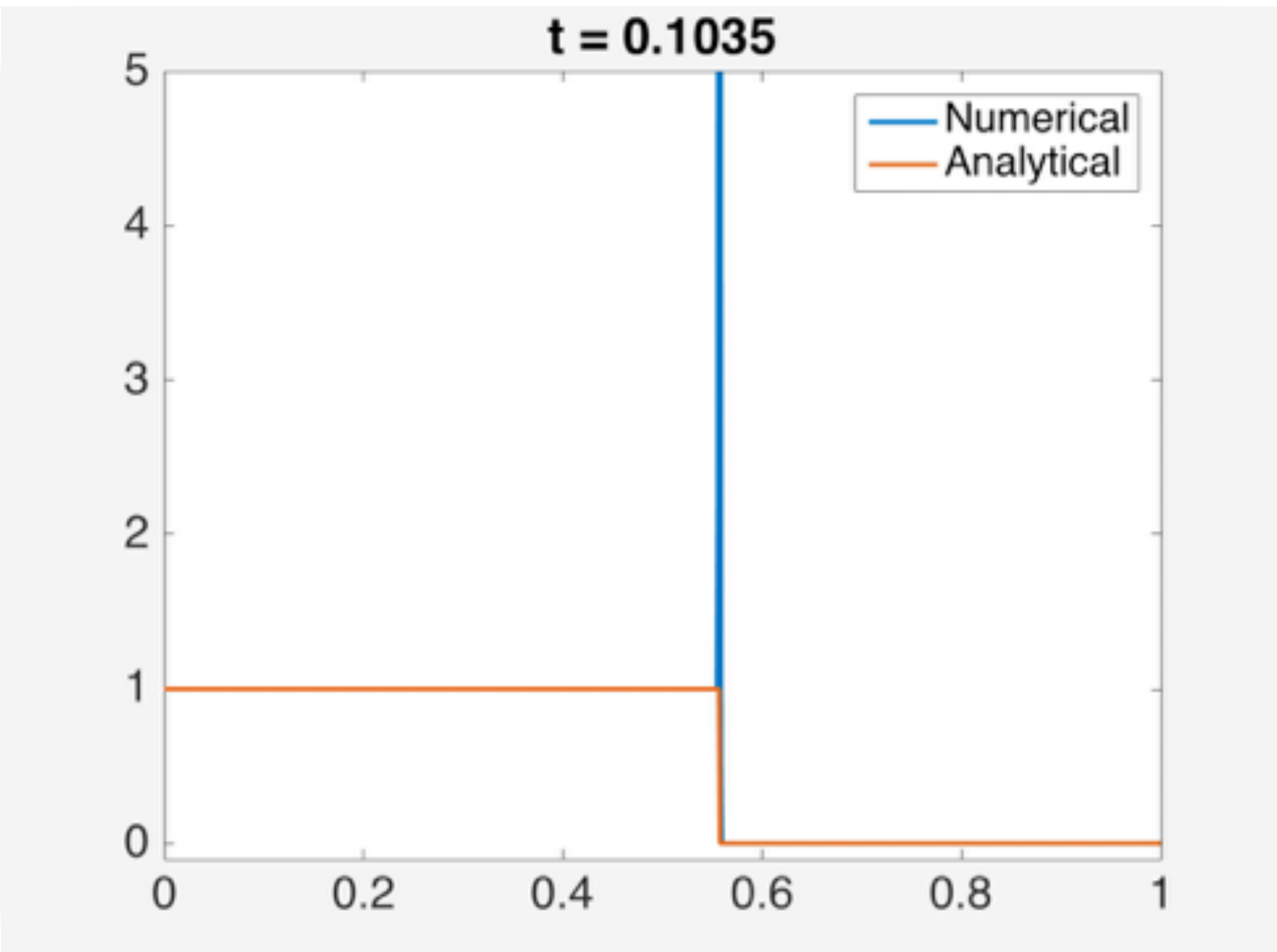
In this case it is easy to define a good shock detector.



Sensitivity **with** source term:

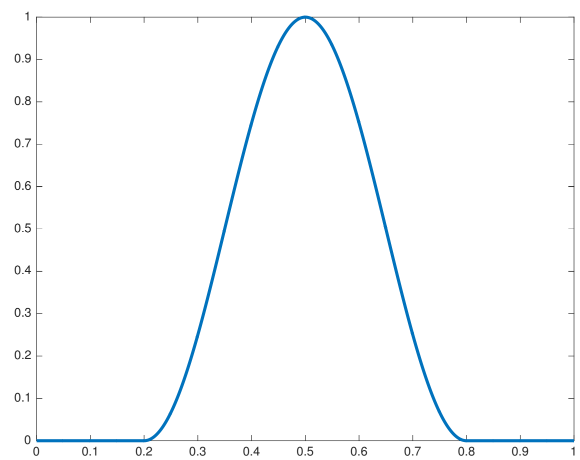


Sensitivity **without** source term:

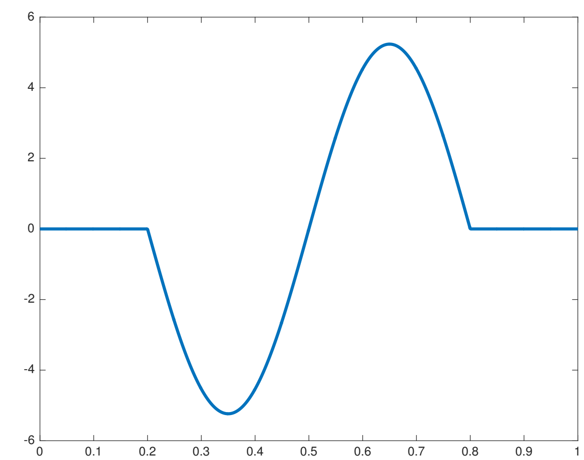


The same shock detector in a less simple case does not work:

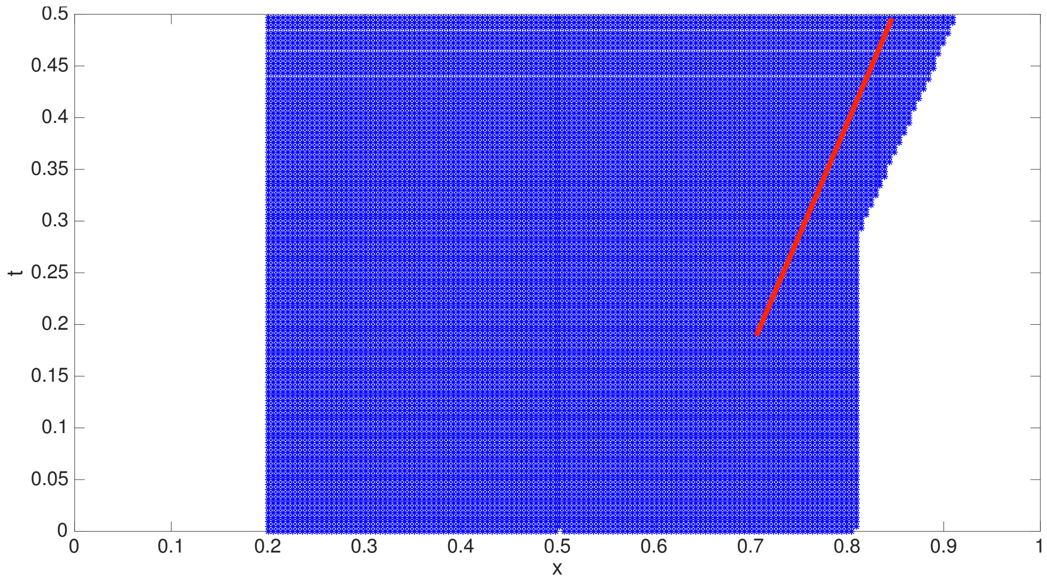
$g(x; \mathbf{p})$

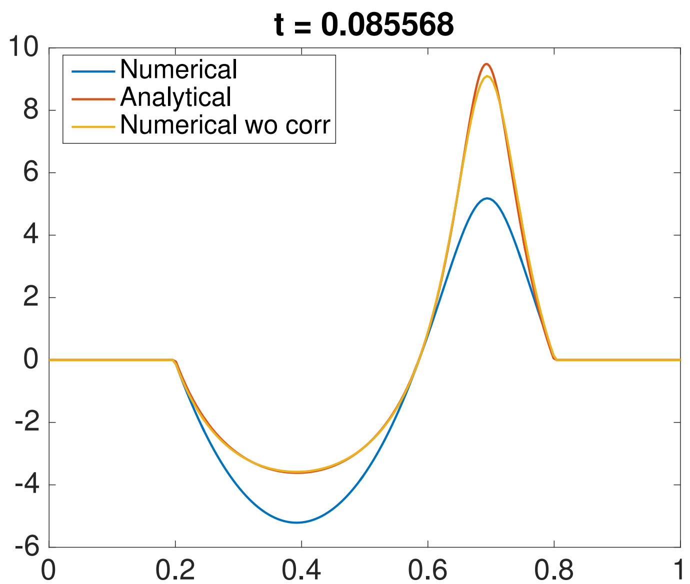


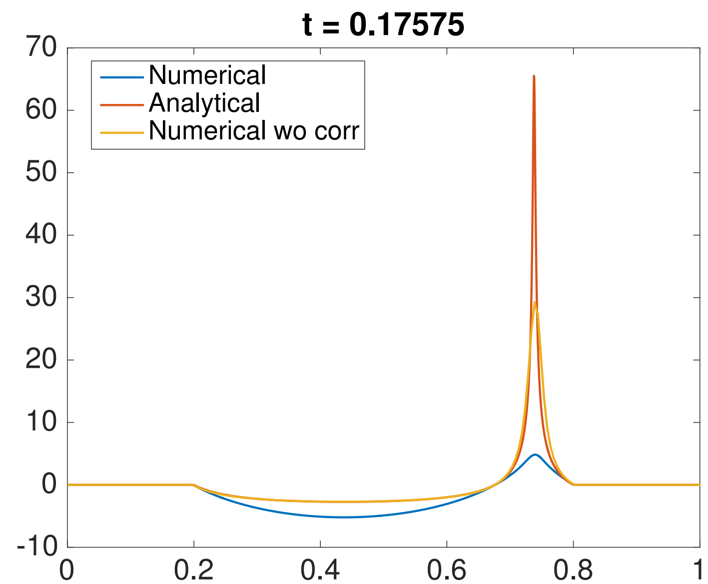
$g_{\mathbf{p}}(x; \mathbf{p})$



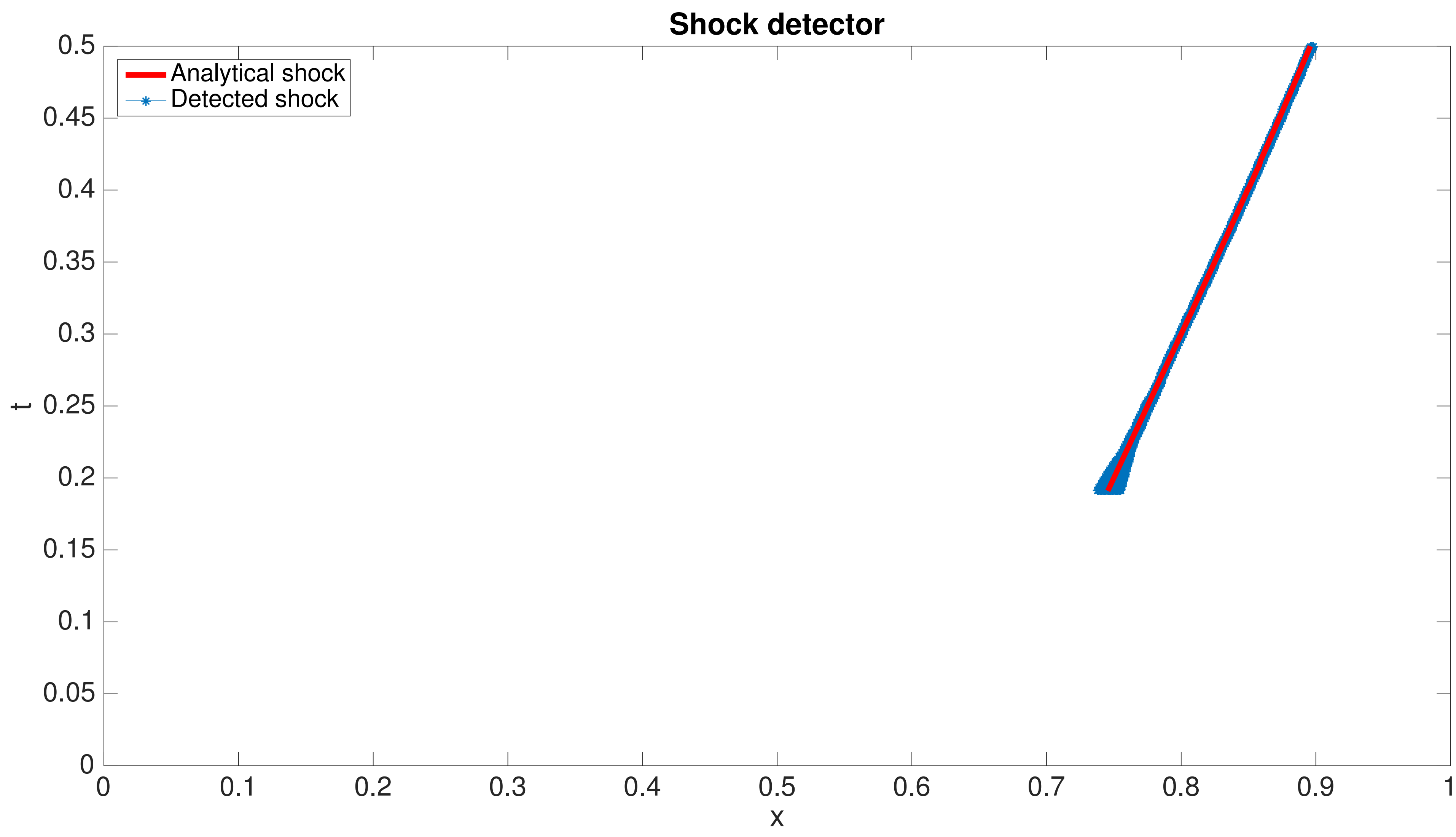
It leads to an **overcorrection**:





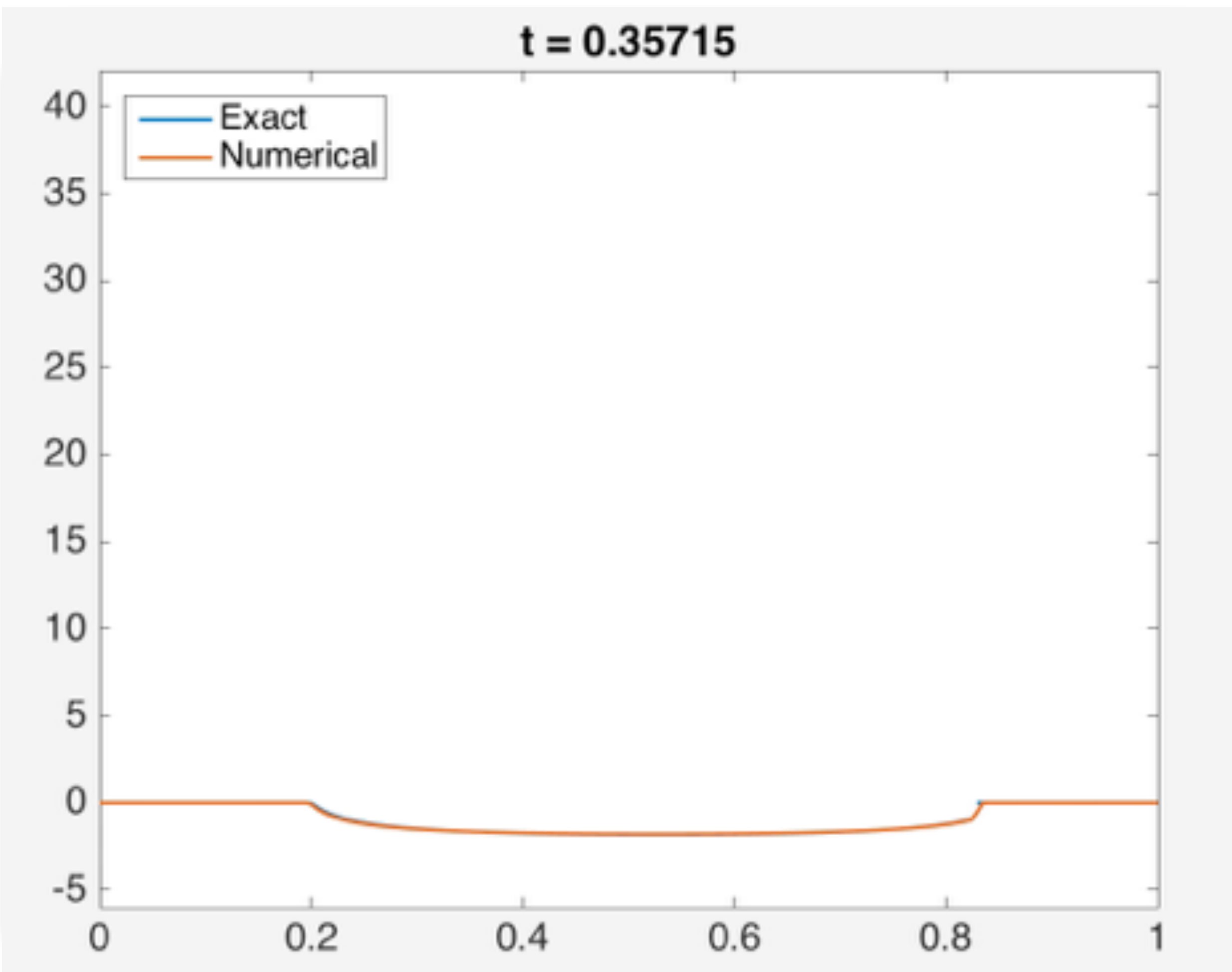


We defined a new shock detector based on the **second derivative** and on the **breaking time**.

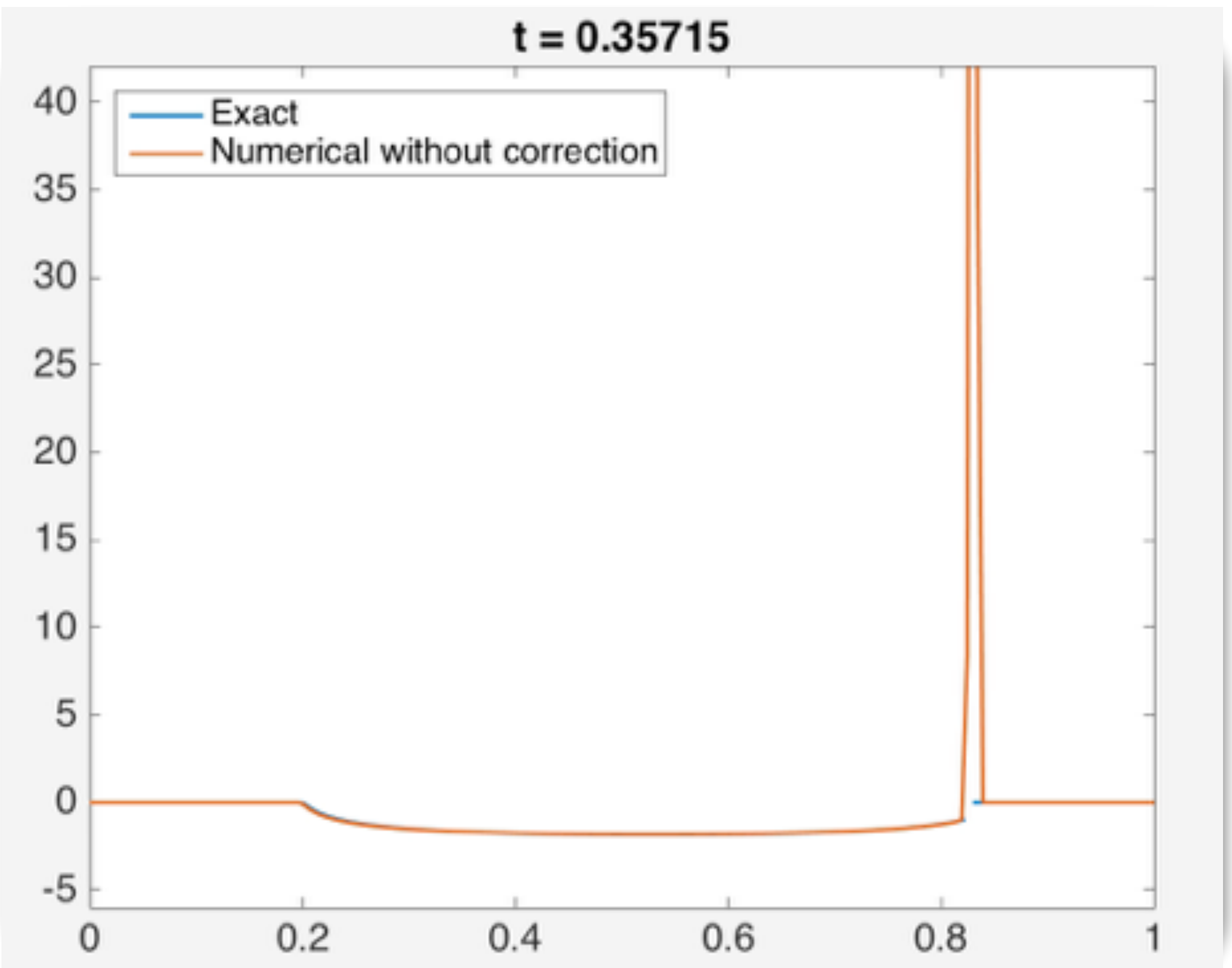


Numerical results

Sensitivity **with** source term:



Sensitivity **without** source term:



Conclusion and future developments

- A general method for sensitivity analysis in case of discontinuities has been developed ;
- Shock detectors are specific to each case;
- The method has been extended to systems (Euler 1D);
- We plan to increase the space dimension (2D or 3D).

Thank you
for your attention!