Uncertainty Quantification of Two-Phase Flow in Heterogeneous Porous Media

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Porous Media

Examples: sponge, subsurface, fuel cells, (human) tissues, etc

Parameters:
- averaged velocity (flux)
- permeability (proportionality constant in Darcys law)
- porosity (ratio of volumes)
- saturation (ratio of fluid phases)
Motivation

Multi-phase flow in heterogeneous porous media

- permeability
- layers
- fractures

Quantify uncertain system parameters, e.g. location of the interfaces

Small stochastic dimension $N$
Outline

Fractional Flow Formulation

Stochastic Galerkin Model

Numerical Experiments

Conclusion and Outlook
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Deterministic model

Capillarity-free fractional flow formulation of two immiscible and incompressible fluid phases \((D \subset \mathbb{R}^2, T > 0, D_T := D \times (0, T))\)

**Unknowns:** saturation \(S\), total velocity \(v\), global pressure \(p\)

\[
\begin{align*}
\mathbf{v} &= -K\lambda(S)\nabla p, \quad \text{and} \quad \text{div} (\mathbf{v}) = q, \quad \text{in} \ D_T, \\
\phi S_t + \text{div} (\mathbf{v} f(x, S)) - \bar{q} &= 0, \quad \text{in} \ D_T, \\
S(\cdot, 0) &= S_0, \quad \text{in} \ D.
\end{align*}
\]

Mixed hyperbolic-elliptic system coupled by \(\mathbf{v}, \lambda(S)\) and \(f(x, S)\).
Fractional flux function with heterogeneity

**Test case:** vertical injection in porous medium with heterogeneous lens

![Diagram showing the test case](image)

Non-linear flux function

\[
f(x, S, \gamma) := \gamma(x) f^1(S) + (1 - \gamma(x)) f^2(S),
\]

with

\[
\gamma(x) := \begin{cases} 
1, & x \in D^1, \\
0, & x \in D^2.
\end{cases}
\]
Vertical injection with heterogeneous lense (deterministic)

- **Saturation** (a) and total **velocity** (b),
- Standing shock wave at the heterogeneity interface,
- (a): central-upwind FV scheme (Kurganov & Petrova, 2005).
- (b): mixed finite elements (*Alberta*-FEM, Schmidt & Siebert, 2005).
Fractional flux function with heterogeneity and uncertainty

Let $\xi(\omega) := \{\xi_1(\omega_1), \ldots, \xi_N(\omega_N)\}$ be a $N$-dimensional random vector on $(\Omega, \mathcal{F})$. The non-linear flux function reads

$$f(x, S, \gamma, \xi) := \gamma(x, \xi) f^1(S) + (1 - \gamma(x, \xi)) f^2(S),$$

with

$$\gamma(x, \xi) := \begin{cases} 1, & x \in D^1(\xi), \quad \xi \in [0, 1]^N, \\ 0, & x \in D^2(\xi), \quad \xi \in [0, 1]^N. \end{cases}$$
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We set $N = 1$. Let $\xi = \xi(\omega)$ be a random variable on $(\Omega, \mathcal{P})$ and \{\varphi_p(\xi)\}_{p \in \mathbb{N}_0}$ a family of polynomials satisfying

$$\langle \varphi_p(\xi), \varphi_q(\xi) \rangle_{L^2(\Omega)} := \int_{\omega} \varphi_p(\xi) \varphi_q(\xi) \, d\mathcal{P}(\omega) = \delta_{pq} \quad \text{for } p, q \in \mathbb{N}_0.$$ 

The random field $S = S(x, t, \xi)$, $(x, t) \in D_T$, with finite variance can be represented by

$$S(x, t, \xi) = \sum_{p=0}^{\infty} S^p(x, t) \varphi_p(\xi),$$

where

$$S^p := S^p(x, t) := \langle S, \varphi_p \rangle_{L^2(\Omega)} \quad \text{for } p \in \mathbb{N}_0,$$

Expectation and variance are given by

$$\mathbb{E}[S] = S^0 \quad \text{and} \quad \text{Var}[S] = \sum_{p=1}^{\infty} (S^p)^2.$$
Stochastic Galerkin (SG) approach

Exemplarily we consider the **hyperbolic** equation for given velocity:

- We test the equation with \( \varphi_0, \ldots, \varphi_{N_\omega} \) and obtain the system

\[
\int_{\Omega} \left( \phi S_t(\xi) + \text{div} \left( \mathbf{v} f(S, \gamma, \xi) \right) - \tilde{q} \right) \varphi_p(\xi) \, d\mathcal{P}(\omega) = 0
\]

for \( p = 0, \ldots, N_\omega \).

- We replace \( S(x, t, \xi) \) by the truncated PC-expansion

\[
\Pi^{N_\omega} [S] (x, t, \xi) := \sum_{p=0}^{N_\omega} S^p (x, t) \varphi_p(\xi).
\]

Introduced by

Stochastic Galerkin

- Orthogonality of $\varphi_p$ yields a truncated *SG system* for the **deterministic** coefficients $S^0, \ldots, S^{N_o}$

\[
\phi S^0_t + \text{div} \left< \mathbf{v} f(\Pi^{N_o}[S]), \varphi_0 \right>_{L^2(\Omega)} - \tilde{q}^0 = 0,
\]

\[
\vdots
\]

\[
\phi S^{N_o}_t + \text{div} \left< \mathbf{v} f(\Pi^{N_o}[S]), \varphi_{N_o} \right>_{L^2(\Omega)} - \tilde{q}^{N_o} = 0.
\]

$\longrightarrow$ Hyperbolic $(N_o + 1)$-dimensional system.

**Numerical challenges:**

- Strongly coupled.
- System complexity increases w.r.t. polynomial order $N_o$.
- Parallel computing requires synchronization in each time-step.
- Huge hyperbolic system and no structural information.
Hybrid stochastic Galerkin (HSG) approach

Multi-element discretization of the stochastic space with $2^{N_r}$ elements,

- R. Bürger, I. Kröker & C. Rohde (2013),
- (J. Trygoen, O. Le Maître & A. Ern (2012))

Let $\xi = \xi(\omega)$ be a random variable. Assume $\xi \sim U(0, 1)$. Define

$$\varphi_{i,l}^{N_r}(\xi) = 2^{N_r/2} \varphi_i(2^{N_r} \xi - l), \quad i = 0, \ldots, N_o, \quad l = 0, \ldots, 2^{N_r} - 1.$$ 

Here $\varphi_i$ is the $i$-th Legendre polynomial.

The polynomials $\varphi_0, 0, \ldots, \varphi_{N_o, 2^{N_r} - 1}$ satisfy

$$\left\langle \varphi_{i,k}^{N_r}, \varphi_{j,l}^{N_r} \right\rangle = \delta_{ij} \delta_{kl}.$$

and their support is $\text{Supp}(\varphi_{i,k}^{N_r}) = [2^{-N_r} k, 2^{-N_r} (k + 1)]$. 
Hybrid stochastic Galerkin approach

The projection of a random field $S(x, t, \cdot) \in L^2(\Omega)$ is defined by

$$
\Pi_{N_o, N_r}^N [S] (x, t, \xi) := \sum_{l=0}^{2^{N_r} - 1} \sum_{i=0}^{N_o} S_{i, l}^{N_r} (x, t) \varphi_{i, l}^{N_r} (\xi),
$$

$$
S_{i, l}^{N_r} (x, t) := \langle S(x, t, \cdot), \varphi_{i, l}^{N_r} \rangle.
$$

For $N_r, N_o \in \mathbb{N}_0$ find $S_{i, l}^{N_r} : D_T \to \mathbb{R}$ such that

$$
\int_{\Omega} \left( \phi \partial_t \Pi_{N_o, N_r}^N [S] (\xi) + \text{div} \left( \mathbf{v} f (\Pi_{N_o, N_r}^N [S], \gamma, \xi) \right) - \tilde{q} \right) \varphi_{i, l}^{N_r} \, d\mathcal{P}(\omega) = 0
$$

for all $(i, l)$, where $i = 0, \ldots, N_o$ and $l = 0, \ldots, 2^{N_r} - 1$.

$\rightarrow$ We obtain a $(N_o + 1)2^{N_r}$-dimensional system.
Hybrid stochastic Galerkin approach

Orthogonality of $\varphi_{i,l}^{N_r}$ leads to the partially decoupled system with $2^{N_r}$ blocks, each of dimension $(N_o + 1)$

\[ \phi \partial_t S_{0,0}^{N_r} + \text{div} \left( v f \left( \Pi^{N_o,N_r} \left[ S \right], \gamma \right), \varphi_{0,0}^{N_r} \right) - \tilde{q}_{0,0}^{N_r} = 0, \]

\[ \vdots \]

\[ \phi \partial_t S_{N_o,0}^{N_r} + \text{div} \left( v f \left( \Pi^{N_o,N_r} \left[ S \right], \gamma \right), \varphi_{N_o,0}^{N_r} \right) - \tilde{q}_{N_o,0}^{N_r} = 0, \]

\[ \vdots \]

\[ \phi \partial_t S_{0,2^{N_r}-1}^{N_r} + \text{div} \left( v f \left( \Pi^{N_o,N_r} \left[ S \right], \gamma \right), \varphi_{0,2^{N_r}-1}^{N_r} \right) - \tilde{q}_{0,2^{N_r}-1}^{N_r} = 0, \]

\[ \vdots \]

\[ \phi \partial_t S_{N_o,2^{N_r}-1}^{N_r} + \text{div} \left( v f \left( \Pi^{N_o,N_r} \left[ S \right], \gamma \right), \varphi_{N_o,2^{N_r}-1}^{N_r} \right) - \tilde{q}_{N_o,2^{N_r}-1}^{N_r} = 0. \]

\[ \rightarrow \] Better for parallel computing

\[ \rightarrow \] Reduction of polynomial order $N_o$
Stochastic Mixed Finite Element Method (SMFEM)

Consideration of the elliptic part

\[
K^{-1} \lambda^{-1} (S) \mathbf{v} + \nabla p = 0 \quad \text{in } D \times (0, T),
\]
\[
\text{div} (\mathbf{v}) = q \quad \text{in } D \times (0, T).
\]

Basis functions of \( \mathbf{v} \) and \( p \) based on lowest order Raviart-Thomas (RT0) elements and HSG expansion

\[
\mathbf{v}(x, \xi) \approx \Pi_{N_0, N_r} [v_h] := \sum_{l=0}^{2^{N_r}-1} \sum_{i=0}^{N_0} \sum_{j=1}^{N_e} v_{j, i, l}^{N_r}(x) \vartheta_{j}(x) \varphi_{i, l}^{N_r}(\xi),
\]
\[
p(x, \xi) \approx \Pi_{N_0, N_r} [p_h] := \sum_{l=0}^{2^{N_r}-1} \sum_{i=0}^{N_0} \sum_{j=1}^{N_t} p_{j, i, l}^{N_r}(x) \chi_{j}(x) \varphi_{i, l}^{N_r}(\xi).
\]

with \( N_e \) number of edges, \( N_t \) number of triangles, deterministic velocity and pressure basis functions \( \vartheta_{j}(x) \) resp. \( \chi_{j}(x) \).

Related work of the mixed elliptic problem by

- O.G. Ernst et al. (2009)
Stochastic Mixed Finite Element Method

This yields a matrix saddle-point problem

\[
\begin{bmatrix}
\hat{A} & -\hat{B}^T \\
\hat{B} & 0 
\end{bmatrix}
\begin{bmatrix}
v \\
p 
\end{bmatrix}
=
\begin{bmatrix}
\bar{p} \\
\bar{q} 
\end{bmatrix},
\]

with block matrices $\hat{A}$ and $\hat{B}$ given by

\[
\hat{A} =
\begin{bmatrix}
A_{0,0} & \cdots & A_{0,\tilde{P}-1} \\
\vdots & \ddots & \vdots \\
A_{\tilde{P}-1,0} & \cdots & A_{\tilde{P}-1,\tilde{P}-1}
\end{bmatrix},
\]

and

\[
\hat{B} = G \otimes B =
\begin{bmatrix}
B_{0,0} & 0 \\
\vdots & \ddots \\
0 & B_{\tilde{P}-1,\tilde{P}-1}
\end{bmatrix},
\]

with $\tilde{P} := (N_0 + 1)2^{N_r}$ and diagonal, stochastic matrix $G$. 
Stochastic Mixed Finite Element Method

and block vectors

\[
\mathbf{v} = \begin{bmatrix}
\mathbf{v}_0 \\
\vdots \\
\mathbf{v}_{N_0 2^{N_r}}
\end{bmatrix} \in \mathbb{R}^{N_e (N_0 + 1) 2^{N_r}}, \quad \mathbf{p} = \begin{bmatrix}
\mathbf{p}_0 \\
\vdots \\
\mathbf{p}_{N_0 2^{N_r}}
\end{bmatrix} \in \mathbb{R}^{N_t (N_0 + 1) 2^{N_r}},
\]

where each vector block is of size \( \dim(\mathbf{v}_l) = N_e \) resp. \( \dim(\mathbf{p}_l) = N_t \) for \( l = 0, \ldots, N_0 2^{N_r} \). Analogous for \( \bar{\mathbf{p}} \) and \( \mathbf{q} \).

**Remark:**

- system matrix is not fully assembled
- degrees of freedom: \( (N_t + N_e)(N_0 + 1) 2^{N_r} \).
- appropriate preconditioning necessary
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Vertical injection with heterogeneous lens & randomness

- **Expectation** (a) and **variance** (b) of the **saturation**,
- $N = 2$ (stoch. dim), $N_r = 1$, $N_o = 1$, (hyp system is 12-dimensional),
- Smereared heterogeneity interface due to uncertainty.
**Vertical injection with heterogeneous lense & randomness**

- **Expectation** (a) and **diag. of covariance** (b) of the velocity \([m/h]\),
- \(N = 2\) (stoch. dim), \(N_r = 1\), \(N_o = 1\), (ellipt dim: \(12 \cdot (N_t + N_e)\)),
- smooth transition at vertical boundaries of the lense due to uncertainty.
Vertical injection with heterogeneous lense & randomness

- **Expectation** (a) and **variance** (b) of the **pressure** [bar],
- \( N = 2 \) (stoch. dim), \( N_r = 1 \), \( N_o = 1 \), (ellipt dim: \( 12 \cdot (N_t + N_e) \)).
Model performance

<table>
<thead>
<tr>
<th>$N_r$/#cores:</th>
<th>0 / 1</th>
<th>1 / 2</th>
<th>2 / 4</th>
<th>3 / 8</th>
<th>4 / 16</th>
<th>5 / 32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_o = 1$, $h$</td>
<td>6.8</td>
<td>8.4</td>
<td>10.0</td>
<td>12.0</td>
<td>14.0</td>
<td>13.5</td>
</tr>
<tr>
<td>$N_o = 2$, $h$</td>
<td>20.5</td>
<td>20.7</td>
<td>26.3</td>
<td>27.8</td>
<td>27.1</td>
<td>28.3</td>
</tr>
<tr>
<td>$N_o = 3$, $h$</td>
<td>36.8</td>
<td>41.6</td>
<td>53.4</td>
<td>54.5</td>
<td>56.8</td>
<td>51.9</td>
</tr>
<tr>
<td>$N_o = 1$, $∅$</td>
<td>28.7</td>
<td>26.6</td>
<td>25.6</td>
<td>22.9</td>
<td>21</td>
<td>19</td>
</tr>
<tr>
<td>$N_o = 2$, $∅$</td>
<td>29.2</td>
<td>28</td>
<td>24.5</td>
<td>22.5</td>
<td>20.4</td>
<td>18.7</td>
</tr>
<tr>
<td>$N_o = 3$, $∅$</td>
<td>30.5</td>
<td>28</td>
<td>23.9</td>
<td>22.6</td>
<td>20.1</td>
<td>18.9</td>
</tr>
<tr>
<td>$N_o = 1$, %</td>
<td>8.5</td>
<td>7</td>
<td>6.4</td>
<td>5.4</td>
<td>5.3</td>
<td>4.3</td>
</tr>
<tr>
<td>$N_o = 2$, %</td>
<td>8.7</td>
<td>7.8</td>
<td>6.5</td>
<td>6.4</td>
<td>6.6</td>
<td>4.9</td>
</tr>
<tr>
<td>$N_o = 3$, %</td>
<td>10.2</td>
<td>8.8</td>
<td>6.7</td>
<td>6.0</td>
<td>5.5</td>
<td>4.7</td>
</tr>
</tbody>
</table>

Table: Computation time (hours $h$), average iteration counts ($∅$) and time ratio of all elliptic solves to global solve (%) with preconditioner ($N = 1$).
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Conclusion

- First uncertainty quantification (UQ) for full (hyperbolic-elliptic) two-phase system in 2D based on SG.
- The HSG approach requires a higher dimensional system than SG, but this system is partially decoupled and needs lower polynomial order.
- Effective parallel computations.

Outlook

- Apply HSG to fractured porous media
- Comparison with other UQ techniques
Thank you for your attention!
Questions