SimTech
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Uncertainty Quantification of Two-Phase Flow in Heterogeneous Porous Media

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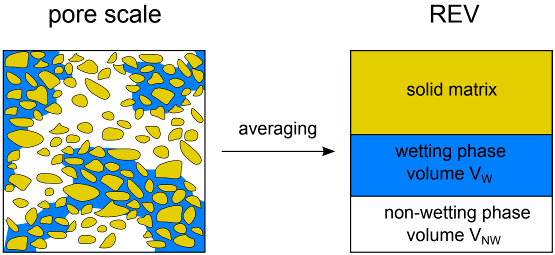
Porous Media

Examples: sponge, subsurface, fuel cells, (human) tissues, etc



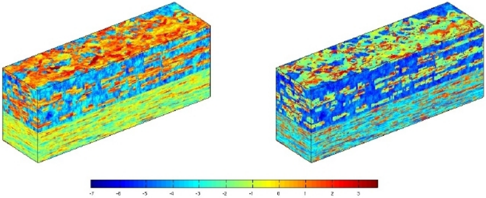
Parameters:

- averaged velocity (flux)
- permeability (proportionality constant in Darcys law)
- porosity (ratio of volumes)
- saturation (ratio of fluid phases)

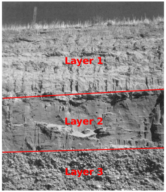
Motivation

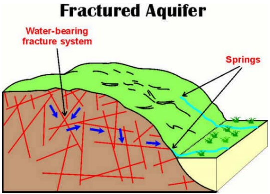
Multi-phase flow in heterogeneous porous media

- permeability
- layers
- fractures



Quantify uncertain system parameters, e.g. location of the interfaces





Small stochastic dimension N

Outline

Fractional Flow Formulation

Stochastic Galerkin Model

Numerical Experiments

Conclusion and Outlook

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Deterministic model

Capillarity-free fractional flow formulation of **two** immiscible and incompressible fluid phases ($D \subset \mathbb{R}^2$, $T > 0$, $D_T := D \times (0, T)$)

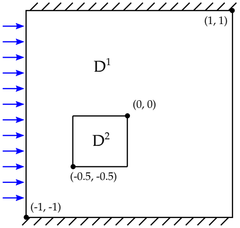
Unknowns: saturation S, total velocity $\mathbf{v}$, global pressure p

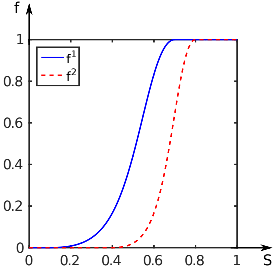
$$\begin{aligned} \mathbf{v} = -\mathbf{K}\lambda(S)\nabla p, \quad \text{and} \quad \text{div}\,(\mathbf{v}) &= q, && \text{in } D_T, \\ \phi S_t + \text{div}\,(\mathbf{v} f(\mathbf{x}, S)) - \tilde{q} &= 0, && \text{in } D_T, \\ S(\cdot, 0) &= S_0, && \text{in } D. \end{aligned}$$

Mixed **hyperbolic-elliptic** system coupled by $\mathbf{v}$, $\lambda(S)$ and $f(\mathbf{x}, S)$.

Fractional flux function with heterogeneity

Test case: vertical injection in porous medium with heterogeneous lense





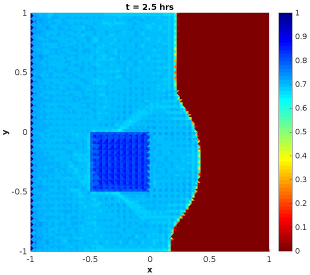
Non-linear flux function

$$f(\mathbf{x}, S, \gamma) := \gamma(\mathbf{x}) f^1(S) + (1 - \gamma(\mathbf{x})) f^2(S),$$

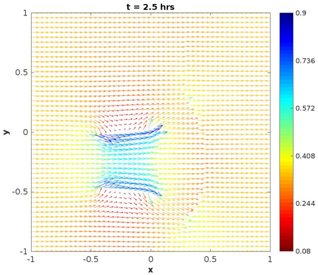
with

$$\gamma(\mathbf{x}) := \begin{cases} 1, & \mathbf{x} \in D^1, \\ 0, & \mathbf{x} \in D^2. \end{cases}$$

Vertical injection with heterogeneous lense (deterministic)



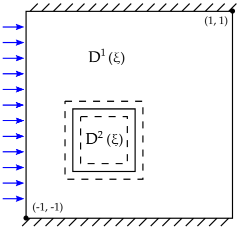
(a)

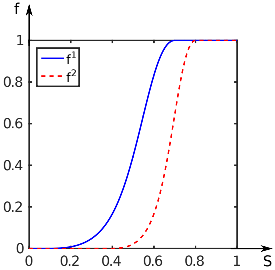


(b)

- **Saturation** (a) and total **velocity** (b),
- Standing shock wave at the heterogeneity interface,
- (a): central-upwind FV scheme (Kurganov & Petrova, 2005).
- (b): mixed finite elements (*Alberta*-FEM, Schmidt & Siebert, 2005).

Fractional flux function with heterogeneity and uncertainty





Let $\boldsymbol{\xi}(\boldsymbol{\omega}) := \{\xi_1(\omega_1), \ldots, \xi_N(\omega_N)\}$ be a N-dimensional random vector on $(\Omega, \mathcal{P})$. The **non-linear** flux function reads

$$f(\mathbf{x}, S, \gamma, \boldsymbol{\xi}) := \gamma(\mathbf{x}, \boldsymbol{\xi}) f^1(S) + (1 - \gamma(\mathbf{x}, \boldsymbol{\xi})) f^2(S),$$

with

$$\gamma(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} 1, & \mathbf{x} \in D^1(\boldsymbol{\xi}), \ \boldsymbol{\xi} \in [0,1]^N, \\ 0, & \mathbf{x} \in D^2(\boldsymbol{\xi}), \ \boldsymbol{\xi} \in [0,1]^N. \end{cases}$$

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Polynomial Chaos

We set $N = 1$. Let $\xi = \xi(\omega)$ be a random variable on $(\Omega, \mathcal{P})$ and $\{\varphi_p(\xi)\}_{p\in\mathbb{N}_0}$ a family of polynomials satisfying

$$\langle \varphi_p(\xi),\, \varphi_q(\xi)\rangle_{L^2(\Omega)} := \int_\omega \varphi_p(\xi)\varphi_q(\xi)\, d\mathcal{P}(\omega) = \delta_{pq} \quad \text{for } p, q \in \mathbb{N}_0.$$

The random field $S = S(\mathbf{x}, t, \xi)$, $(\mathbf{x}, t) \in D_T$, with finite variance can be represented by

$$S(\mathbf{x}, t, \xi) = \sum_{p=0}^{\infty} S^p(\mathbf{x}, t)\varphi_p(\xi),$$

where

$$S^p := S^p(\mathbf{x}, t) := \langle S,\, \varphi_p\rangle_{L^2(\Omega)} \quad \text{for } p \in \mathbb{N}_0,$$

Expectation and variance are given by

$$\mathbb{E}\,[S] = S^0 \quad \text{and} \quad \mathrm{Var}[S] = \sum_{p=1}^{\infty} (S^p)^2.$$

Stochastic Galerkin (SG) approach

Exemplarily we consider the **hyperbolic** equation for given velocity:

- We test the equation with $\varphi_0, \ldots, \varphi_{N_o}$ and obtain the system

$$\int_\Omega \left(\phi S_t(\xi) + \mathrm{div}\,(\mathbf{v} f(S, \gamma, \xi)) - \tilde{q} \right) \varphi_p(\xi)\, \mathrm{d}\mathcal{P}(\omega) = 0$$

for $p = 0, \ldots, N_o$.

- We replace $S(\mathbf{x}, t, \xi)$ by the truncated PC-expansion

$$\Pi^{N_o}[S](\mathbf{x}, t, \xi) := \sum_{p=0}^{N_o} S^p(\mathbf{x}, t) \varphi_p(\xi).$$

Introduced by

- R.G. Ghanem and P.D. Spanos (1989) (elliptic problem),
- R. Abgrall (2008), B. Després et al. (2008) (hyperbolic problem)

Stochastic Galerkin

- Orthogonality of φ_p yields a truncated *SG system* for the **deterministic** coefficients $S^0, \ldots, S^{N_o}$

$$\begin{aligned}
\phi S_t^0 + \operatorname{div} \left\langle \mathbf{v} f(\Pi^{N_o}[S]), \varphi_0 \right\rangle_{L^2(\Omega)} - \tilde{q}^0 &= 0, \\
&\vdots \\
\phi S_t^{N_o} + \operatorname{div} \left\langle \mathbf{v} f(\Pi^{N_o}[S]), \varphi_{N_o} \right\rangle_{L^2(\Omega)} - \tilde{q}^{N_o} &= 0.
\end{aligned}$$

$\longrightarrow$ Hyperbolic $(N_o + 1)$-dimensional system.

Numerical challenges:

- Strongly coupled.
- System complexity increases w.r.t. polynomial order N_o.
- Parallel computing requires synchronization in each time-step.
- Huge hyperbolic system and no structural information.

Hybrid stochastic Galerkin (HSG) approach

Multi-element discretization of the stochastic space with 2^{N_r} elements,

- R. Bürger, I. Kröker & C. Rohde (2013),
- (J. Trygoen, O. Le Maître & A. Ern (2012))

Let $\xi = \xi(\omega)$ be a random variable. Assume $\xi \sim \mathcal{U}(0,1)$. Define

$$\varphi_{i,l}^{N_\mathrm{r}}(\xi) = 2^{N_\mathrm{r}/2}\varphi_i(2^{N_\mathrm{r}}\xi - l), \quad i = 0, \dots, N_\mathrm{o}, \quad l = 0, \dots, 2^{N_\mathrm{r}} - 1.$$

Here φ_i is the i-th Legendre polynomial.

The polynomials $\varphi_{0,0}^{N_\mathrm{r}}, \dots, \varphi_{N_\mathrm{o},2^{N_\mathrm{r}}-1}^{N_\mathrm{r}}$ satisfy

$$\left\langle \varphi_{i,k}^{N_\mathrm{r}},\, \varphi_{j,l}^{N_\mathrm{r}} \right\rangle = \delta_{ij}\delta_{kl}.$$

and their support is $\mathrm{Supp}(\varphi_{i,k}^{N_\mathrm{r}}) = [2^{-N_\mathrm{r}}k, 2^{-N_\mathrm{r}}(k+1)]$.

Hybrid stochastic Galerkin approach

The projection of a random field $S(\mathbf{x},t,\cdot) \in L^2(\Omega)$ is defined by

$$\Pi^{N_\mathrm{o},N_\mathrm{r}}\left[S\right](\mathbf{x},t,\xi) := \sum_{l=0}^{2^{N_\mathrm{r}}-1} \sum_{i=0}^{N_\mathrm{o}} S_{i,l}^{N_\mathrm{r}}(\mathbf{x},t)\varphi_{i,l}^{N_\mathrm{r}}(\xi),$$

$$S_{i,l}^{N_\mathrm{r}}(\mathbf{x},t) := \left\langle S(\mathbf{x},t,\cdot),\ \varphi_{i,l}^{N_\mathrm{r}} \right\rangle.$$

For $N_\mathrm{r}, N_\mathrm{o} \in \mathbb{N}_0$ find $S_{i,l}^{N_\mathrm{r}} : D_T \to \mathbb{R}$ such that

$$\int_\Omega \left(\phi\partial_t \Pi^{N_\mathrm{o},N_\mathrm{r}}\left[S\right](\xi) + \mathrm{div}\left(\mathbf{v}f(\Pi^{N_\mathrm{o},N_\mathrm{r}}\left[S\right],\gamma,\xi)\right) - \tilde{q} \right) \varphi_{i,l}^{N_\mathrm{r}}\, d\mathcal{P}(\omega) = 0$$

for all (i,l), where $i = 0,\ldots,N_\mathrm{o}$ and $l = 0,\ldots,2^{N_\mathrm{r}}-1$.

$\longrightarrow$ We obtain a $(N_\mathrm{o}+1)2^{N_\mathrm{r}}$- dimensional system

Hybrid stochastic Galerkin approach

Orthogonality of $\varphi_{i,l}^{N_\mathrm{r}}$ leads to the **partially decoupled** system with 2^{N_r} blocks, each of dimension $(N_\mathrm{o}+1)$

$$
\begin{aligned}
\phi\partial_t S_{0,0}^{N_\mathrm{r}} + \operatorname{div}\left\langle \mathbf{v} f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S],\gamma\right), \varphi_{0,0}^{N_\mathrm{r}}\right\rangle - \tilde{q}_{0,0}^{N_\mathrm{r}} &= 0,\\
&\vdots\\
\phi\partial_t S_{N_\mathrm{o},0}^{N_\mathrm{r}} + \operatorname{div}\left\langle \mathbf{v} f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S],\gamma\right), \varphi_{N_\mathrm{o},0}^{N_\mathrm{r}}\right\rangle - \tilde{q}_{N_\mathrm{o},0}^{N_\mathrm{r}} &= 0,\\
&\vdots\\
\phi\partial_t S_{0,2^{N_\mathrm{r}}-1}^{N_\mathrm{r}} + \operatorname{div}\left\langle \mathbf{v} f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S],\gamma\right), \varphi_{0,2^{N_\mathrm{r}}-1}^{N_\mathrm{r}}\right\rangle - \tilde{q}_{0,2^{N_\mathrm{r}}-1}^{N_\mathrm{r}} &= 0,\\
&\vdots\\
\phi\partial_t S_{N_\mathrm{o},2^{N_\mathrm{r}}-1}^{N_\mathrm{r}} + \operatorname{div}\left\langle \mathbf{v} f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S],\gamma\right), \varphi_{N_\mathrm{o},2^{N_\mathrm{r}}-1}^{N_\mathrm{r}}\right\rangle - \tilde{q}_{N_\mathrm{o},2^{N_\mathrm{r}}-1}^{N_\mathrm{r}} &= 0.
\end{aligned}
$$

$\longrightarrow$ Better for parallel computing
$\longrightarrow$ Reduction of polynomial order N_o

Stochastic Mixed Finite Element Method (SMFEM)

Consideration of the elliptic part

$$\mathbf{K}^{-1}\lambda^{-1}(S)\,\mathbf{v} + \nabla p = 0 \qquad \text{in } D \times (0,T),$$
$$\operatorname{div}\,(\mathbf{v}) = q \qquad \text{in } D \times (0,T).$$

Basis functions of $\mathbf{v}$ and p based on lowest order Raviart-Thomas (RT0) elements and HSG expansion

$$\mathbf{v}(\mathbf{x},\xi) \approx \Pi^{N_o,N_r}[\mathbf{v}_h] \quad := \quad \sum_{l=0}^{2^{N_r}-1}\sum_{i=0}^{N_o}\sum_{j=1}^{N_e} v_{j,i,l}^{N_r}(\mathbf{x})\boldsymbol{\vartheta}_j(\mathbf{x})\varphi_{i,l}^{N_r}(\xi),$$
$$p(\mathbf{x},\xi) \approx \Pi^{N_o,N_r}[p_h] \quad := \quad \sum_{l=0}^{2^{N_r}-1}\sum_{i=0}^{N_o}\sum_{j=1}^{N_t} p_{j,i,l}^{N_r}(\mathbf{x})\chi_j(\mathbf{x})\varphi_{i,l}^{N_r}(\xi).$$

with N_e number of edges, N_t number of triangles, deterministic velocity and pressure basis functions $\boldsymbol{\vartheta}_j(\mathbf{x})$ resp. $\chi_j(\mathbf{x})$.

Related work of the mixed elliptic problem by

- L. Traverso, T.N. Phillips, Y. Yang (2014)
- O.G. Ernst et al. (2009)

Stochastic Mixed Finite Element Method

This yields a matrix saddle-point problem

$$\begin{bmatrix} \hat{A} & -\hat{B}^T \\ \hat{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{p}} \\ \mathbf{q} \end{bmatrix},$$

with block matrices $\hat{A}$ and $\hat{B}$ given by

$$\hat{A} = \begin{bmatrix} A_{0,0} & \cdots & A_{0,\tilde{P}-1} \\ \vdots & \ddots & \vdots \\ A_{\tilde{P}-1,0} & \cdots & A_{\tilde{P}-1,\tilde{P}-1} \end{bmatrix},$$

and

$$\hat{B} = G \otimes B = \begin{bmatrix} B_{0,0} & & 0 \\ & \ddots & \\ 0 & & B_{\tilde{P}-1,\tilde{P}-1} \end{bmatrix},$$

with $\tilde{P} := (N_\mathrm{o} + 1)2^{N_\mathrm{r}}$ and diagonal, stochastic matrix G,

Stochastic Mixed Finite Element Method

and block vectors

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_{N_\mathrm{o}2^{N_\mathrm{r}}} \end{bmatrix} \in \mathbb{R}^{N_\mathrm{e}(N_\mathrm{o}+1)2^{N_\mathrm{r}}}, \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_{N_\mathrm{o}2^{N_\mathrm{r}}} \end{bmatrix} \in \mathbb{R}^{N_\mathrm{t}(N_\mathrm{o}+1)2^{N_\mathrm{r}}},$$

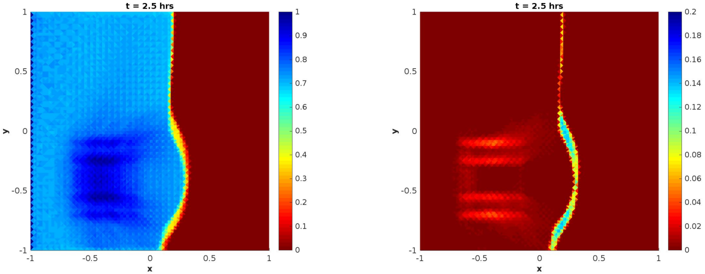
where each vector block is of size $\dim(\mathbf{v}_l) = N_\mathrm{e}$ resp. $\dim(\mathbf{p}_l) = N_\mathrm{t}$ for $l = 0, \ldots, N_\mathrm{o}2^{N_\mathrm{r}}$. Analogous for $\bar{\mathbf{p}}$ and $\mathbf{q}$.

Remark:

- system matrix is not fully assembled
- degrees of freedom: $(N_\mathrm{t} + N_\mathrm{e})(N_\mathrm{o} + 1)2^{N_\mathrm{r}}$.
- appropriate preconditioning necessary

Outline

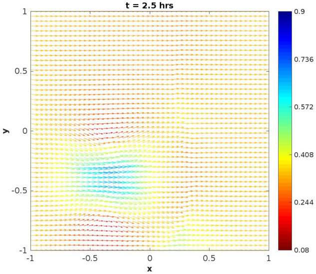
Vertical injection with heterogeneous lense & randomness



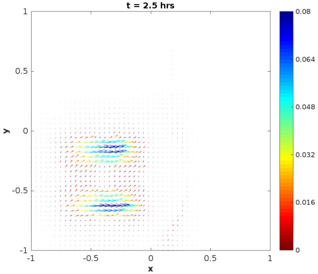
(a) (b)

- **Expectation** (a) and **variance** (b) of the **saturation**,
- $N = 2$ (stoch. dim), $N_{\mathrm{r}} = 1$, $N_{\mathrm{o}} = 1$, (hyp system is 12-dimensional),
- Smeared heterogeneity interface due to uncertainty.

Vertical injection with heterogeneous lense & randomness



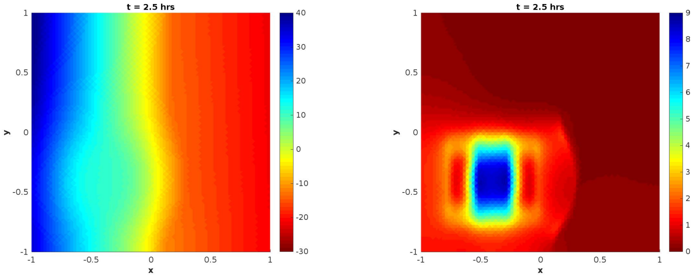
(a)



(b)

- **Expectation** (a) and **diag. of covariance** (b) of the **velocity** $[m/h]$,
- $N = 2$ (stoch. dim), $N_\mathrm{r} = 1$, $N_\mathrm{o} = 1$, (ellipt dim: $12 \cdot (N_\mathrm{t} + N_\mathrm{e})$),
- smooth transition at vertical boundaries of the lense due to uncertainty.

Vertical injection with heterogeneous lense & randomness



(a) (b)

- **Expectation** (a) and **variance** (b) of the **pressure** $[bar]$,
- $N = 2$ (stoch. dim), $N_\mathrm{r} = 1$, $N_\mathrm{o} = 1$, (ellipt dim: $12 \cdot (N_\mathrm{t} + N_\mathrm{e})$).

Model performance

N_r/#cores:	0 / 1	1 / 2	2 / 4	3 / 8	4 / 16	5 / 32
$N_o = 1,\ h$	6.8	8.4	10.0	12.0	14.0	13.5
$N_o = 2,\ h$	20.5	20.7	26.3	27.8	27.1	28.3
$N_o = 3,\ h$	36.8	41.6	53.4	54.5	56.8	51.9
$N_o = 1,\ \varnothing$	28.7	26.6	25.6	22.9	21	19
$N_o = 2,\ \varnothing$	29.2	28	24.5	22.5	20.4	18.7
$N_o = 3,\ \varnothing$	30.5	28	23.9	22.6	20.1	18.9
$N_o = 1,\ \%$	8.5	7	6.4	5.4	5.3	4.3
$N_o = 2,\ \%$	8.7	7.8	6.5	6.4	6.6	4.9
$N_o = 3,\ \%$	10.2	8.8	6.7	6.0	5.5	4.7

Table: Computation time (hours h), average iteration counts ($\varnothing$) and time ratio of all elliptic solves to global solve ($\%$) with preconditioner ($N = 1$).

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Conclusion

- First uncertainty quantification (UQ) for full (hyperbolic-elliptic) two-phase system in 2D based on SG.
- The HSG approach requires a higher dimensional system than SG, but this system is partially decoupled and needs lower polynomial order.
- Effective parallel computations.

Outlook

- Apply HSG to fractured porous media
- Comparsion with other UQ techniques

Thank you for your attention!

Questions