On estimating the algebraic error in numerical PDEs

Jan PAPEŽ

group ALPINES, office A328

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Linear algebraic problem

We are solving the system of linear algebraic equations

$$\mathbf{Ax} = \mathbf{b}$$

Natural questions

- How to measure the error?
- Which solver should be used?
- When should we stop the iterations?

Linear algebraic problem

Where does the system come from? What properties does it have?

How to measure the error?

The problems from which the system originates from have (usually) a natural measure for the error. We should respect this also in the matrix computations.

Which solver should be used?

The choice of the particular method should be justified (rigorously or heuristically) and it should, if possible, be related to the measure of the error.

Complex view

When should we stop the iterations?

An appropriate stopping criterion for iterative algebraic solver cannot be set within the matrix computations only. Effective numerical solution requires comparison of the algebraic error with errors of other origin (discretization error, error of the model, etc.).

Linking various fields

It is necessary to link all the stages of the real-world problem solution.

This is a general requirement!
We will now focus on the numerical solution of the PDEs.

Phases of the solution process in numerical PDEs

Real-world problem

↓ modeling
error of the model

Mathematical model (PDE + BC)

↓ discretization
discretization error

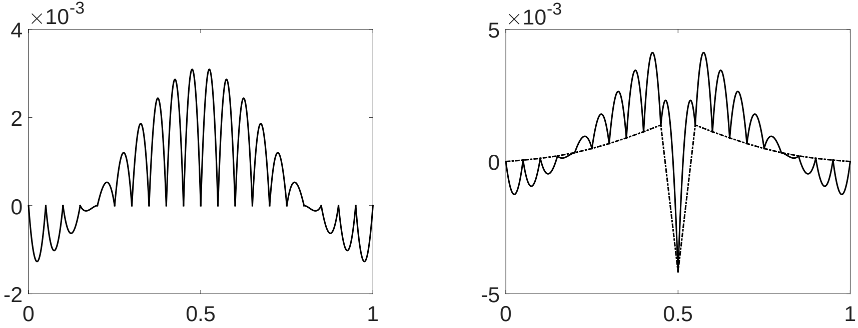
Algebraic problem

↓ algebraic solution
algebraic error (truncation + rounding)

Approximation to the solution

Spatial distribution of the errors of different origin

1D example

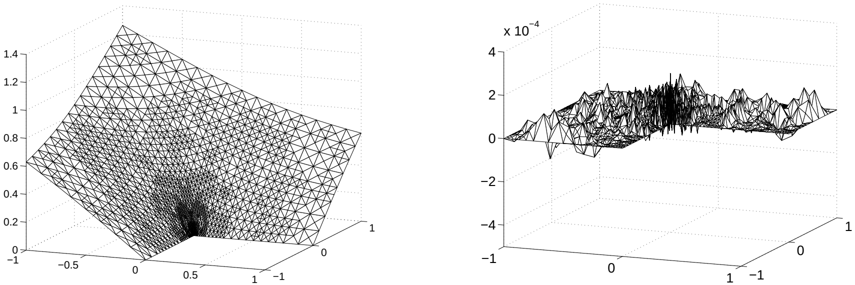


1D Poisson problem, uniform partition with 19 nodes, P1 FEM. Left: discretization error $u - u_h$. Right: algebraic error $u_h - u_h^9$ (dashed-dotted line) and total error $u - u_h^9$ (solid line).

$$\|u_h - u_h^9\|_a = 1.23 \times 10^{-3} < 6.81 \times 10^{-3} = \|u - u_h\|_a$$

Spatial distribution of the errors of different origin

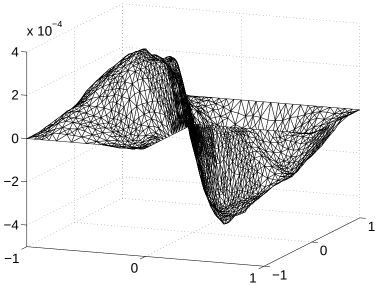
L-shaped domain (2D)

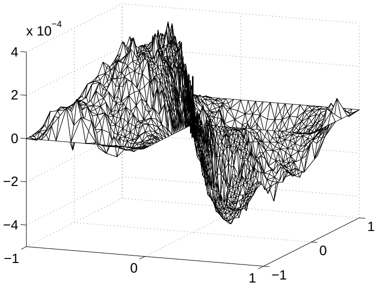


Exact solution u (left) and the discretization error $u - u_h$ (right) in the Poisson model problem on the L-shaped domain.

Spatial distribution of the errors of different origin

L-shaped domain (2D)

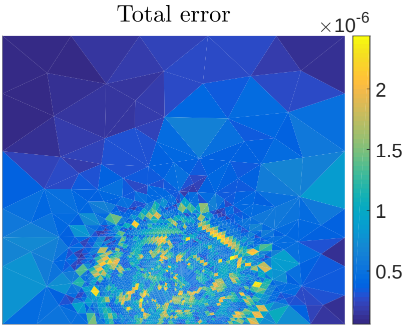


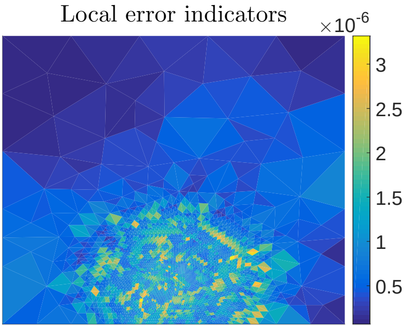


Algebraic error $u_h - u_h^i$ (left) and the total error $u - u_h^i$ (right). Here

$$\|u_h - u_h^i\|_a < 0.1\,\|u - u_h\|_a.$$

Estimating the total error using flux reconstruction

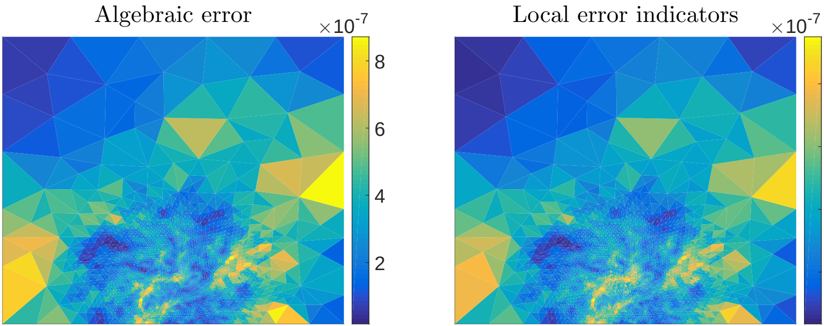




Elementwise distribution of the total error (left) and the local error indicators (right).

This technique allows to provide guaranteed (global) upper bounds on the total and algebraic errors, and the indicators to estimate the local distribution of the errors.

Estimating the algebraic error using flux reconstruction



Elementwise distribution of the algebraic error (left) and the local error indicators (right).

However, in this case, many additional algebraic iterations are necessary to evaluate the error indicators!

Adaptive mesh refinement

One step of the procedure is

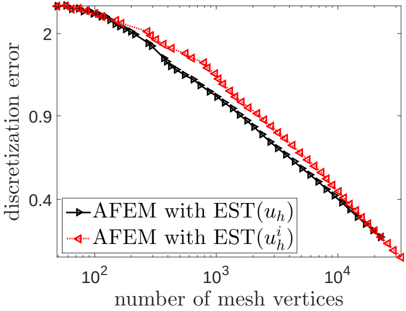
$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

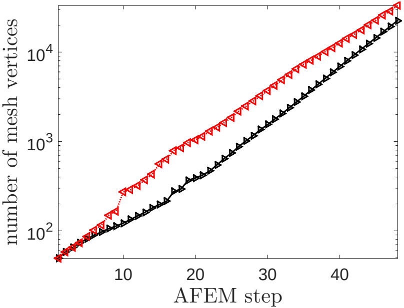
Most of the results in the literature are based on the assumption that the SOLVEr is exact, i.e., no algebraic error is allowed.

For ESTIMATing the (discretization) error, often a simple and cheap *residual-based error estimator* is considered.

What happens if we plug in a computed approximation instead of the (unavailable) exact algebraic solution?

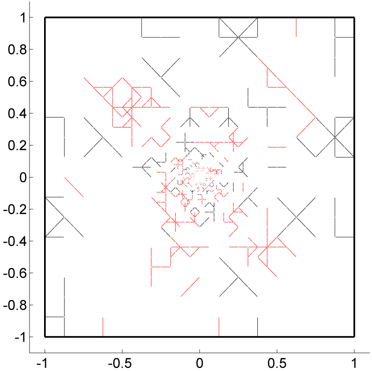
Adaptive mesh refinement based on $\mathrm{EST}(u_h^i)$

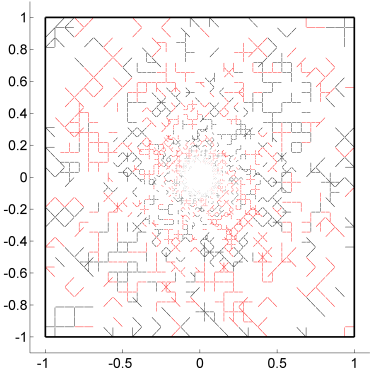




Left: the decrease of the discretization error norm in adaptive FEM that is based on $\mathrm{EST}(u_h)$ (black) and $\mathrm{EST}(u_h^i)$ (red), respectively. Right: the corresponding number of degrees of freedom in refinement steps.

Adaptive mesh refinement based on $\mathrm{EST}(u_h^i)$





The difference of the adaptively refined meshes after 35, respectively 48 refinement steps.

Conclusions and summary

- Solution of the algebraic problem should be considered an indivisible part of the overall solution process.
- In problems stemming from real-world applications, the exact algebraic solution is unaffordable.
- Algebraic error can significantly affect the computed approximation.
- The techniques used for numerical solution of PDEs must incorporate the algebraic error, or clearly state that they are based on assumptions that are not valid and that they may provide inaccurate results.
- There are still many challenges ahead of us.

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Thank you for your attention