Formalizing Asymptotic Complexity Claims via Deductive Program Verification

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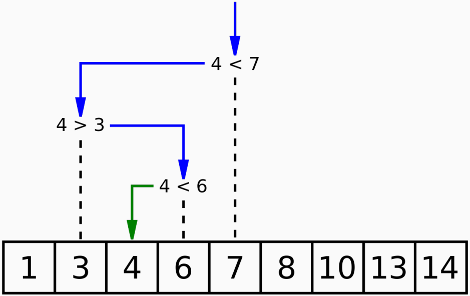
Gallium

Formalizing **Asymptotic Complexity** Claims via Deductive Program Verification

Recall our undergrad algorithm courses...

"Is the value 4 present in this sorted array?"



"Binary search finds the element in time $O(\log n)$"

Homework: implement binary search

```
(* Requires arr to be a sorted array of integers.
   Returns k such that i <= k < j and arr.(k) = v
   or -1 if there is no such k. *)
let rec bsearch (arr: int array) v i j =
  if j <= i then -1 else
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```
# bsearch [|1;3;4;6;7;8;10;13;14|] 4 0 9;;
- : int = 2
```

It works! We could even prove that it **always** works.

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But there is a complexity bug…

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Complexity bugs can be critical

`http://ocert.org/advisories/ocert-2011-003.html`

"Denial of Service via Algorithmic Complexity Attacks", S. Crosby, D. Wallach

One of the things we do at Gallium...

Machine-checked proofs of programs

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Machine-checked proofs of programs

including their algorithmic complexity

Formalizing Asymptotic Complexity Claims via Deductive Program Verification

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Formal verification is a set of techniques for:

- Writing a wishlist about a program (aka specification)
- Checking the program against this wishlist ("is the program correct?")

Formalizing Asymptotic Complexity Claims via Deductive Program Verification

Deductive verification

From the code of the program and the specification, deduce a set of proof obligations, and try to prove those.

- Automated proofs: FramaC (C), Verifast (C, Java), Infer (C, C++, Obj-C, Java), Why3 (OCaml)
- Interactive proofs, using “proof assistants”: Coq, Isabelle

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- The program does not leak secrets (e.g. for crypto primitives)
- **The program does not use too much time**
- The program does not use too much space, network bandwidth...

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Asymptotic time guarantees

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- Not "binary search terminates in less than 5ms"
- Rather "binary search runs in $O(\log n)$ steps"

Typical paper proofs rely on informal reasoning principles – which can easily be abused

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let rec bsearch arr v i j =
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- $j - i \leq 0$: line 2 is $O(1)$. OK!

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- $j - i \leq 0$: line 2 is $O(1)$. OK!
- $j - i > 0$: $O(1)$ (l.3-5) $+ O(1)$ (l.6) $+ O(1)$ (l.8) $= O(1)$. OK!

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"By induction on $j - i$" …but which statement are we proving?

$\forall n,\ \exists c,$" `bsearch` costs c" $\neq$ $\exists c,\ \forall n,$" `bsearch` costs c"

Reasoning about O in a proof assistant

Using a proof assistant steers us clear of these abuses...

but maybe also from the simplicity of paper proofs.

Formally, what are we trying to prove?

"`bsearch arr v i j` runs in $O(\log(j-i))$ steps."

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"there exists a cost function $f \in O(\log n)$ such that
for every `arr`, `v`, `i`, `j`,
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First step of the proof: exhibit a concrete cost function?

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Concrete cost function?

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Concrete cost function? $2\log(j-i)+1$? $3\log(j-i)+4$?

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- Interactive proofs (using Coq)
- Convince Coq to postpone the moment where the concrete cost function is provided
- Start proving the program and its invariants without knowing the cost function
- At the same time, infer the cost function from the code of the program

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```
cost (j-i) = 1 + …
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cost (j-i) = 1 + (if (j-i) <= 0 then … else …)
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               if (j-i) <= 0 then 0 else
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                           (cost ((j-i) - (j-i)/2 - 1))
                 )
           )
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```
cost n     = 1 + (
               if n <= 0 then 0 else
                 0 + 1 + max 0 (
                   1 + max (cost (n/2))
                           (cost (n - n/2 - 1))
                 )
             )
```

To finish the proof

Solve this equation, and prove that $cost(n)$ is $O(\log n)$: by hand, or using the "Master theorem".

Conclusion

Machine-checked proofs of the asymptotic complexity of programs.

- Implemented as a Coq library, to verify OCaml programs
- Similar approach implemented in Isabelle at TUM (Munich)
- The approach could be applied to other languages (eg. C, C++, Java)