Long time stability of Markov and branching processes: a pedagogical introduction.

INRIA – Séminaire des doctorants

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CERMICS - ENPC, INRIA & Labex BEZOUT

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Outline

1. What are we talking about ? What for ?

2. Markov chains and their stability

3. Branching processes

1. What are we talking about ? What for ?

A classical description of matter

In classical mechanics, a common way of describing a system at a fixed temperature T is to consider:

- a **set of atoms** with positions x;
- a **potential energy** given by a Hamiltonien $H(x)$;
- a **force** F acting on the system, typically typically $F(x) = -\nabla H(x)$;
- a thermal agitation at temperature T, generally represented by a **random noise**.

Mathematical formulation

In practice, the latter elements are resumed in the following time evolution, for $n \in \mathbb{N}$:

$$x_{n+1} = x_n + F(x_n) + \sigma G_n,$$

where:

- n corresponds to time;
- $F(x_n)$ is the force at location x_n;
- σ is the intensity of the noise, proportional to $\sqrt{T}$;
- G_n is a sequence of Gaussian random variables.

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When $F = -\nabla H$, we have

$$x_{n+1} = x_n - \nabla H(x_n) + \sigma G_n,$$

hence x_{n+1} «reduces» the energy H in average.

Some motivations and questions

This basic framework can be used to model many different systems:

- molecules, chemical reactions;
- metals, phase transition;
- surface interactions, etc.

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The dynamics of the system is encoded in the transition kernel P, for a position x and a set S,

$$P(x, S) = \mathbb{P}\left[x_{n+1} \in S \,\middle|\, x_n = x \right],$$

that is **the probability of reaching a set** S starting from x. There are natural questions about such systems:

- their long time behavior (Sec. 2.);
- the probabilities of fluctuations (Sec. 3.);
- decorrelation times, linear response theory, etc.

2. Markov chains and their stability

What is stability?

Importance of the long time behavior for the computation of averages of functions φ (like a pressure), as:

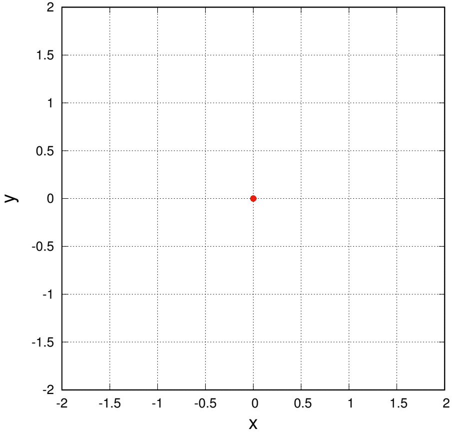
$$\frac{1}{N_{\mathrm{iter}}} \sum_{n=0}^{N_{\mathrm{iter}}-1} \varphi(x_n) \xrightarrow[N_{\mathrm{iter}} \to +\infty]{} \text{average of } \varphi.$$

For this, an important condition to hold is stability. By this, we mean somehow that the system does not get lost at infinity. Two useful applications:

- more physical insight into the considered system;
- stability of numerical schemes.

A basic example

Brownian motion over $\mathbb{R}^2$: $F = 0$, so (x_n) is a sum of $2d$-Gaussian random variables.



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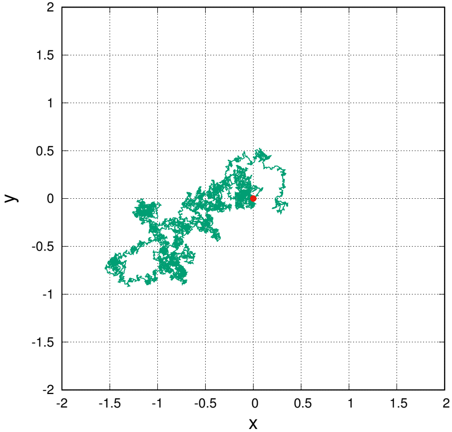

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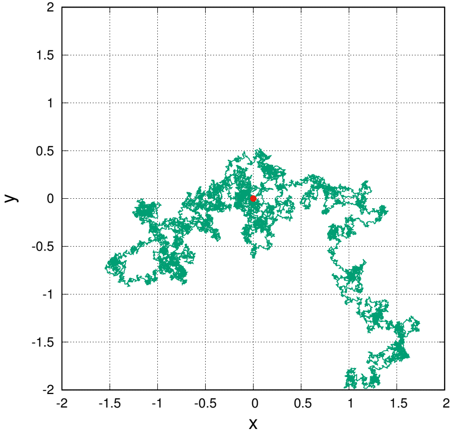
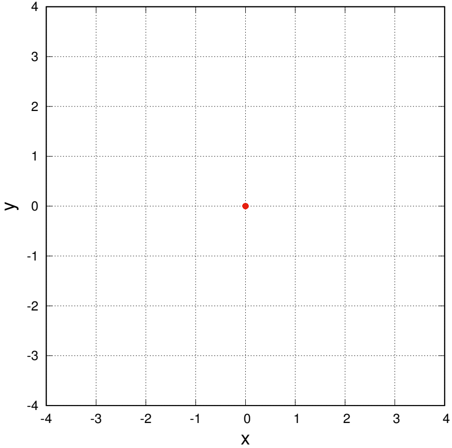


The process starts at $(0,0)$... and goes away without restriction.

Partially confined particle

Dynamics over $\mathbb{R}^2$ with $F(x;y) = (-5x;0)$.



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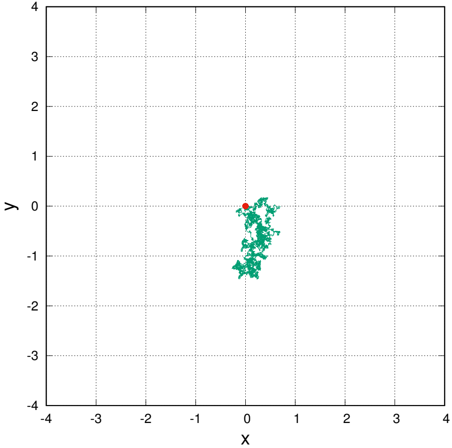

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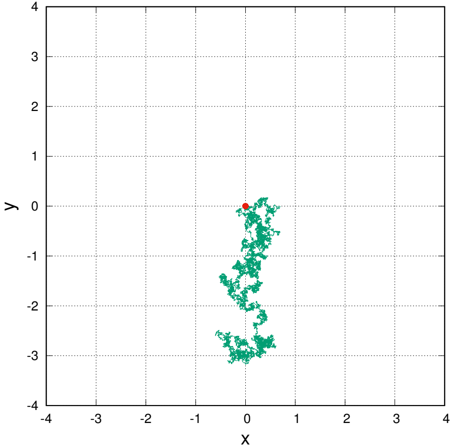

The process is confined in the x direction... but moves without restriction in the y direction.

Ornstein-Uhlenbeck process

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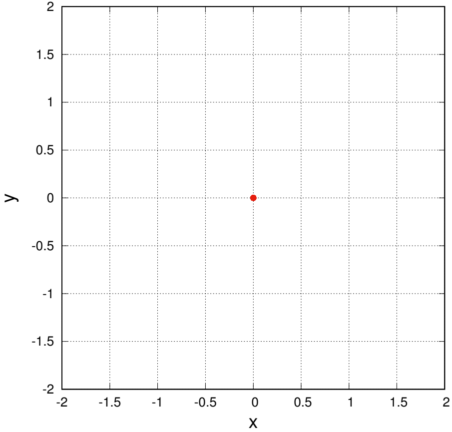

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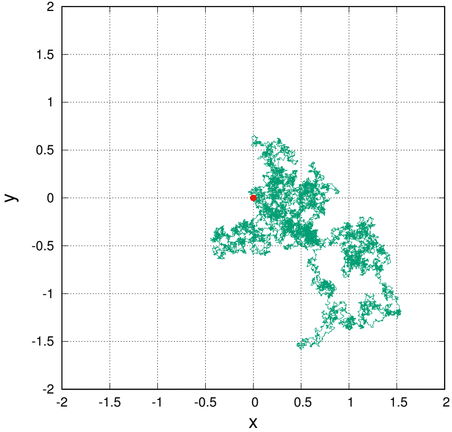

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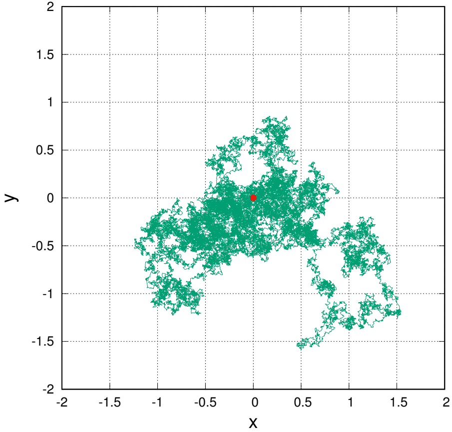

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Hand waving theorem

How can we understand this stability ?

Hand waving theorem

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A possible solution: show the existence of an energy that decreases in average.

Idea of theorem

If there exists a function $W \geqslant 0$ and constants $\gamma \in (0,1)$, $b \geqslant 0$ such that

$$PW(x) \leqslant \gamma W(x) + b,$$

and $W(x) \to +\infty$ as $|x| \to +\infty$, then the dynamics is stable.

Back to the Ornstein-Uhlenbeck process

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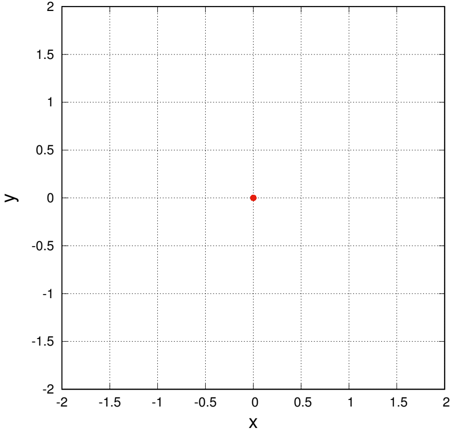

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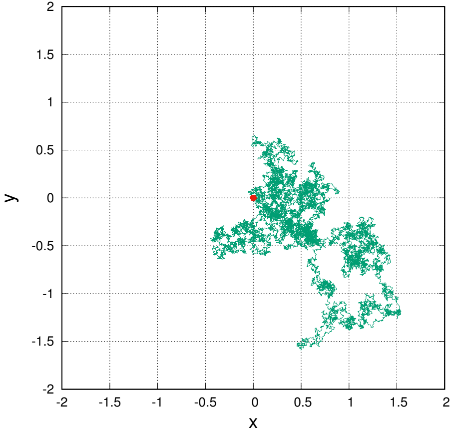

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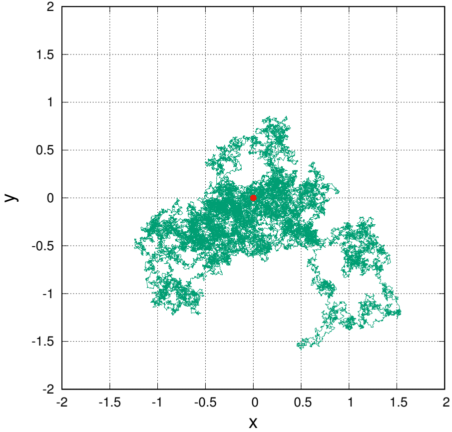

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In practice, one can choose

$$W(x,y) = H(x,y),$$

the energy of the system.

3. Branching processes

What is a branching process?

A branching system consists of:

- a set of M replicas $(x^m)_{m=1}^M$;
- each replica follows a Markovian dynamics;
- each replica m is assigned a weight w_m depending on its past trajectory;
- the total weight is $\bar{w} = \sum_{m=1}^M w_m$;
- one computes a probability of surviving $p_m = w_m / \bar{w}$;
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In practice, the weight over k steps of the m^{th} replica is computed with

$$w_m^k = \exp\left(\sum_{i=0}^{k-1} V(x_i^m)\right).$$

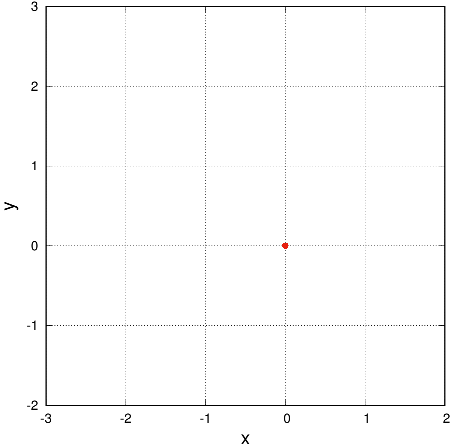
Some motivations

There are (at least) three motivations for studying such dynamics:

- in quantum physics;
- in rare event simulation;
- in filtering.

A quantum mechanics example

Brownian dynamics (x_n) over $\mathbb{R}^2$ with $V(x) = -|x|^2$.



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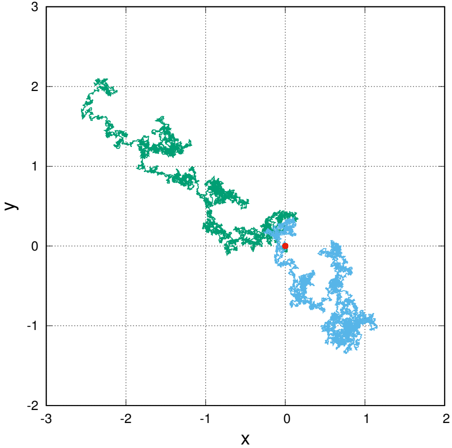

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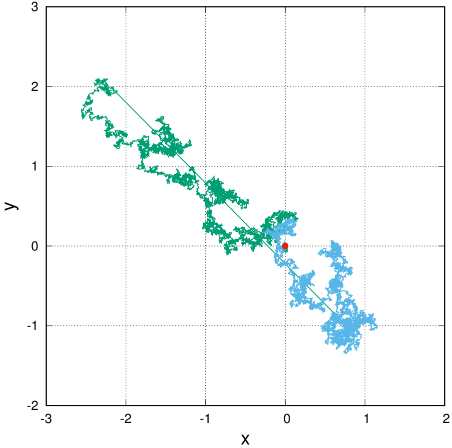

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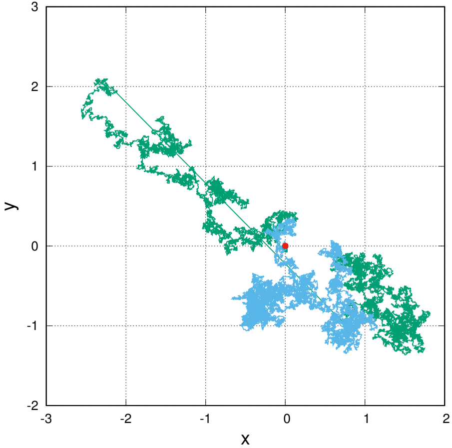

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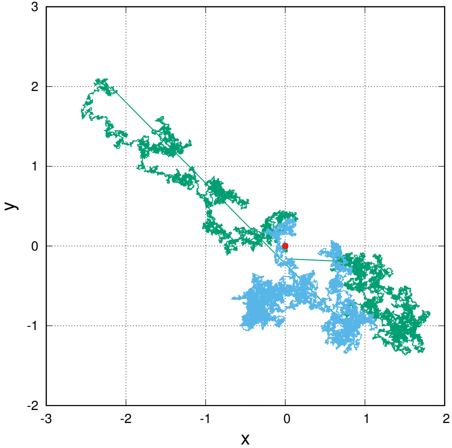

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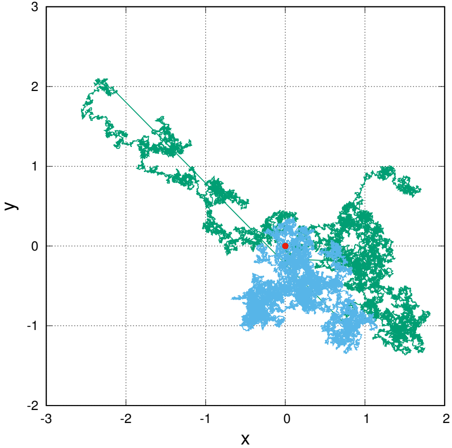

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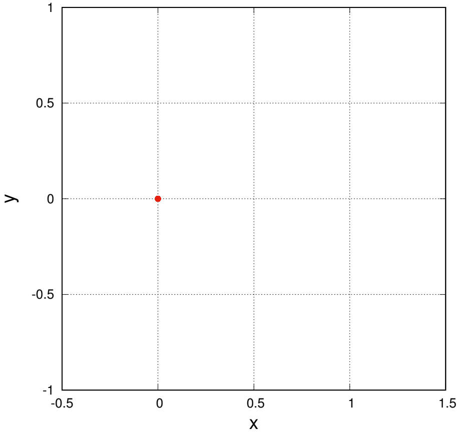
Conclusion: there is a confinement by selection.

A rare event example

Ornstein-Uhlenbeck dynamics with $F(x,y) = (-2x, -2y)$. Weight function $V(x,y) = x$.



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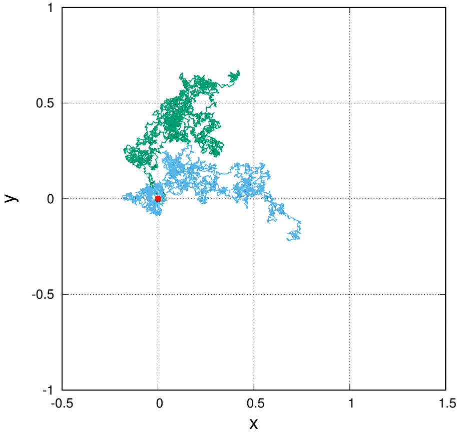

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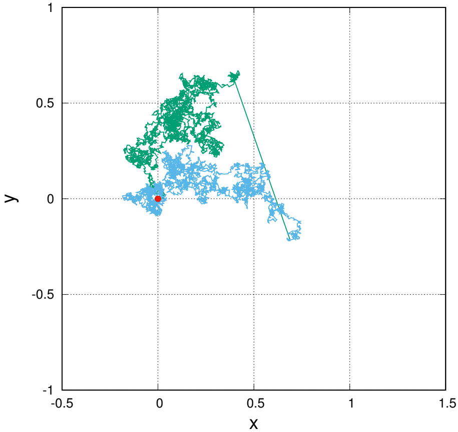

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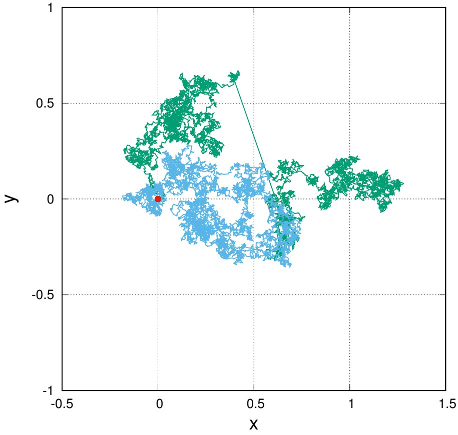

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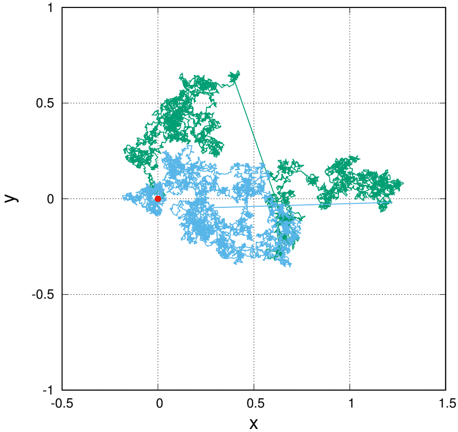

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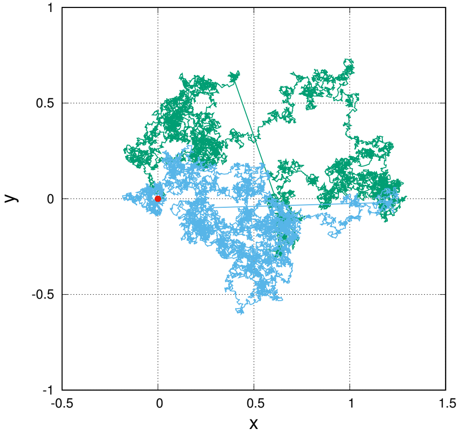

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Conclusion: the particles are selected towards the right.

Some heuristic

First observations:

- there are two phenomena: the dynamics itself and the selection rule;
- the confinement can be provided by the dynamics or by the selection;

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Two natural questions:

- unstable Markov dynamics (first example) $\rightarrow$ stable branching process ?
- stable Markov dynamics (stable example) $\rightarrow$ unstable branching process ?

Hand waving new theorem

Condition of stability based on the decrease of some energy ? The answer is yes [G.F., M. Rousset, G. Stoltz, in prep.]. For this, define

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Question: what the ❄ does that mean?

Hand waving of hand waving etc

Idea: the energy decrease must take into account the weight V.

Hand waving of hand waving etc

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Result in a nutshell

More generally,

- if $V(x) \to -\infty$ as $|x| \to +\infty$, the dynamics is **always stable**, since replicas always end up dying;
- if not, there is a trade-off: too many kids away may **blow up to your face**.

Conclusion

Take home message:

- random systems for modeling physics;
- problems of stability, long time behavor and energy;
- branching processes, a **funny problem** (as far as mathematics can be funny);
- new results, «energy» for branching processes.