How to classify things?

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Let's go back to primary school!

Problem 1

Léa buys six cacti. Each cactus costs eight euros. How much does she pay?

Problem 2

I rent a spacious apartment in the center of Paris. It is three meters wide and four meters long. What is the surface of my apartment?

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To solve these two very different concrete problems, we use the same abstract mathematical notions: numbers and multiplication.

Theorem

Every positive integer can be decomposed, in an essentially unique way, as a product of prime numbers.

To prove this, we use:

Lemma (Existence of the Euclidean division)

For every $a \in \mathbb{N}$ and $b \in \mathbb{N}^*$, there exist $q \in \mathbb{N}$ and $0 \leqslant b < r$ with $a = bq + r$.

Recall that a polynomial is an expression of the form $P(X) = a_n X^n + \ldots + a_1 X + a_0$, and that its degree is $\deg(P) = n$. A polynomial P is irreducible if for every Q, R such that $P(X) = Q(X)R(X)$, then either Q or R is constant.

Mathematics are a story of structures

First example: Euclidean rings

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Every nonzero polynomial can be decomposed, in an essentially unique way, as a product of irreducible polynomials.

To prove this, we use:

Lemma (Existence of the Euclidean division)

For every polynomials A and $B \neq 0$, there exist polynomials Q and R with $A = BQ + R$ and $\deg(R) < \deg(B)$.

The proof of the theorem from the lemma is exactly the same in both cases.

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The set of integers, and the set of polynomials, are particular cases of Euclidean rings. From this fact follow immediately the two previous theorems.

- Vectors of the plane can be summed and multiplied by a real number.

  If R is a rotation of the plane, $\vec{u}$ and $\vec{v}$ two vectors, and λ a real number, then $R(\vec{u}+\vec{v}) = R(\vec{u}) + R(\vec{v})$, and $R(\lambda\vec{u}) = \lambda R(\vec{u})$.

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- $\mathcal{C}^\infty$ functions $\mathbb{R} \longrightarrow \mathbb{R}$ can be summed and multiplied by a real number.

  We can derivate these functions. If f and g two such functions, and λ a real number, then $(f+g)' = f' + g'$, and $(\lambda f)' = \lambda f'$.

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We can define the abstract notion of a vector space, a set with two operation: an internal addition and a multiplication by scalar numbers; and the notion of a linear mapping, a mapping between vector spaces that preserves these operations. The two last examples are particular cases of vector spaces.

Isomorphism

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- $(\mathbb{R}, +)$ and $((0, +\infty), \times)$ are isomorphic, because the exponential function maps bijectively $\mathbb{R}$ to $(0, +\infty)$ and it "transforms" the addition into the multiplication.
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If we prove a theorem for one of these structure, then it is also true for the other one.

What is classification?

Classifying a certain type of structures, it's finding a list of structures $\mathcal{L}$ of this type such that:

- the structures in $\mathcal{L}$ are pairwise non-isomorphic;
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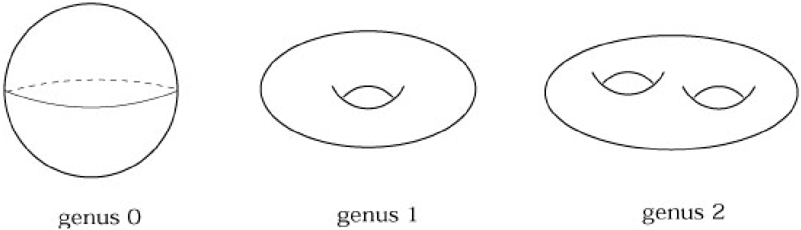
Sometimes we can study weaker notions of equivalence between structures than isomorphism.

Examples of classification

First example: orientable compact surfaces



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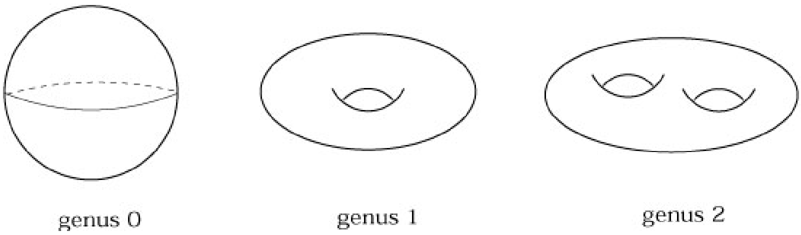

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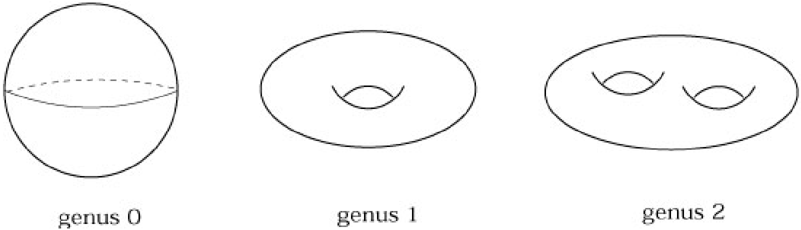

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They can be classified by their genus, i.e. the number of holes. Two surfaces are isomorphic if and only if they have the same genus.

- Finite-dimensional vector spaces (over $\mathbb{R}$) can be classified by their dimension. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

Examples of classification

Example 2: finite-dimensional vector spaces

- Finite-dimensional vector spaces (over $\mathbb{R}$) can be classified by their dimension. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

- All standard Borel spaces with a non-atomic probability measure are isomorphic. Thus, when you want to prove a result on these spaces, you only need to prove it on you favourite example, $[0, 1]$ with the Lebesgue measure.

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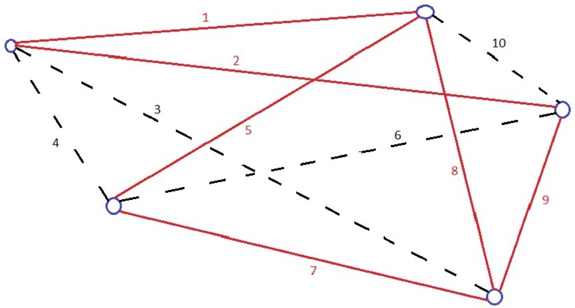
In general, classifying a class of structures means associating to each structure a characteristic (which is a real number or a sequence of real numbers), such that two structures are isomorphic iff they have the same characteristic.

An example of bad classification

A countably infinite graph is an infinite graph whose vertices can be enumerated by integers. In this case, pairs of vertices can also be enumerated by integers.



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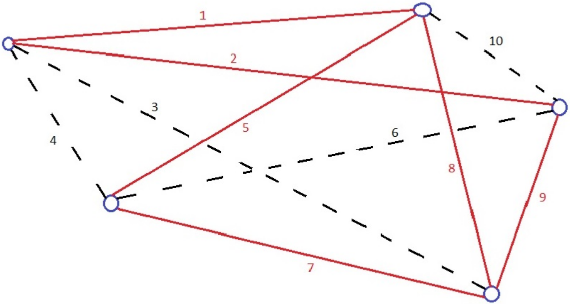

To such a graph, we can associate a real number between 0 and 1 in the following way: its n^{th} digit (in base 3) is 1 if the n^{th} pair of vertices is linked by an edge, and 0 otherwise.

For this graph we get $0,1100101110\ldots$

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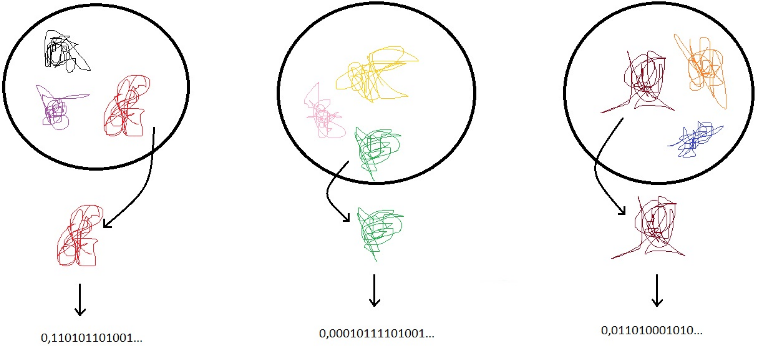

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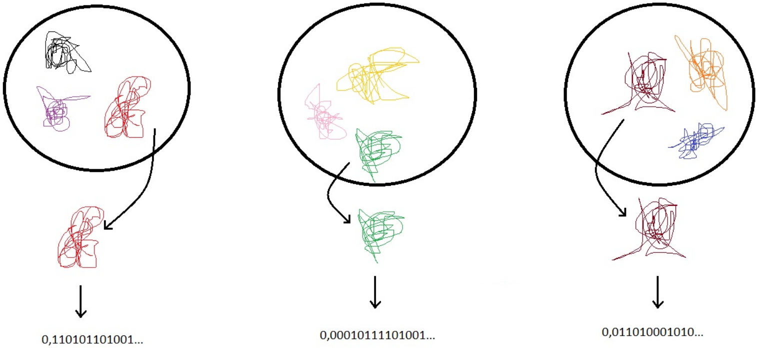

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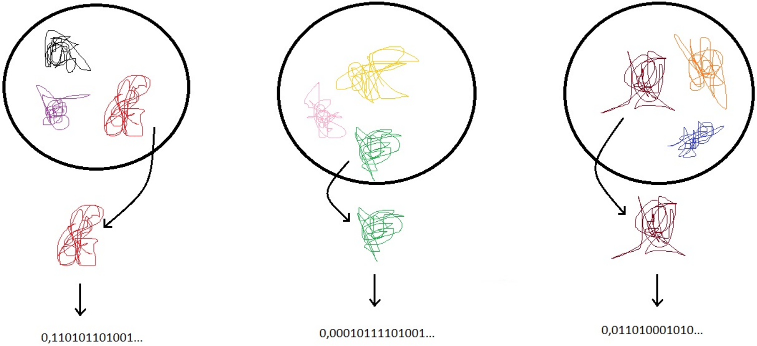

1. Gather graphs in isomorphism classes;
2. Pick one graph in each class;
3. Associate to the whole class the number of this graph.

NO DON'T DO THAT, THIS IS BAD :-(((((

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A formalism

To prove something, we need to formalise it.

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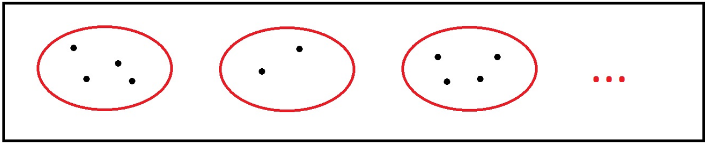

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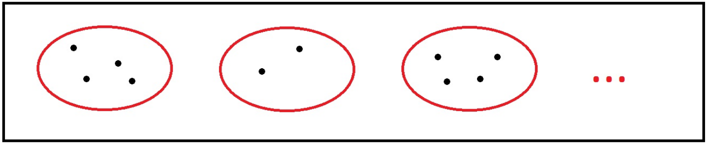

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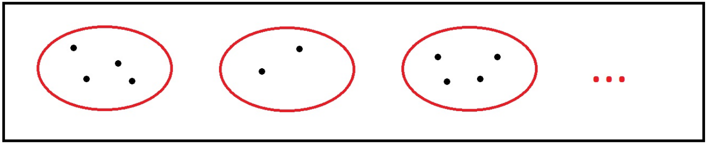

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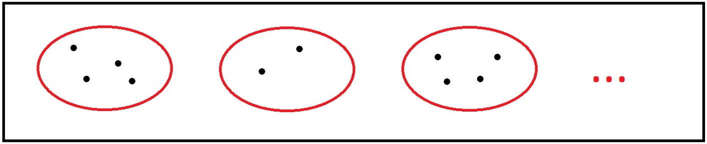
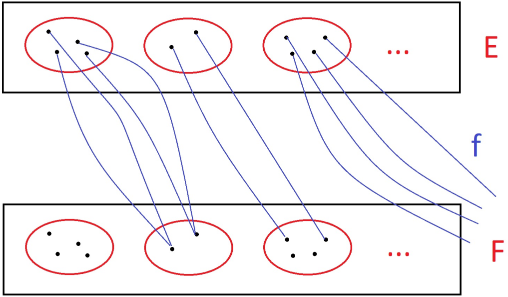


Here, we consider equivalence relations on standard Borel spaces. These are spaces where we can define a continuous analogue of calculability. The morphisms between these spaces are called Borel mappings; they can be seen as computable functions, in a continuous way. Classes of infinite structures can often be endowed with a structure of standard Borel space.

A formalism

We say that an equivalence relation E on a standard Borel space X is reducible to an equivalence relation F on a standard Borel space Y if there exists a mapping $f : X \to Y$ that maps each class of E to exactly one class of F.



The idea is that if we know F, then we can compute E.

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There exists an equivalence relation E_0 which is not reducible to the equality on $\mathbb{R}$. We say that two reals number x and y are E_0-equivalents if and only if they have the same writing in basis 2 from some rank.

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Theorem (Louveau – Rosendal, 2001)

The isomorphism relation between countably infinite graphs is analytic complete, that is, every analytic equivalence relation can be reduced to it. In particular, it is not reducible to the equality on the real numbers.

Thank you for your attention!

"C'est bien plus beau lorsque c'est inutile !"

"It's much more beautiful when it's useless!"

Cyrano de Bergerac, acte V, scène 6