Some multi-scale techniques to solve elliptic equations with highly oscillating coefficients

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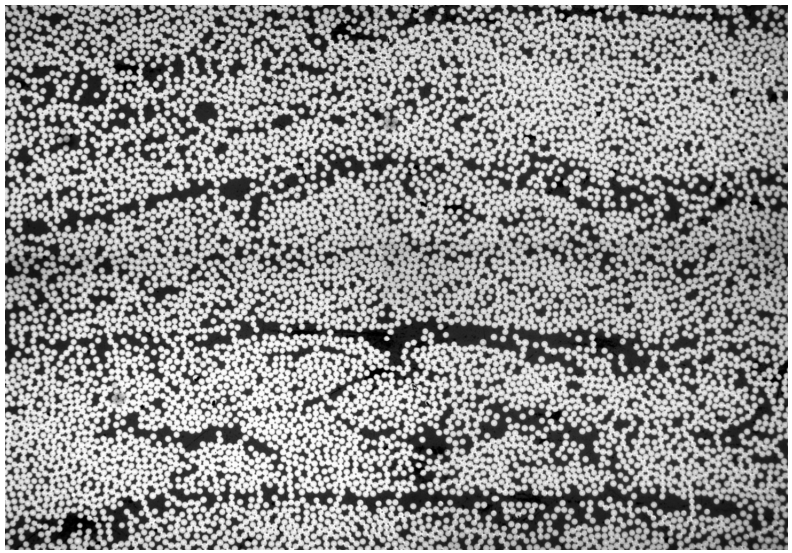
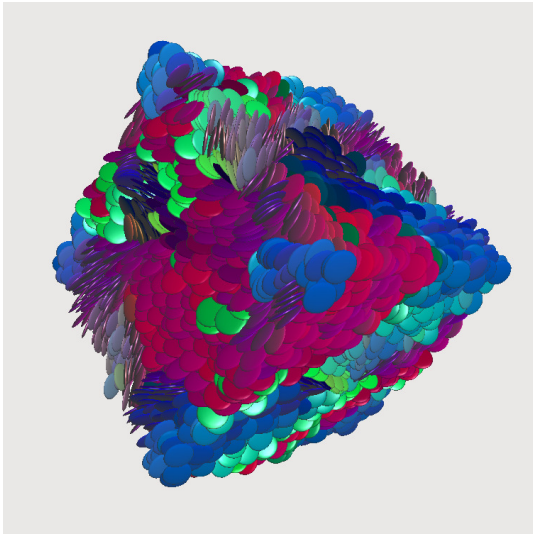
Outline

Motivation

More and more complex materials are used in industry:





Courtesy M. Thomas (Airbus) and S. Brisard (ENPC)

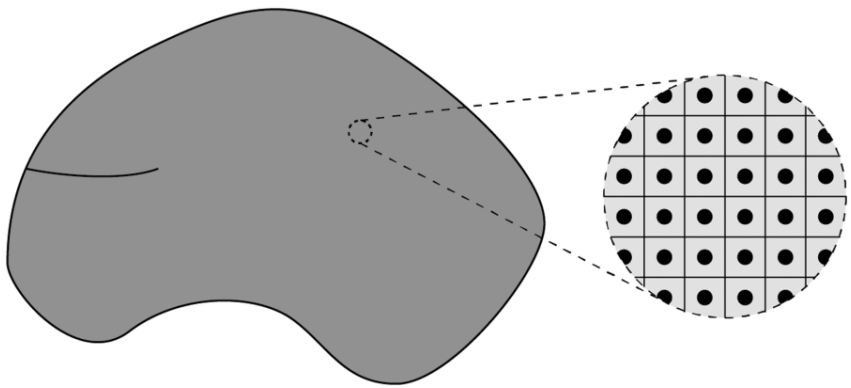
For such complex materials => need to efficiently compute the electrical, temperature or mechanical response to some input.

Problem

Goal: Compute u_ε, the response of a material whose properties are defined by A_ε.

Elliptic equation with highly oscillating coefficients

$$\begin{cases} -\operatorname{div}\left[A_\varepsilon(x)\,\nabla u_\varepsilon(x)\right] = f(x) \quad \text{in } D, \\ u_\varepsilon(\cdot) = 0 \quad \text{on } \partial D. \end{cases}$$



where $D \subset \mathbb{R}^d$, $\varepsilon \ll \operatorname{diam}(D)$, and the matrix A_ε is elliptic, symmetric, bounded and is varying at the small scale ε.

Variational approach

The problem

Equation with highly oscillating coefficients

$$\begin{cases} -\operatorname{div}\left[A_\varepsilon(x)\,\nabla u_\varepsilon(x)\right] = f(x) \quad \text{in } D, \\ u_\varepsilon(\cdot) = 0 \quad \text{on } \partial D. \end{cases}$$

is equivalent to

Variational formulation

Find $u_\varepsilon \in H_0^1(D)$ such that

$$a_\varepsilon(u_\varepsilon, v) = b(v), \quad \forall v \in H_0^1(D)$$

where

$$\begin{cases} a_\varepsilon(u,v) = \int_D (A_\varepsilon(x)\nabla u(x)) \cdot \nabla v(x)\, dx, \\ b(v) = \int_D f(x)v(x)\, dx. \end{cases}$$

Galerkine approach

Instead of solving the variational formulation on $H_0^1(D)$
-> Solve the problem on a finite dimensional space $V_H \subset H_0^1(D)$.

Galerkine variational problem

Find $u_H \in V_H$ such that

$$a_\varepsilon(u_H, v_H) = b(v_H), \quad \forall v_H \in V_H$$

Denoting $\{\phi_i^H\}_{\{i=1..N_H\}}$ a basis of V_H, it is equivalent to solve the linear system

Matrix problem

Find $U \in \mathbb{R}^n$ such that

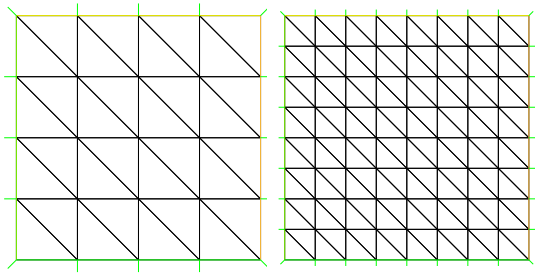
$$KU = B,$$

with $K_{i,j} = a_\varepsilon(\phi_i^H, \phi_j^H)$, and $B = b(\phi_i^H)$.
The Galerkine approached solution: $u_H(x) = \sum_{i=1}^{N_H} U_i \phi_i^H(x)$

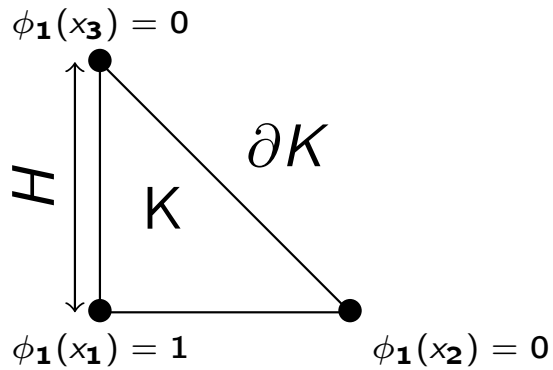
Finite Element Method (FEM) - 1

Finite Element Method (P1):

1. Mesh the domain D in N_K elements: edges (1D), triangles (2D) or tetrahedron (3D)



2. Define $\phi_{i\{i=1..Nb_{nodes}\}}$ such that : $\phi_i(x_j) = \delta_{i,j}$ and ϕ_i is linear on each element K.



3. Write Galerkine approach for $V_H = \mathrm{Span}(\phi_{i\{i=1..Nb_{nodes}\}})$
4. Solve the system to have $u_{FEM} = \sum_{i=1}^{Nb_{nodes}} U_i \phi_i(x)$

Finite Element Method (FEM) - 2

Figure : 2D FE basis for node $x_i = (0, 1)$ (left) and for node $x_j = (1/4, 3/4)$ (right)

Computational cost: solve linear system of size $O(1/H^2)$

Approximation result

Denoting H the maximal diameter of the elements, we have $\|u_\varepsilon - u_{FEM}\|_{H^1(D)} \leq CH\|u_\varepsilon\|_{H^2(D)}$

Toy example in 1D

1D Toy problem

$$\begin{cases} -\left(a_\varepsilon(x)u'_\varepsilon(x)\right)' = 1 \quad \text{in } [0,1], \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases}$$

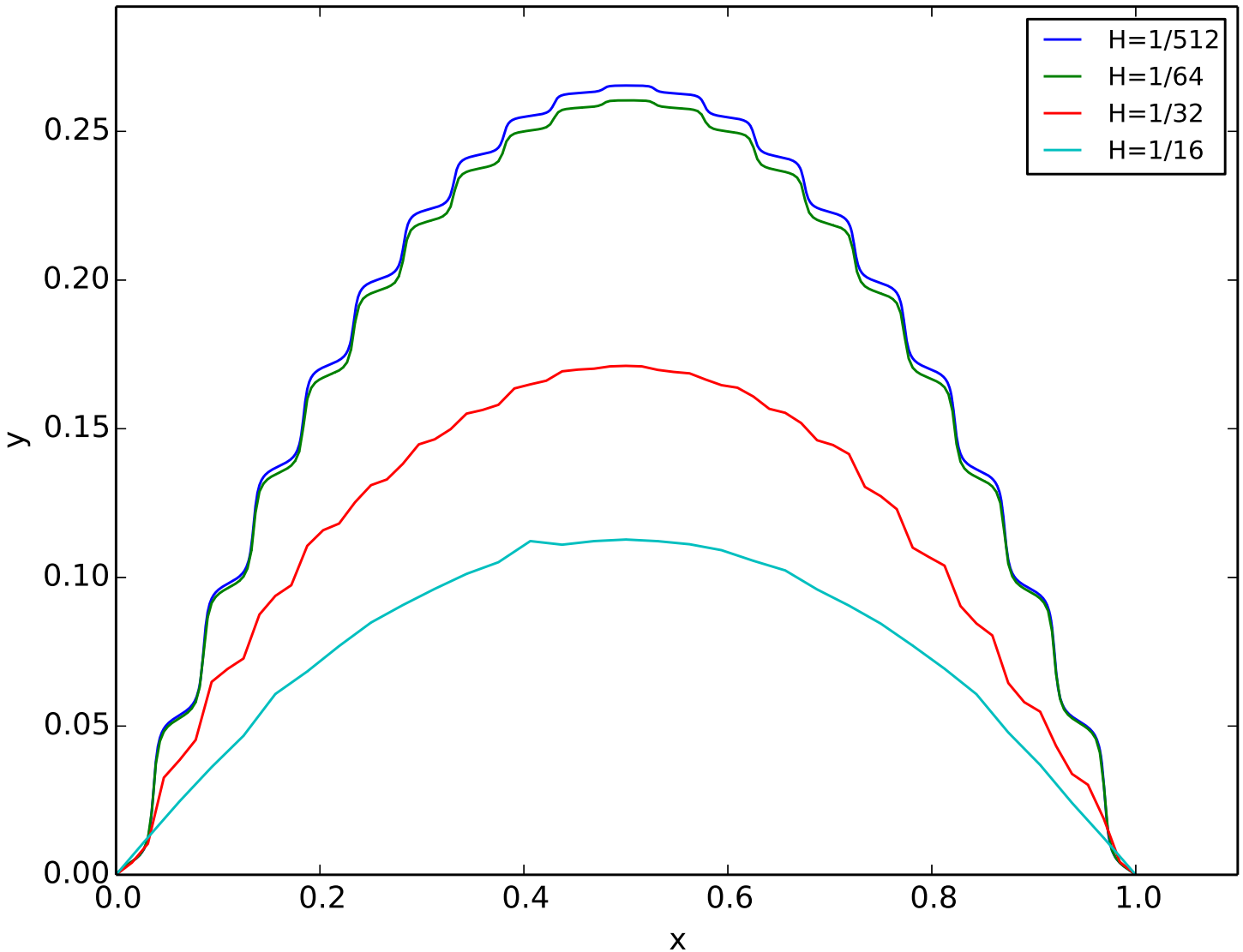
with $a_\varepsilon(x) = a(\frac{x}{\varepsilon}) = 1.1 + \sin(\frac{x}{\varepsilon})$.

Is the FEM efficient when coefficient a_ε is highly oscillating ?

Numerical experiment:

- Set $\varepsilon = \frac{1}{128}$ -> u_ε is periodic with $T_\varepsilon \simeq \frac{1}{20}$
- Compute FEM method for $H = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$
- Compare to FEM reference solution obtained for $H = \frac{1}{512}$

Oscillating problem in 1D - FEM Results



Limitations of FEM method for equation highly oscillating coefficients

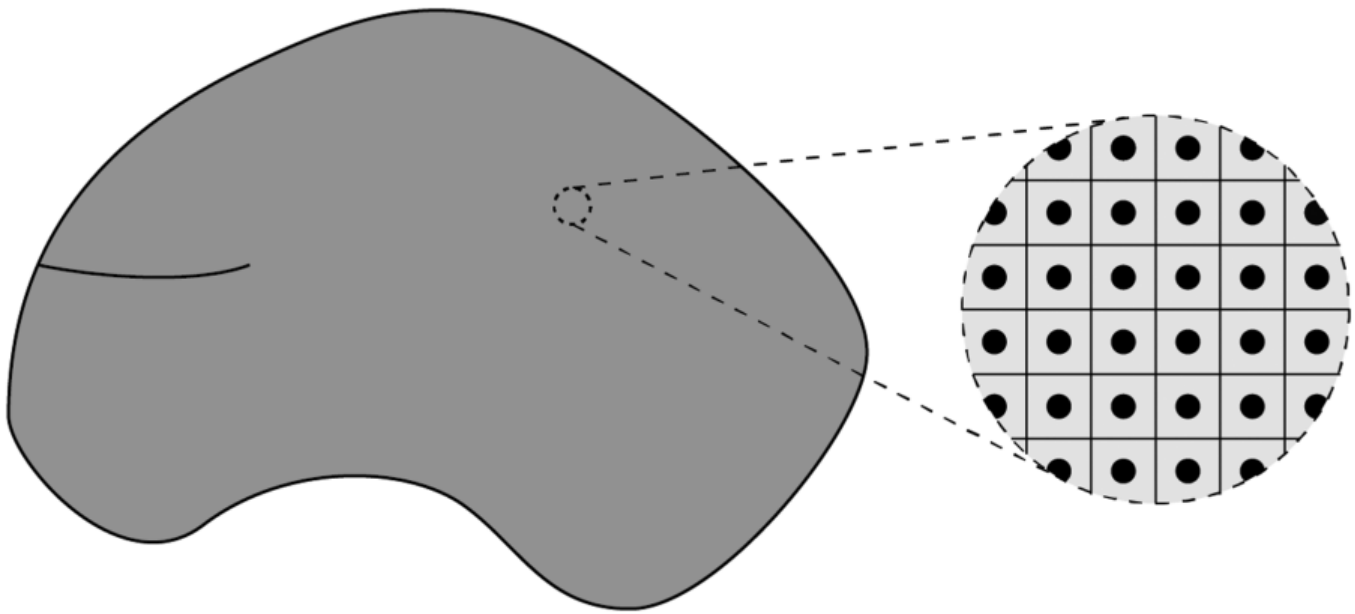
- FEM performs well in the regime $H \ll \varepsilon$ when oscillations are captured, however when $\varepsilon \ll 1$, the computational cost can be too expensive.
- In the regime $H \gg \varepsilon$, the FEM approximation is not accurate at all because it fails to encrypt the oscillations of the coefficient.
- Need to find methods that can encrypt the knowledge of the micro-structure of the coefficient to get more accurate approximations for an acceptable computational cost.

Multi-scale methods: bottom up approach

In our case, we apply a bottom-up approach: we have some micro-scale knowledge of the problem and we want to use it to compute solutions accurately at coarser scale.

Multi-scale methods: Different ways to exploit this knowledge to enrich and design accurate approximations at the macro-scale.

Usually, local computations at a fine scale are used to improve global accuracy on a coarser scale.



Some examples of multi-scale methods

Some examples of such methods:

- Homogenization techniques [Bensoussan, Lions and Papanicolaou, 1978],
- Multi-scale Finite Element Method (MSFEM) [Hou and Wu, 1997]
- The Heterogeneous Multi-scale Method (HMM) [E, Engquist, 2003]
- Decomposition domain methods ...

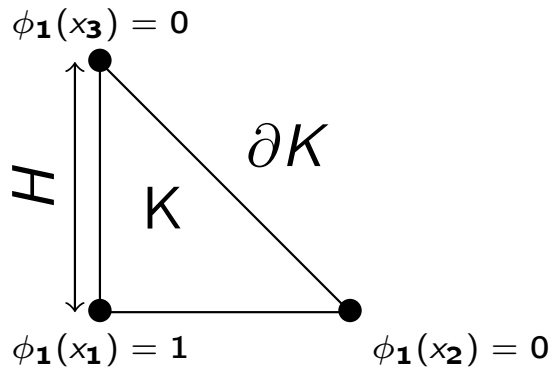
We will focus only on the MsFEM method.

Review on MsFEM method - 1

Let us consider a coarse mesh (mesh size H) of D

- Offline step: Create a nodal basis (similar to the Finite elements) on each element K adapted to the problem.

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla \phi_i) = 0 \quad \text{in } K \\ \phi_i \text{ is linear on the edges} \\ \phi_i(x_j) = \delta_{i,j} \end{cases}$$



- Online step: Solve the coarse Galerkine problem for any source term f

$$a_\varepsilon(u_{MsFEM}, v) = < f, v >, \quad \forall v \in V_{MsFEM} = \mathrm{Span}(\{\phi_i\})$$

Then we have $u_{MsFEM} = \sum_{j=1}^{J} U_j \phi_j$

Approximation result, periodic case [Hou and Wu, 1999]

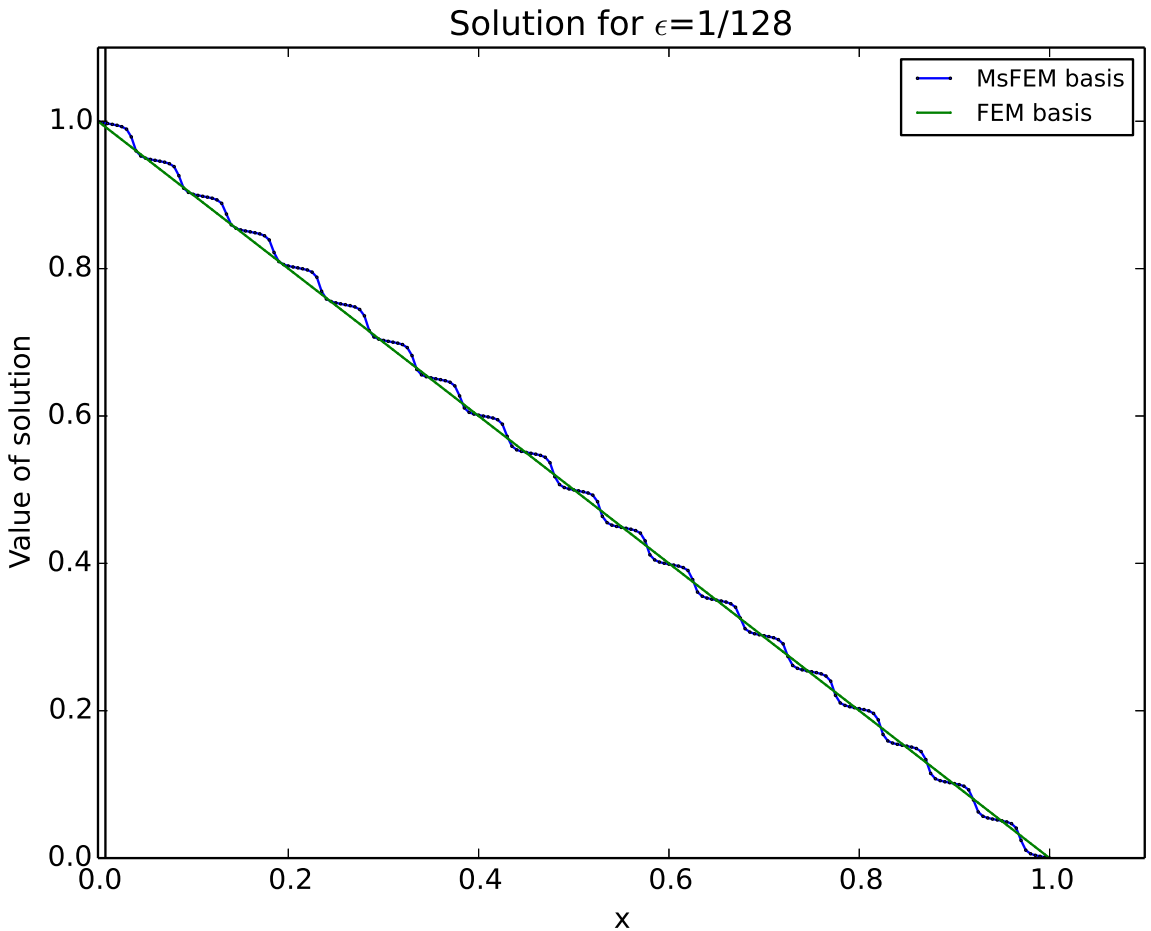
$$\|u_\varepsilon - u_{MsFEM}\|_{H^1(D)} \leq C \left(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}} \right)$$

Review on MsFEM method - 2

- The MsFEM basis ϕ_i is defined by local problems that can be solved in parallel (usually solved with FEM on a finer embedded grid)
- The offline stage is independent from the source term f, hence you can use the same basis for multiple f.
- The Stiffness matrix $K_{i,j} = a_\varepsilon(\phi_i, \phi_j)$ is precomputed to speed up the online phase.
- Online phase cost: assemble the right-hand side term $B_i = b(\phi_i)$ and solve the coarse linear system $KU = B$.

Example of MsFEM basis in 1D

We consider the former 1D example with a coefficient $a_\varepsilon(x) = a(\frac{x}{\varepsilon}) = 1.1 + \sin(\frac{x}{\varepsilon})$, and we compare the basis of Finite Elements with the MsFEM basis



2D example - Reference solution

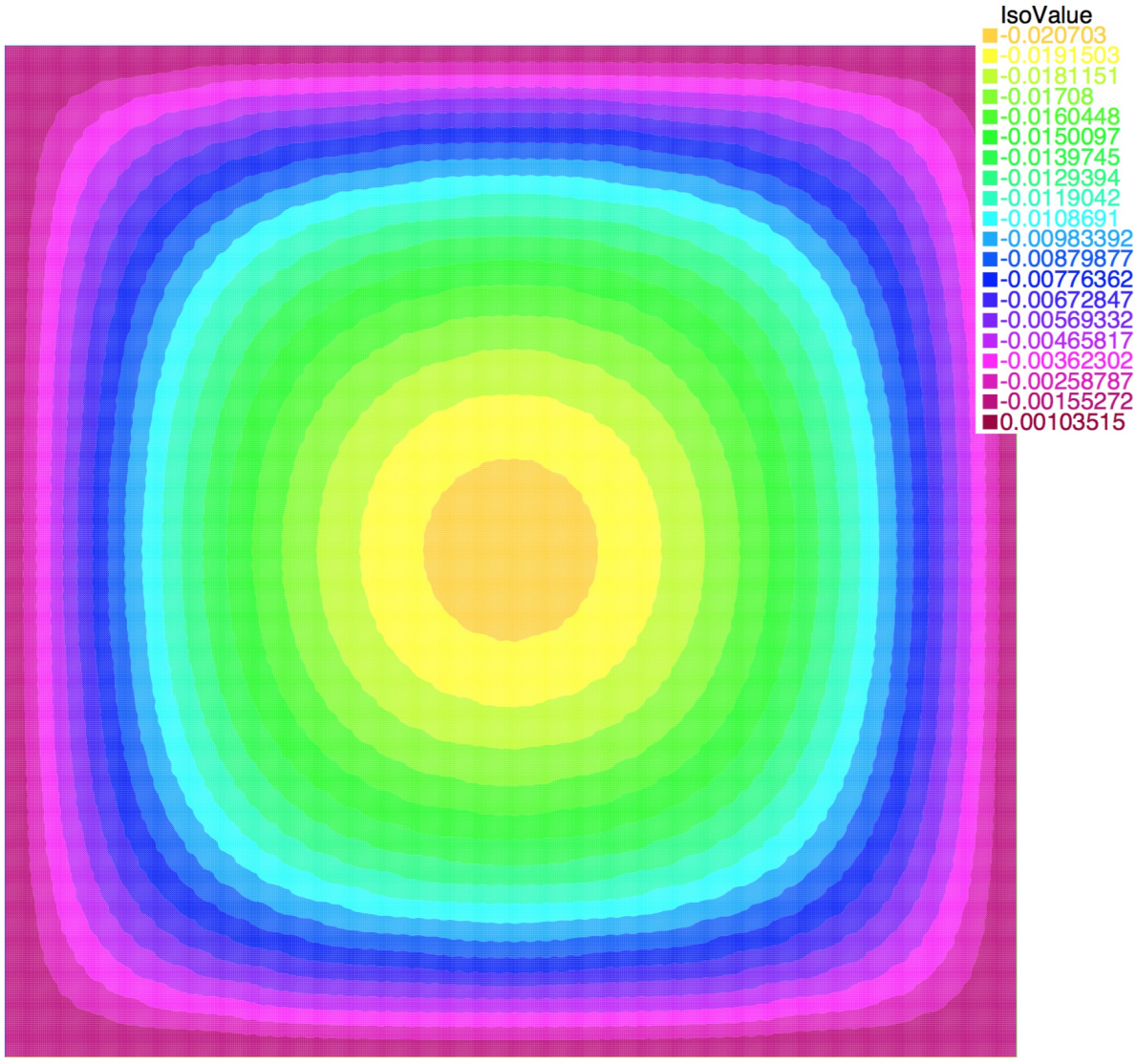


Figure : Q1-FEM solution H=1/512

2D example - Q1 solution

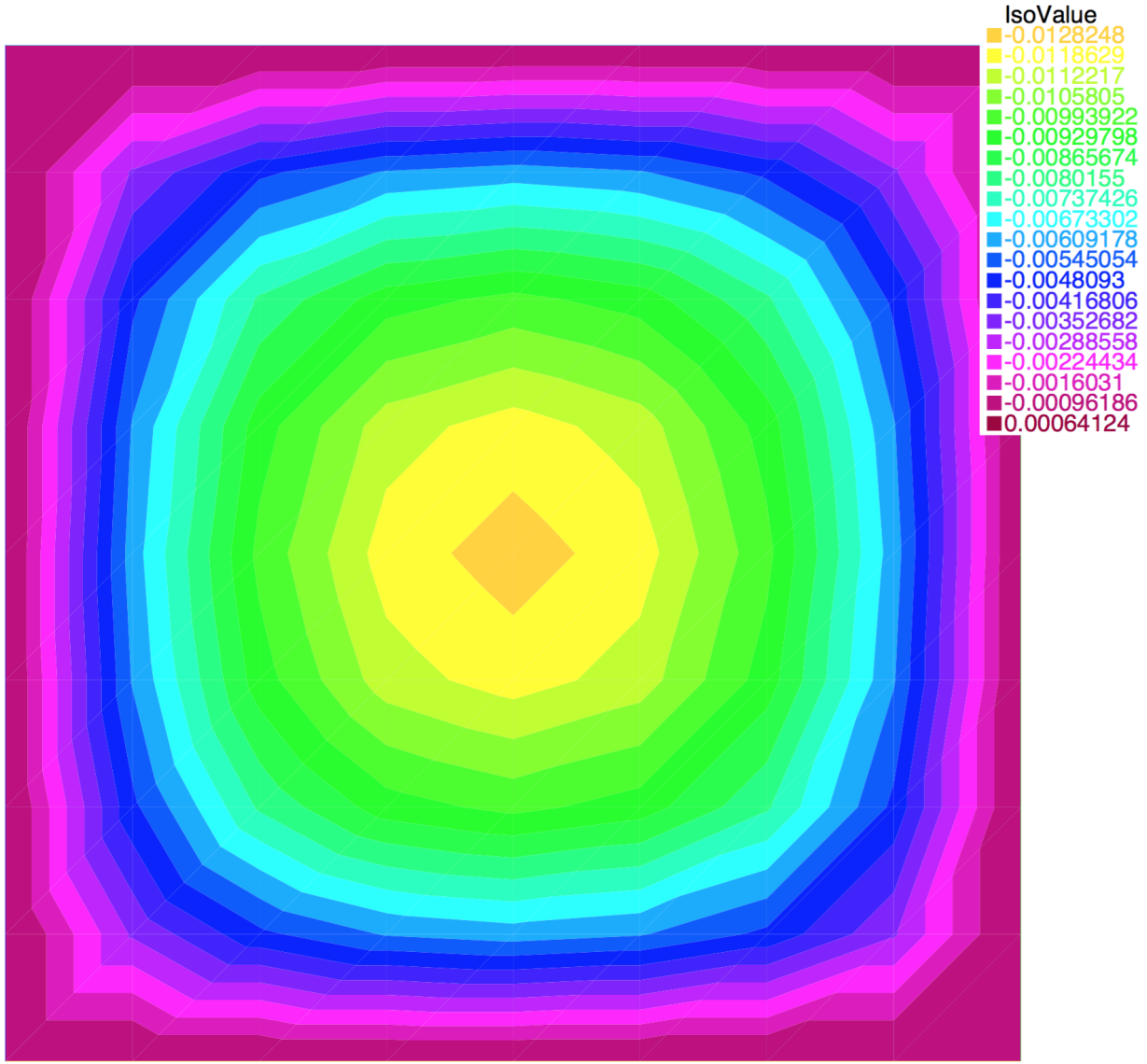


Figure : Q1-FEM solution H=1/8

2D example - MSFEM solution

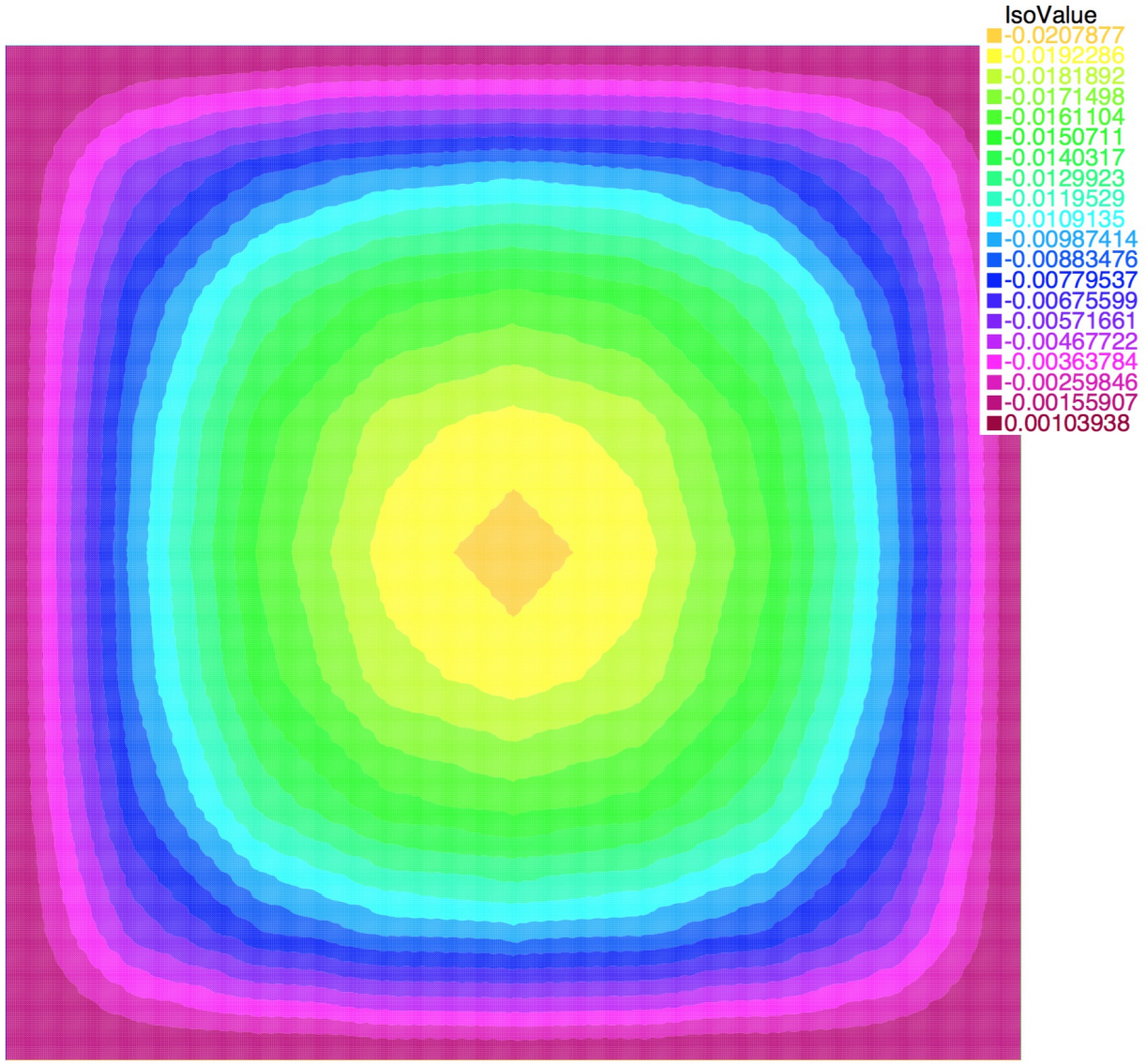


Figure : MsFEM solution H=1/8

The "resonance" regime

Recalling the result

Approximation result, periodic case [Hou and Wu, 1999]

$$\|u_\varepsilon - u_{MsFEM}\|_{H^1(D)} \leq C\left(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\right)$$

In the regime $H \simeq \varepsilon$, the error bound can be huge and seems to be sharp as shown in the following numerical experiments.

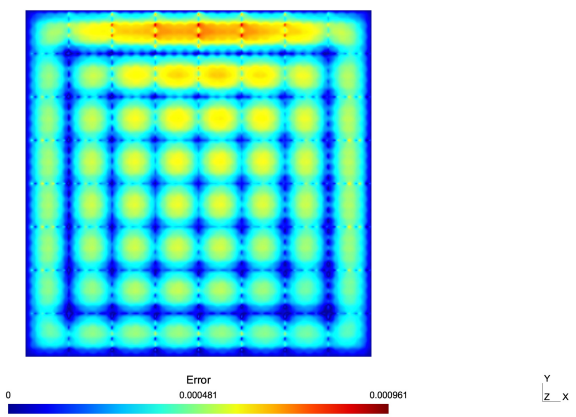


Figure : MsFEM error map
$H = 4\varepsilon = 1/8$

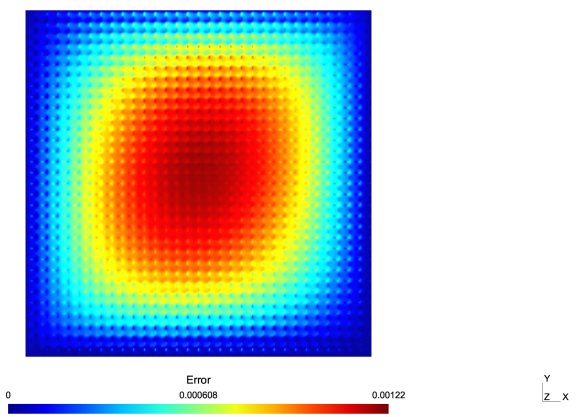


Figure : MsFEM error map
$H = \varepsilon = 1/32$

Pros and cons of the method

Pros:

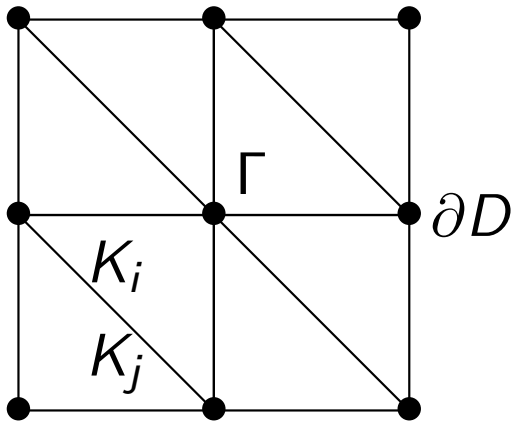
- Offline step can be parallelized as only local problems are considered
- Good behavior for coarse discretization H compared to FEM
- Conformal approach (the solution is continuous)
- Useful when computation have to be repeated for multiple source term f (optimization process, inverse problem...)

Cons:

- Resonance regime -> large error when $H \simeq \varepsilon$
- The fine scale resolution h has to be chosen carefully ($\frac{\varepsilon}{10}$)
- Less useful if only one computation to do

MsFEM enriched method - Setting

Consider the coarse mesh of size H



with Γ the interface between elements K_i.
We want an approximation of the solution to

Variational formulation

Find $u_\varepsilon \in H_0^1(D)$ such that

$$a_\varepsilon(u_\varepsilon, v) = b(v), \quad \forall v \in H_0^1(D)$$

MsFEM enriched method - Motivation: Orthogonal decomposition - 1

$a_\varepsilon(u, v)$ -> defines a scalar product Hence the following orthogonal decomposition:

$$H_0^1(D) = \{\oplus\{V_i\}_{i=1..NbTri}\} \oplus V_\Gamma$$

- V_i: space of functions in $H_0^1(K_i)$
- V_Γ: space of functions in $H_0^1(D)$ such that $a_\varepsilon(u, v) = 0$ for all $v \in H_0^1(K_i)$ that are not vanishing on the interface Γ

Define the energy error norm:

$$E(u_\varepsilon - u) = a_\varepsilon(u_\varepsilon - u, u_\varepsilon - u),$$

We have the solution $u_\varepsilon = u_\varepsilon^K + u_\varepsilon^\Gamma$ -> decomposition of the error between bubble terms in V_i and interface terms V_Γ

MsFEM enriched method - Motivation: Orthogonal decomposition - 2

Bubble a priori error

$E(u_\varepsilon^K) \leq C\|f\|_{L^2(D)}H^2$ with C independent of ε

Problem: when $H = \mathrm{Diam}(K) \searrow$ then $\mathrm{Diam}(\Gamma) \nearrow$
-> Interface error do not converge in H independently of ε.

We recall that $u_{MsFEM} \in V_\Gamma$

$$E(u_\varepsilon - u_{MsFEM}) = E(u_\varepsilon^K) + E(u_\varepsilon^\Gamma - u_{MsFEM})$$

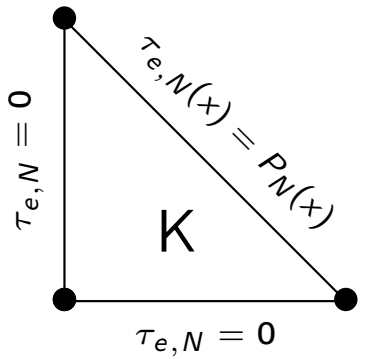
$E(u_\varepsilon^\Gamma - u_{MsFEM})$ -> Error term responsible for resonance effect

Idea -> Enrich the *MsFEM* approximation only in the V_Γ space

Construction of MsFEM enriched basis

- Offline step: Consider the same nodal basis ϕ_j. Compute the edge enrichments $\tau_{e,k}$ for all $1 < k \leq N$ such that on each element K containing the edge e

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla \tau_{e,N}) = 0 \text{ in } K \\ \tau_{e,k} = P_k \text{ on } e \\ \tau_{e,k} = 0 \text{ on } \partial K \setminus e \end{cases}$$



with P_k the k^{th} Legendre polynomial

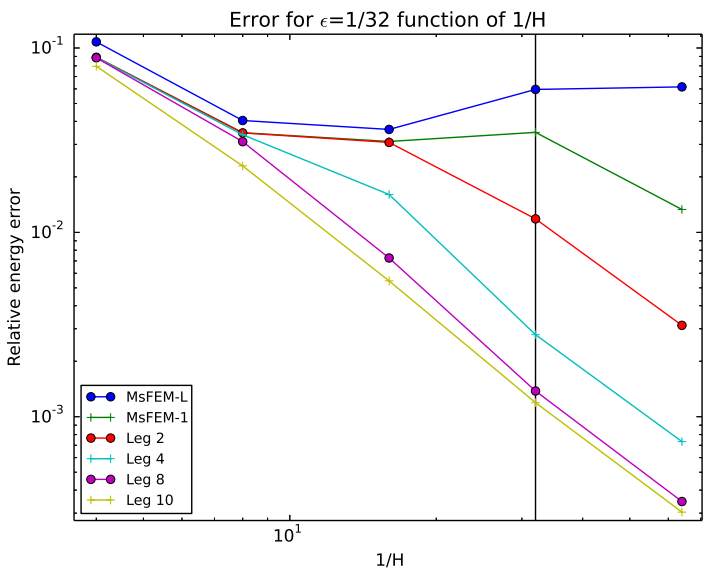
- Online step: Solving the Galerkine problem with the basis $V_{MsFEM,N} = \mathrm{Span}(\{\{\phi_j\}_{j=1..\#Nodes}, \{\tau_{e_i,l}\}_{i=1..\#edges, l=1..N}\})$

$$a_\varepsilon(u_{MsFEM,N}, v) =< f, v >, \quad \forall v \in V_{MsFEM,N}$$

Interface a priori error

$$E(u_\Gamma^\varepsilon - u_{MsFEM,N}) \leq C \frac{H^2}{N^2} \| u_\varepsilon^\Gamma \|_{H^2}$$

$\| u_\varepsilon^\Gamma \|_{H^2} \simeq \frac{C}{\varepsilon^2}$ -> when $H \simeq \varepsilon$ adjust polynomial degree N to decrease resonance error



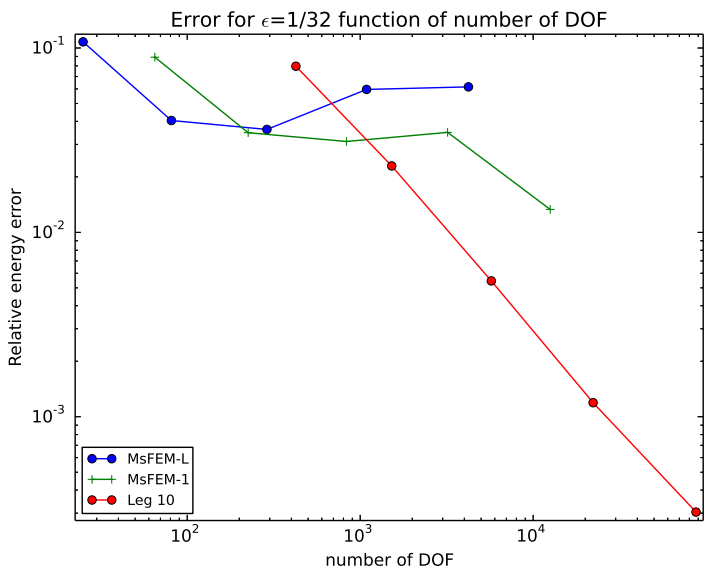


Figure : Error (as a function of the coarse discretization H (left) and the number of DOF (right)) for linear MsFEM (MSFEM-L) and MsFEM enriched with polynomials of degree N ($Leg\ N$)

Error maps at the resonance

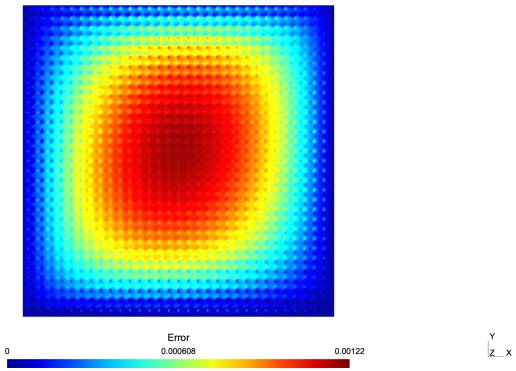


Figure : MsFEM solution and gradient error maps
$H = \varepsilon = 1/32$

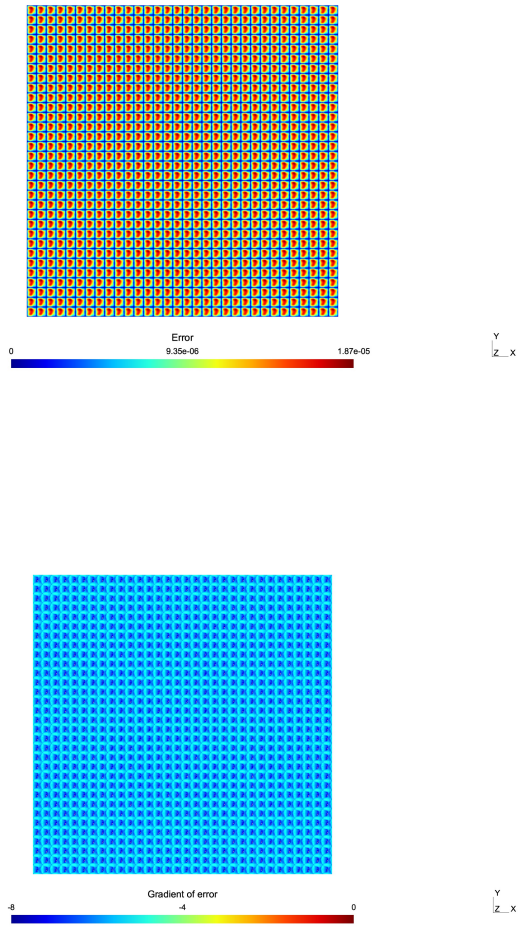


Figure : Legendre 10 solution and gradient error maps
$H = \varepsilon = 1/32$

Conclusions and perspectives

- Summary:
  - Design of an enriched MsFEM method with Legendre polynomials -> convergence proved with an a priori error bound
  - Numerical experiments show a cancellation of the resonance error with N large enough and the error is greatly decreased even with small N
  - Reasonable additional time cost compared to MsFEM online phase
  - Same pros as the MsFEM
- Perspectives:
  - Develop an adaptive method by choosing the polynomial degree according to the local fluctuations of the coefficient A.
  - An a posteriori error estimator has already been designed -> numerical tests has to be performed to assess its efficiency.
  - The a priori bound is pessimistic when $H \gg \varepsilon$, hence need to use different approach to get a sharper result in this case.