The Martingale Optimal Transport (MOT) problem

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Inria’s Junior Seminar

February 18, 2020
1 The optimal transport problem

2 The Martingale Optimal Transport Problem

3 A new family of martingale couplings
1. The optimal transport problem

2. The Martingale Optimal Transport Problem

3. A new family of martingale couplings
Monge’s Formulation

- $\mu$ and $\nu$: two probability measures on $\mathbb{R}$
- $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$: a nonnegative cost function
- $X$: a random variable distributed according to $\mu$
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Optimal Transport problem (Monge’s Formulation):

$$\inf_T \mathbb{E}[c(X, T(X))]$$

where $T$ is such that $T(X) \sim \nu$. 
Kantorovich’s Formulation

Figure: Leonid Kantorovich (1912-1986)


Kantorovich’s Formulation

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Optimal Transport problem (Kantorovich’s Formulation):

$$\inf_{X,Y} \mathbb{E}[c(X, Y)]$$

where $X \sim \mu$ and $Y \sim \nu$. 
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Robust finance: model-free pricing bounds on derivative financial products

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The Martingale Optimal Transport Problem:

$$\inf_{X, Y} \mathbb{E}[c(X, Y)],$$

where $X \sim \mu$, $Y \sim \nu$ and $\mathbb{E}[Y|X] = X.$
Application: Option Pricing

- \((S_t)_{t \in [0, T]}\): value of a financial asset between 0 and \(T\)
- \(c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+:\) a nonnegative function payoff of an option which only depends on \(S_0\) and \(S_T\)
- \(\mu\) and \(\nu\): respective distributions of \(S_0\) and \(S_T\)
- \(r\): interest rate (suppose \(r = 0\) for convenience)

An option is a contract which gives the buyer the right to buy or sell an underlying asset at a specified strike price on a specified date.
Hypothesis: the payoff is of the form $c(S_0, S_T)$. Examples:

- Payoff from buying a call: $(S_T - K)^+$
- Payoff from buying a put: $(K - S_T)^+$
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Fundamental theorem of asset pricing:

$$\text{Price of the option} = \mathbb{E}^*[c(S_0, S_T)].$$
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$$\inf_{X \sim \mu, \ Y \sim \nu} \mathbb{E}^*[c(X, Y)] \leq \mathbb{E}^*[c(S_0, S_T)] \leq \sup_{X \sim \mu, \ Y \sim \nu} \mathbb{E}^*[c(X, Y)]$$
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$$\mathbb{E}^*[Y|X] = X \quad \mathbb{E}^*[Y|X] = X$$
How to solve the Martingale Optimal Transport problem?

Hypothesis: \( \mu = \sum_{i=1}^{I} p_i \delta_{x_i}, \nu = \sum_{j=1}^{J} q_j \delta_{y_j} \)

Then the MOT problem becomes

\[
\inf \sum_{i=1}^{I} \sum_{j=1}^{J} r_{i,j} c(x_i, y_j),
\]

subject to

- \( r_{i,j} \geq 0 \)
- \( \sum_{i=1}^{I} r_{i,j} = q_j \)
- \( \sum_{j=1}^{J} r_{i,j} = p_i \)
- \( \sum_{j=1}^{J} r_{i,j} y_j = p_i x_i \)
The optimal transport problem

The Martingale Optimal Transport Problem

A new family of martingale couplings
The convex order

Two $\mu, \nu$ probability measures on $\mathbb{R}$ are said to be in the convex order, denoted $\mu \leq_{cx} \nu$, if

$$\forall \varphi : \mathbb{R} \to \mathbb{R} \text{ convex}, \quad \int_{\mathbb{R}} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}} \varphi(y) \nu(dy).$$

Theorem (Strassen (1964))

Let $\mu, \nu$ be two probability measures on $\mathbb{R}$. Then

$$\exists X \sim \mu, Y \sim \nu, \; \mathbb{E}[Y|X] = X \; \text{a.s.} \iff \mu \leq_{cx} \nu.$$
Main Theorem

- \( \mu \) and \( \nu \) : two probability measures on \( \mathbb{R} \) such that \( \mu \leq_{cx} \nu \)
- Wasserstein distance:

\[
W_1(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|] = \mathbb{E}[|F_\mu^{-1}(U) - F_\nu^{-1}(U)|],
\]

where \( U \sim \mathcal{U}((0, 1)) \) and \( F_\eta^{-1}(u) = \inf\{x \in \mathbb{R} \mid \eta((-\infty, x]) \geq u\} \).
- "Martingale-Wasserstein" distance:

\[
M_1(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|].
\]

Theorem (Stability inequality)

\[
W_1(\mu, \nu) \leq M_1(\mu, \nu) \leq 2W_1(\mu, \nu).
\]
A new family of martingale couplings

- Family \( \{(X^Q, Y^Q) \mid Q \in Q\} \) parametrised by a set \( Q \) of probability measures on \((0, 1)^2\)

**Proposition**

\[
\forall Q \in Q, \quad \begin{cases} 
X^Q \sim \mu \\
Y^Q \sim \nu \\
\mathbb{E}[Y^Q|X^Q] = X^Q \\
\mathbb{E}[|X^Q - Y^Q|] \leq 2\mathcal{W}_1(\mu, \nu)
\end{cases}
\]
Some Properties

- $|Q| = +\infty$
- $|\text{supp } \nu| \leq 2 \implies$ unique coupling
- $\mu$ and $\nu$ with densities $\implies$ infinitely many couplings
- generalisation to the sub- and supermartingale case
The Inverse Transform Martingale Coupling

- \( \Psi_\pm(u) = \int_0^u (F^{-1}_\mu - F^{-1}_\nu) \pm(v) \, dv \)
- \( \varphi(u) = \begin{cases} 
\Psi_-(\Psi_+(u)) & \text{if } F^{-1}_\mu(u) > F^{-1}_\nu(u) \\
\Psi_+(\Psi_-(u)) & \text{if } F^{-1}_\mu(u) < F^{-1}_\nu(u) \\
u & \text{if } F^{-1}_\nu(u) = F^{-1}_\mu(u) 
\end{cases} \)
- \( U, V \sim \mathcal{U}((0,1)) \) independent

\[
X^Q = F^{-1}_\mu(U) \\
Y^Q = F^{-1}_\nu(U)1_{\{V \geq \frac{F^{-1}_\mu(U) - F^{-1}_\nu(U)}{F^{-1}_\nu(\varphi(U)) - F^{-1}_\nu(U)}\}} + F^{-1}_\nu(\varphi(U))1_{\{V \leq \frac{F^{-1}_\mu(U) - F^{-1}_\nu(U)}{F^{-1}_\nu(\varphi(U)) - F^{-1}_\nu(U)}\}}
\]