The Martingale Optimal Transport (MOT) problem

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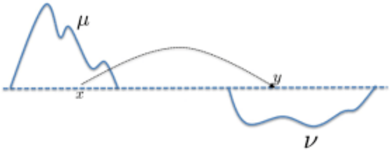
Monge's Formulation

Figure: Gaspard Monge (1746-1818)

Monge's Formulation

- μ and ν: two probability measures on $\mathbb{R}$
- $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$: a nonnegative cost function
- X: a random variable distributed according to μ



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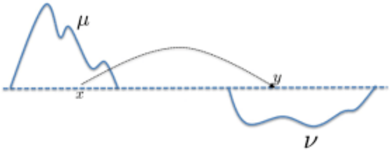

Optimal Transport problem (Monge's Formulation):

$$\inf_T \mathbb{E}[c(X, T(X))]$$

where T is such that $T(X) \sim \nu$.

Kantorovich's Formulation

Figure: Leonid Kantorovich (1912-1986)

Kantorovich's Formulation

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Optimal Transport problem (Kantorovich's Formulation):

$$\inf_{X,Y} \mathbb{E}[c(X,Y)]$$

where $X \sim \mu$ and $Y \sim \nu$.

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Motivation and definition

Robust finance : model-free pricing bounds on derivative financial products

- μ and ν : two probability measures
- $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$: a nonnegative cost function

The Martingale Optimal Transport Problem:

$$\inf_{X,Y} \mathbb{E}[c(X,Y)],$$

where $X \sim \mu$, $Y \sim \nu$ and $\mathbb{E}[Y|X] = X$.

Application: Option Pricing

- $(S_t)_{t\in[0,T]}$: value of a financial asset between 0 and T
- $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$: a nonnegative function payoff of an option which only depends on S_0 and S_T
- μ and ν: respective distributions of S_0 and S_T
- r: interest rate (suppose $r = 0$ for convenience)

An option is a contract which gives the buyer the right to buy or sell an underlying asset at a specified strike price on a specified date.

Hypothesis: the payoff is of the form $c(S_0, S_T)$. Examples:

- Payoff from buying a call: $(S_T - K)^+$
- Payoff from buying a put: $(K - S_T)^+$

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Fundamental theorem of asset pricing:

$$\text{Price of the option} = \mathbb{E}^*[c(S_0, S_T)].$$

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$$\inf_{\substack{X\sim\mu \\ Y\sim\nu}} \mathbb{E}^*[c(X,Y)] \leq \mathbb{E}^*[c(S_0, S_T)] \leq \sup_{\substack{X\sim\mu \\ Y\sim\nu}} \mathbb{E}^*[c(X,Y)]$$

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$$\text{Price of the option} = \mathbb{E}^*[c(S_0, S_T)].$$

$$\inf_{\substack{X \sim \mu \\ Y \sim \nu \\ \mathbb{E}^*[Y|X]=X}} \mathbb{E}^*[c(X,Y)] \le \mathbb{E}^*[c(S_0, S_T)] \le \sup_{\substack{X \sim \mu \\ Y \sim \nu \\ \mathbb{E}^*[Y|X]=X}} \mathbb{E}^*[c(X,Y)]$$

Problem

How to solve the Martingale Optimal Transport problem?

- Hypothesis: $\mu = \sum_{i=1}^{I} p_i \delta_{x_i}$, $\nu = \sum_{j=1}^{J} q_j \delta_{y_j}$

Then the MOT problem becomes

$$\inf \sum_{i=1}^{I} \sum_{j=1}^{J} r_{i,j} c(x_i, y_j),$$

subject to

- $r_{i,j} \geq 0$
- $\sum_{i=1}^{I} r_{i,j} = q_j$
- $\sum_{j=1}^{J} r_{i,j} = p_i$
- $\sum_{j=1}^{J} r_{i,j} y_j = p_i x_i$

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The convex order

Two μ, ν probability measures on $\mathbb{R}$ are said to be in the convex order, denoted $\mu \leq_{cx} \nu$, if

$$\forall \varphi : \mathbb{R} \to \mathbb{R} \text{ convex}, \quad \int_{\mathbb{R}} \varphi(x)\, \mu(dx) \leq \int_{\mathbb{R}} \varphi(y)\, \nu(dy).$$

Theorem (Strassen (1964))

Let μ, ν be two probability measures on $\mathbb{R}$. Then

$$\exists X \sim \mu, Y \sim \nu, \ \mathbb{E}[Y|X] = X \ a.s. \iff \mu \leq_{cx} \nu.$$

Main Theorem

- μ and ν : two probability measures on $\mathbb{R}$ such that $\mu \leq_{cx} \nu$
- Wasserstein distance:

$$\mathcal{W}_1(\mu,\nu) = \inf_{X\sim\mu, Y\sim\nu} \mathbb{E}[|X - Y|] = \mathbb{E}[|F_\mu^{-1}(U) - F_\nu^{-1}(U)|],$$

where $U \sim \mathcal{U}((0,1))$ and $F_\eta^{-1}(u) = \inf\{x \in \mathbb{R} \mid \eta((-\infty, x]) \geq u\}$.

- "Martingale-Wasserstein" distance:

$$\mathcal{M}_1(\mu,\nu) = \inf_{\substack{X\sim\mu, Y\sim\nu \\ \mathbb{E}[Y|X]=X}} \mathbb{E}[|X - Y|].$$

Theorem (Stability inequality)

$$\mathcal{W}_1(\mu,\nu) \leq \mathcal{M}_1(\mu,\nu) \leq 2\mathcal{W}_1(\mu,\nu).$$

A new family of martingale couplings

- Family $\{(X^Q, Y^Q) \mid Q \in \mathcal{Q}\}$ parametrised by a set $\mathcal{Q}$ of probability measures on $(0,1)^2$

Proposition

$$\forall Q \in \mathcal{Q}, \quad \begin{cases} X^Q \sim \mu \\ Y^Q \sim \nu \\ \mathbb{E}[Y^Q | X^Q] = X^Q \\ \mathbb{E}[|X^Q - Y^Q|] \leq 2\mathcal{W}_1(\mu, \nu) \end{cases}$$

Some Properties

- $|Q| = +\infty$
- $|\text{supp } \nu| \leq 2 \implies$ unique coupling
- μ and ν with densities $\implies$ infinitely many couplings
- generalisation to the sub- and supermartingale case

The Inverse Transform Martingale Coupling

- $\Psi_\pm(u) = \int_0^u (F_\mu^{-1} - F_\nu^{-1})^\pm(v)\, dv$
- $\varphi(u) = \begin{cases} \Psi_-^{-1}(\Psi_+(u)) & \text{if} \quad F_\mu^{-1}(u) > F_\nu^{-1}(u) \\ \Psi_+^{-1}(\Psi_-(u)) & \text{if} \quad F_\mu^{-1}(u) < F_\nu^{-1}(u) \\ u & \text{if} \quad F_\nu^{-1}(u) = F_\mu^{-1}(u) \end{cases}$
- $U, V \sim \mathcal{U}((0,1))$ independent
-

$$\begin{aligned} X^Q &= F_\mu^{-1}(U) \\ Y^Q &= F_\nu^{-1}(U) 1_{\left\{V \geq \frac{F_\mu^{-1}(U) - F_\nu^{-1}(U)}{F_\nu^{-1}(\varphi(U)) - F_\nu^{-1}(U)}\right\}} \\ &\quad + F_\nu^{-1}(\varphi(U)) 1_{\left\{V \leq \frac{F_\mu^{-1}(U) - F_\nu^{-1}(U)}{F_\nu^{-1}(\varphi(U)) - F_\nu^{-1}(U)}\right\}} \end{aligned}$$