Limit Theorems for Default Contagion and Systemic Risk

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Motivation

- Financial institutions are becoming more and more connected to each other.
- The size and diversity of the financial system are also becoming larger and larger.
- This leads to a significant systemic risk.
- **Financial default contagion:** The bankruptcies of some institutions bring loss to its neighbors, might leading new insolvency and propagating through the network.
- Even a small part of institutions' defaults can trigger a large default cascade.

Main Issues

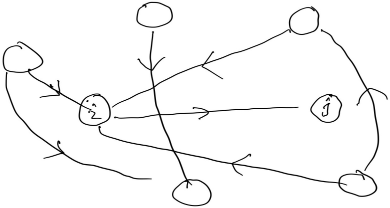
- In many cases, the size of financial network is large.
- heterogeneity (diversity) is high in the financial network.
- Partial informations available:
    - we do not know the structure of linkages in the network;
    - we know partial characteristics of the institutions: the total assets and liabilities, the total number of out-links and in-links...

Our concerns: Using limit theorems to

- Analyse the network structure at the end of the contagion
- Quantify the systemic risk of the network
- Try to minimize the systemic risk

Overview

Financial Network



- **Interbank liability:** For two financial institutions $i, j \in [n]$, $\ell_{ij} \geq 0$ denotes the cash-amount that bank i owes bank j.
- A link from i to j means that there is interbank liability from i to j, i.e. $\ell_{ij} > 0$.

Financial Network

The **capital structure** of institution i in the network:

- **Assets**: Interbank asset $a_i = \sum_{j \in [n]} \ell_{ji}$, external assets e_i, cash h_i.
- **Liabilities**: Interbank liability $\ell_i = \sum_{j \in [n]} \ell_{ij}$.
- **Shock scenario**: a fraction of loss ϵ_i in external assets.

Capital before shock: $c_i(\epsilon) = e_i + a_i + h_i - \ell_i$.

Capital after shock: $c_i(\epsilon) = (1 - \epsilon_i)e_i + a_i + h_i - \ell_i$.

Default cascade

Shock scenario $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in [0,1]^n$ for size n network, the set of fundamental defaults:

$$\mathcal{D}_0(\boldsymbol{\epsilon}) = \{i \in [n] : c_i(\epsilon_i) < 0\}.$$

Liability recovery rates matrix $\mathcal{R} = R_{ij}$, satisfying:

$$h_i + (1 - \epsilon_i)e_i + \sum_{j=1}^{n} R_{ji}\ell_{ji} \geq \sum_{j=1}^{n} R_{ij}\ell_{ij}.$$

Default cascade: evolution of the defaulted set at k step

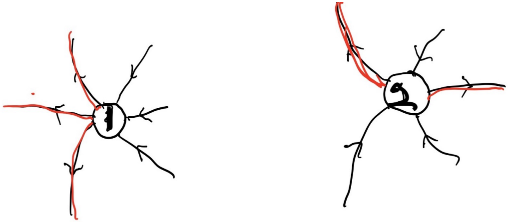
$$\mathcal{D}_k = \mathcal{D}_k(\boldsymbol{\epsilon}, \mathcal{R}) = \Big\{i \in [n] : c_i(\epsilon_i) < \sum_{j \in \mathcal{D}_{k-1}} (1 - R_{ji})\ell_{ji}\Big\}.$$

$\mathcal{D}_k \nearrow$, can not be larger than $[n]$. There is a final set of defaulted institutions $\mathcal{D}^\star$.

Configuration Model

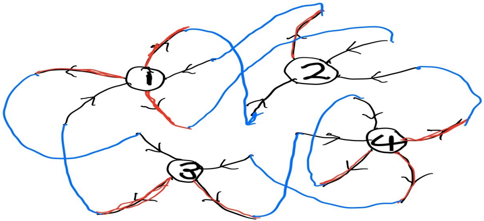
It is natural to model financial network as a random graph, where all institutions are nodes and they connect to others uniformly at random through directed edges.

- In-degree sequence $\mathbf{d}_n^+ = (d_1^+, \ldots, d_n^+)$ and out-degree sequence $\mathbf{d}_n^- = (d_1^-, \ldots, d_n^-)$.
- $\sum_{i\in[n]} d_i^+ = \sum_{i\in[n]} d_i^-$.



In the above figure, $(d_1^+, d_1^-) = (3, 3)$, $(d_2^+, d_2^-) = (3, 2)$.

- A **configuration** is a matching of all out half-edges with all in half-edges.
- The **configuration model** is the random directed multigraph which is uniformly distributed across all possible configurations.



The above figure is a configuration between four nodes with degree $(d_1^+, d_1^-) = (2,3)$, $(d_2^+, d_2^-) = (3,1)$, $(d_3^+, d_3^-) = (2,2)$, $(d_4^+, d_4^-) = (2,3)$.

Default threshold: For a node i, the default threshold is the maximum number of defaulted neighbours that i can tolerate before becoming defaulted, provided that its counterparties default in an order that is uniformly at random.

- Similar to Amini, Cont, Minca 2016, the information regarding assets, liabilities, capital after exogenous shocks and recovery rates (distributions) could all be encoded in a single probability threshold function.
- Each node i has a **random threshold** $\Theta^{(n)}(i)$ with certain distribution.
- to reduce the dimensionality, consider a classification of financial institutions into a countable (finite or infinite) set of characteristics $\mathcal{X}$.
- For each $x \in \mathcal{X}$, it contains all observable informations for the financial institutions:

$$x = (d_x^+, d_x^-, e_x, h_x, \ldots).$$

- any institutions belongs to one charateristics,

$$x_i^{(n)} = (d_i^+, d_i^-, e_i, h_i, \ldots) \in \mathcal{X}.$$

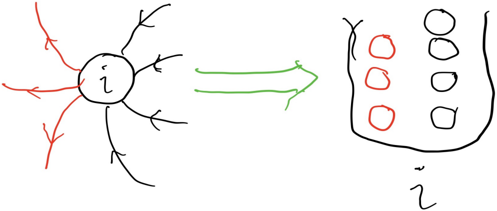
Finding the final solvent institutions

We call a link coming from a defaulted node as infected link. Using the default threshold, the default set $\mathcal{D}_k$ can be identified by

$$\mathcal{D}_k = \left\{ i \in [n] : \sum_{j : j \to i} \mathbb{1}\{j \in \mathcal{D}_{k-1}\} \geq \Theta_i \right\},$$

The dynamics of contagion:

- We reveal the infected links one by one.
- We can set the duration between two successive reveals as we want.



Death process of balls and bins

Regard all nodes as bins and all half-edges as balls. We have the following types:

- Bins: **D** (defaulted), **S** (solvent).
- Balls: $\mathbf{H}^{+}$ (healthy in), $\mathbf{H}^{-}$ (healthy out), $\mathbf{I}^{+}$ (infected in), $\mathbf{I}^{-}$ (infected out).

Balls' death and colouring:

- Initially, all $\mathbf{I}^{-}$ balls white, all $\mathbf{H}^{+} \cup \mathbf{I}^{+}$ (in) balls alive, and randomly recolor a white ball red.
- From time 0 on, in balls start to die randomly.
- If there are ℓ in balls remaining, next random death for an in ball is after a exponential time with mean $1/\ell$;
- we recolor a white ball red randomly at the same moment when an in ball dies.

Death process of balls and bins

- Denote by $W_n(t)$ the number of white balls at time t.
- The contagion stops when all infected links are revealed, denote the stopping time by $\tau_n^\star$.
- Infected links $\longrightarrow$ White balls   Institutions $\longrightarrow$ Bins
  Reveal an infected link $\longrightarrow$ An in ball's death + Coloring a white ball
- Let $S^{(n)}_{x,\theta,\ell}(t)$ be the number of solvent institutions (bins) with type x, threshold θ and ℓ defaulted neighbors at time t.
- Let $S_n(t)$ and $D_n(t)$ be the number of solvent (defaulted) institutions et time t respectively,

$$S_n(t) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \sum_{\ell=0}^{\theta-1} S^{(n)}_{x,\theta,\ell}(t).$$

LLN of the default contagion

Assumptions

- The institutions in the same characteristic class have the same threshold distribution function independently. Namely, for all $x \in \mathcal{X}$, $\theta = 0, 1, \dots, d_x^+$:

$$\mathbb{P}(\Theta_x^{(n)} = \theta) = q_x^{(n)}(\theta).$$

- For some probability distribution functions μ and q_x over the set of characteristics $\mathcal{X}$ and independent of n, we have $\mu_x^{(n)} \to \mu_x$ and $q_x^{(n)}(\theta) \to q_x(\theta)$ as $n \to \infty$, for all $x \in \mathcal{X}$ and $\theta = 0, 1, \dots, d_x^+$.
- As $n \to \infty$, the average degree converges:

$$\lambda^{(n)} := \sum_{x \in \mathcal{X}} d_x^+ \mu_x^{(n)} = \sum_{x \in \mathcal{X}} d_x^- \mu_x^{(n)} \longrightarrow \lambda := \sum_{x \in \mathcal{X}} d_x^+ \mu_x \in (0, \infty).$$

LLN of the default contagion

Theorem 1

Let $\tau_n \leq \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then for all (x, θ, ℓ), we have (as $n \to \infty$)

$$\sup_{t \leq \tau_n} \left| \frac{S_{x,\theta,\ell}^{(n)}(t)}{n} - \mu_x q_x(\theta) b\left(d_x^+, 1 - e^{-t}, \ell\right) \right| \xrightarrow{p} 0,$$

$$\sup_{t \leq \tau_n} \left| \frac{S_n(t)}{n} - f_S(e^{-t}) \right| \xrightarrow{p} 0, \quad \sup_{t \leq \tau_n} \left| \frac{D_n(t)}{n} - f_D(e^{-t}) \right| \xrightarrow{p} 0,$$

$$\sup_{t \leq \tau_n} \left| \frac{W_n(t)}{n} - f_W(e^{-t}) \right| \xrightarrow{p} 0.$$

The limit function of $S_n(t)/n$, $D_n(t)/n$ and $W_n(t)/n$ are given by:

$$f_S(e^{-t}) := \sum_{x\in\mathcal{X}} \mu_x \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, e^{-t}, d_x^+ - \theta + 1\big), \quad f_D(e^{-t}) = 1 - f_S(e^{-t}),$$

$$f_W(e^{-t}) := \lambda e^{-t} - \sum_{x\in\mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, e^{-t}, d_x^+ - \theta + 1\big).$$

where

$$b(d,z,\ell) := \mathbb{P}(\mathrm{Bin}(d,z) = \ell) = \binom{d}{\ell} z^{\ell}(1-z)^{d-\ell},$$

$$\beta(d,z,\ell) := \mathbb{P}(\mathrm{Bin}(d,z) \geq \ell) = \sum_{r=\ell}^{d} \binom{d}{r} z^{r}(1-z)^{d-r},$$

and $\mathrm{Bin}(d,z)$ denotes the binomial distribution with parameters d and z.
In fact, we obtained LLN for all quantities regarding the network structure, even for the numbers of balls of four different types $\mathbf{H}^+$, $\mathbf{H}^-$, $\mathbf{I}^+$, $\mathbf{I}^-$.

Stopping time

Define further,

$$z^\star := \sup\{z \in [0,1] : f_W(z) = 0\}.$$

Since the white ball process $W_n(t)$ control the stopping time of the contagion dynamics and $f_W(e^{-t})$ is the limit function of $W_n(t)/n$, $z^\star$ should be the limit of $e^{-\tau_n^\star}$.

In fact, we show that (as $n \to \infty$):

(i) If $z^\star = 0$ then $\tau_n^\star \xrightarrow{p} \infty$.

(ii) If $z^\star \in (0,1]$ and $z^\star$ is a stable solution, i.e. $f'_W(z^\star) > 0$, then $\tau_n^\star \xrightarrow{p} -\ln z^\star$.

LLN of the final structure

Theorem 2

The final fraction of defaulted institutions satisfies:

(i) If $z^\star = 0$ then asymptotically almost all institutions default during the cascade and

$$D_n(\tau_n^\star) = n - o_p(n).$$

(ii) If $z^\star \in (0, 1]$ and $f_W'(z^\star) > 0$, then

$$\frac{D_n(\tau_n^\star)}{n} \xrightarrow{p} f_D(z^\star).$$

Further,

$$\frac{S_{x,\theta,\ell}^{(n)}(\tau_n^\star)}{n} \xrightarrow{p} \mu_x q_x(\theta) b\left(d_x^+, 1 - z^\star, \ell\right).$$

Quantifying Systemic Risk

Use some aggregation functions to measure the systemic risk:

- **Number of solvent banks**: $\Gamma_n^{\#}(t) := S_n(t) = n - D_n(t)$.
- **External wealth**: Let $\bar{\Gamma}_n^{\odot}$ denotes the total external wealth to society if there is no default in the financial system.

$$\Gamma_n^{\odot}(t) := \bar{\Gamma}_n^{\odot} - \sum_{x \in \mathcal{X}} \bar{L}_x^{\odot} D_x^{(n)}(t),$$

where we assume a bounded constant type-dependent societal loss $\bar{L}_x^{\odot}$ over each defaulted institution.

- **System-wide wealth** : Let $\bar{\Gamma}_n^{\diamondsuit}$ denotes the total wealth in the financial system if there is no default in the system. We define the system-wide aggregation function as

$$\Gamma_n^{\diamondsuit}(t) := \bar{\Gamma}_n^{\diamondsuit} - \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} D_x^{(n)}(t) - \sum_{x\in\mathcal{X}} \bar{L}_x^{\diamondsuit} \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell S_{x,\theta,\ell}^{(n)}(t),$$

where we assume a bounded fixed (host institutions' type-dependent) cost $\bar{L}_x^{\diamondsuit}$ over each defaulted links. Assume that $\bar{\Gamma}_n^{\diamondsuit}/n \to \bar{\Gamma}^{\diamondsuit}$ when the size of network $n \to \infty$.
The corresponding limit function should be:

$$f_{\diamondsuit}(z) := \bar{\Gamma}^{\diamondsuit} - \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} f_{D_x}(z) - \sum_{x\in\mathcal{X}} \bar{L}_x^{\diamondsuit} \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}(z),$$

where $f_{D_x}(z) := \mu_x\big(1 - \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta(d_x^+, e^{-t}, d_x^+ - \theta + 1)\big)$.

Theorem 6 (LLN for systemic risk)

Let Assumptions before hold and $\tau_n \leq \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then, as $n \to \infty$,

$$\sup_{t \leq \tau_n} \left| \frac{\Gamma_n^\diamondsuit(t)}{n} - f_\diamondsuit(e^{-t}) \right| \xrightarrow{p} 0.$$

Further, the final (system-wide) aggregation functions satisfy:

(i) If $z^\star = 0$ then asymptotically almost all institutions default during the cascade and

$$\frac{\Gamma_n^\diamondsuit(\tau_n^\star)}{n} \xrightarrow{p} \bar{\Gamma}^\diamondsuit - \sum_{x \in \mathcal{X}} \mu_x \bar{L}_x^\odot.$$

(ii) If $z^\star \in (0, 1]$ and $z^\star$ is a stable solution, i.e. $f_W'(z^\star) > 0$, then

$$\frac{\Gamma_n^\diamondsuit(\tau_n^\star)}{n} \xrightarrow{p} f_\diamondsuit(z^\star).$$

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Intervention Mechanism

Consider a planner (lender of last resort or government) who seeks to minimize the systemic risk at the beginning of the financial contagion, after an exogenous macroeconomic shock ϵ, subject to a budget constraint.

- The planner only has information regarding the type of each institution and, consequently, the institutions' threshold distributions.
- The planner' decision is only based on the type of each institution.
- Intervene an infected link means save an infected link (or remove this link from the financial network).
- These interventions will be type-dependent and at random over all defaulted links leading to the same type institutions.
- Denote by $\alpha_x^{(n)}$ the planner intervention decision on the fraction of the saved links leading to any institution of type $x \in \mathcal{X}$.
- The cost to save an infected link of type x is C_x.

Let us define

$$f_W^{(\boldsymbol{\alpha})}(z) := \lambda z - \sum_{x \in \mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta) \beta\big(d_x^+, \alpha_x + (1-\alpha_x) z, d_x^+ - \theta + 1\big),$$

and,

$$z_{\boldsymbol{\alpha}}^\star := \sup\big\{z \in [0,1] : f_W^{(\boldsymbol{\alpha})}(z) = 0\big\}.$$

- Intuitively, in the intervened network, we have a different probability that a link is infected compared with the original network. Namely, in the original network, the links that come from a defaulted institution is infected with probability 1, while in the intervened one, the probability is $(1-\alpha_x^{(n)})$.
- Save an infected link means that we let an in ball which should have died remain alive.
- each in ball has a probability $\alpha_x^{(n)} + (1-\alpha_x^{(n)})e^{-t}$ to stay alive before time t.

LLN for planner

Let Assumptions hold and $\boldsymbol{\alpha}_n \to \boldsymbol{\alpha}$ as $n \to \infty$. If $z^\star_{\boldsymbol{\alpha}}$ is a stable solution,

(i) For all $x \in \mathcal{X}, \theta = 1, \ldots, d^+_x$ and $\ell = 0, \ldots, \theta - 1$, the **final fraction** of solvent institutions with type x, threshold θ and ℓ defaulted neighbors under intervention $\boldsymbol{\alpha}_n$ converges to

$$\frac{S^{(n)}_{x,\theta,\ell}(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} s^{(\boldsymbol{\alpha})}_{x,\theta,\ell}(z^\star_{\boldsymbol{\alpha}}) := \mu_x q_x(\theta) b\left(d^+_x, (1-\alpha_x)(1-z^\star_{\boldsymbol{\alpha}}), \ell\right).$$

(ii) The **total number of defaulted institutions** under intervention $\boldsymbol{\alpha}_n$ converges to:

$$\frac{D_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} f^{(\boldsymbol{\alpha})}_D(z^\star_{\boldsymbol{\alpha}}) := 1 - \sum_{x\in\mathcal{X}} \mu_x \sum_{\theta=1}^{d^+_x} q_x(\theta)\beta\left(d^+_x, \alpha_x + (1-\alpha_x)z^\star_{\boldsymbol{\alpha}}, d^+_x - \theta + 1\right)$$

(iii) The **system-wide wealth** under the intervention decision $\boldsymbol{\alpha}_n$ converges to

$$\frac{\Gamma^\diamondsuit_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} f^{(\boldsymbol{\alpha})}_\diamondsuit(z^\star_{\boldsymbol{\alpha}}) := \bar{\Gamma}^\diamondsuit - \sum_{x\in\mathcal{X}} \bar{L}^\odot_x f^{(\boldsymbol{\alpha})}_D(z^\star_{\boldsymbol{\alpha}}) - \sum_{x\in\mathcal{X}} \bar{L}^\diamondsuit_x \sum_{\theta=1}^{d^+_x} \sum_{\ell=1}^{\theta-1} \ell s^{(\boldsymbol{\alpha})}_{x,\theta,\ell}(z^\star_{\boldsymbol{\alpha}}).$$

(iv) The **total cost of interventions** $\boldsymbol{\alpha}_n$ for the planner converges to

$$\frac{\Phi_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} \phi(z^\star_{\boldsymbol{\alpha}}) := \sum_{x\in\mathcal{X}} \mu_x \alpha_x C_x \sum_{\ell=1}^{d^+_x} \ell b\left(d^+_x, 1 - z^\star_{\boldsymbol{\alpha}}, \ell\right).$$

Planner optimal decision

$$\max_{\boldsymbol{\alpha}} f_{\Diamond}^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^{\star}) := \bar{\Gamma}^{\Diamond} - \sum_{x \in \mathcal{X}} \bar{L}_x^{\odot} f_D^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^{\star}) - \sum_{x \in \mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^{\star}),$$

$$\text{subject to} \quad \phi(z_{\boldsymbol{\alpha}}^{\star}) := \sum_{x \in \mathcal{X}} \mu_x \alpha_x C_x \sum_{\ell=1}^{d_x^+} \ell b\left(d_x^+, 1 - z_{\boldsymbol{\alpha}}^{\star}, \ell\right) \leq C,$$

for some budget constraint $C > 0$.

Thank you