Quantum binary quadratic form reduction

A $O(n \log n)$ depth circuit and application to lattice reduction

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Human Context

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Symmetric Cryptography and
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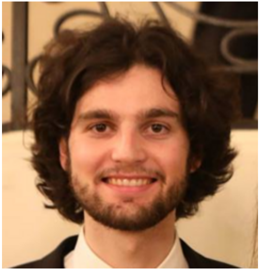

Thomas Espitau
Postdoc at NTT in Tokyo
Lattice Based Crytpography

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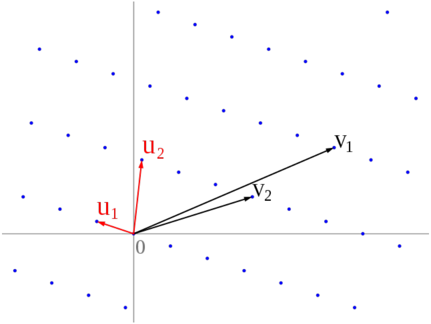
Symmetric Cryptography and
Quantum Cryptography

Lattice



- A lattice is a discrete subgroup of $\mathbf{R}^n$
- u_1, u_2 is a reduced basis

Scientific Context

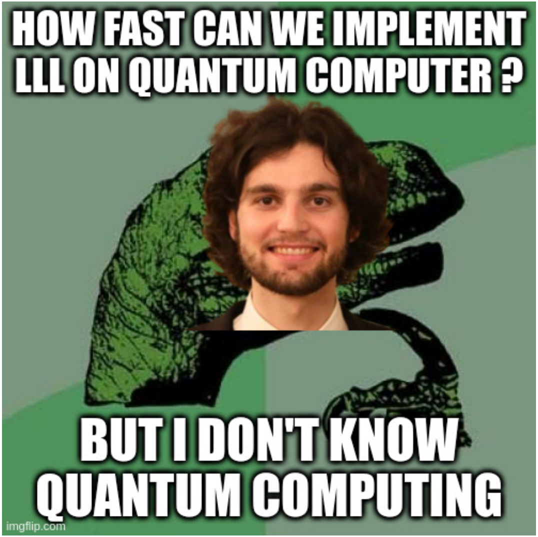
Lattice

- Lattice based cryptography is based on the difficulty of finding a reduced basis
- LLL is an algorithm that computes a reduced basis

Quantum Cryptanalysis

- With the help of quantum computer it is possible to break some cryptographic primitives

How it all started



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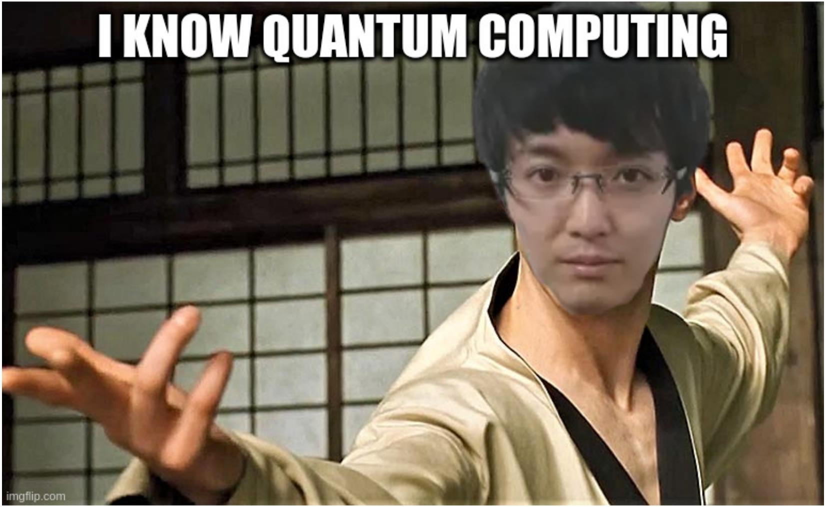

Efficient reversible shallow circuits

Designing efficient shallow circuits is a new domain of research. It mainly consists in

- Designing efficient circuit for basic operations
- Give its precise complexity allowing to derive precise costs in applications

State of the art

Some previous work

- Addition
- Multiplication
- GCD

Today we will look into Binary quadratic form reduction

- first circuit designed for quadratic form reduction
- Application to LLL

[0]Yasuhiro Takahashi and Noboru Kunihiro. "A fast quantum circuit for addition with few qubits. Quantum Information Computation"
Srijit Dutta, Debjyoti Bhattacharjee, and Anupam Chattopadhyay. "Quantum circuits for toom-cook multiplication." Mehdi Saeedi and Igor L Markov. "Quantum Circuits for GCD Computation with O(nlogn) Depth and O(n) Ancillae."

Outline

1. **Preliminaries**
   - Quantum Circuits
   - Binary quadratic form
2. **Quantum Quadratic form reduction**
3. **Quantum multiplication by 2^x**

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Quantum circuit

Qubit

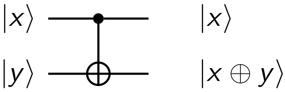
A qubit is the basic unit of quantum information

- pure states : $|0\rangle, |1\rangle$ (extended to $|00\rangle, |01\rangle, |11\rangle$ etc.)
- superposition : $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbf{C}$ and $|\alpha|^2 + |\beta|^2 = 1$

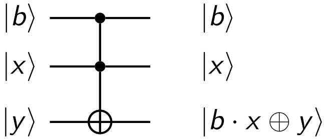
Some Quantum Gates

- NOT.

- Bitwise addition (CNOT).

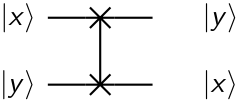


- Toffoli (CCNOT).

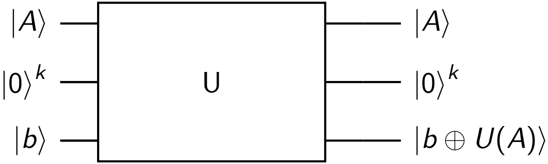


Some Quantum Gates

- Swap.

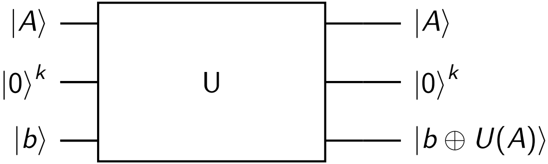


Quantum circuit and complexity measures



$|0\rangle^k$ are called Ancilae qubits

Quantum circuit and complexity measures



$|0\rangle^k$ are called Ancilae qubits

- Depth : Related to the speed of the execution.
- Width : Related to the memory ($\sharp$ Ancilae).
- Volume: Total number of operations.

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Binary quadratic form : Definition

Definition

An (integral) Binary quadratic form $\mathcal{Q} = [A, B, C]$ is a polynomial $(A\,X^2 + B\,XY + C\,Y^2) \in [X, Y]$. The integer $\Delta = B^2 - 4AC$ is called the *discriminant* of $\mathcal{Q}$. The form is said to be:

- **Degenerate** when $\Delta = 0$
- **Positive (resp. Negative) Definite** when $\Delta < 0$ and $\mathcal{Q}(x, y) \geq 0$ (resp. $\mathcal{Q}(x, y) \leq 0$) for any $(x, y) \in \mathbf{R}^2$.

Associated matrix : $Q = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$

Binary quadratic form : Class

Definition (Class)

Let $\mathcal{Q}$ be a Binary quadratic form, the following set define the class of $\mathcal{Q}$

$$< \mathcal{Q} >= \{S^T QS : S \in \mathrm{Sl}(2, \mathbf{Z})\}$$

Some notes:

- The determinant is invariant
- Every class contains a **reduced** binary quadratic form

Reduced Binary quadratic form

Definition (Reduced form)

A binary quadratic form $[A, B, C]$ is reduced if

$$\begin{Bmatrix} |B| \leq A \leq C \\ B \geq 0 \text{ if } |B| = A \text{ or } C \end{Bmatrix} \quad \text{when } [A, B, C] \text{ is positive definite}$$

Quadratic form: Link with lattices

	Lattice formalism	Quadratic form formalism
Object	Basis $M = (u, v)$	Gram matrix $G = M^t M$
Step operation	$M \leftarrow MS_\lambda$	$G \leftarrow S_\lambda^t G S_\lambda$
Reduceness condition	$\|\|v\|\|^2 \geq \|\|u\|\|^2 \geq 2\|\langle u, v\rangle\|$	$C \geq A \geq \|B\|$

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Gauss Reduction

Algorithm 1 Gauss reduction

Compute Δ
while Q is not reduced **do**
  if $|C| \leq \sqrt{|\Delta|}$ **then**
    $t \leftarrow -sgn(C) \cdot \left\lfloor \frac{B}{2|C|} \right\rceil$
  else
    $t \leftarrow -sgn(C) \cdot \left\lfloor \frac{\sqrt{|\Delta|}+B}{2|C|} \right\rceil$
  end if
  $S \leftarrow \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}$
  $Q \leftarrow S^T\, Q\, S$
end while
return Q

From Multiplication to 2^x operations

Algorithm 2 Positive Definite Reduction

$m \leftarrow 0$,$\epsilon \leftarrow \mathrm{sgn}(B)$
if $C < A$ **then**
    $(C, A) \leftarrow (A, C)$
end if
if $\neg(|B| \leq 2A)$ **then**
    $m \leftarrow 2^{\lfloor \log_2 |B| \rfloor - \lfloor \log_2 A \rfloor - 1}$
else
    $m \leftarrow \left\lfloor \frac{|B|}{2A} \right\rceil$
end if
if $m = 0$ **then**
    return $[A, (-1)^{\delta(A=-B)} B, C]$
else
    Reduce$(C - \epsilon m B + m^2 A, B - \epsilon 2mA, A)$
end if

More global optimisations

Some more optimisations can be done:

- Iterative version : independance from the input
- Fewer conditions branches : reduce the quantum cost

We now focus on implementing the following subroutines:

- 2^x multiplication
- (Integer logarithm)

Outline

Aimed Circuit

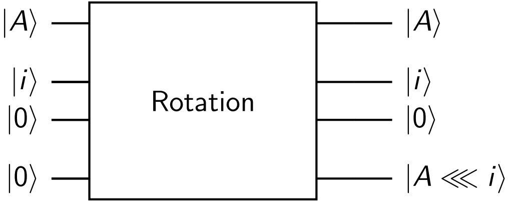


Figure: The quantum circuit for bit rotation.

Simplified circuit

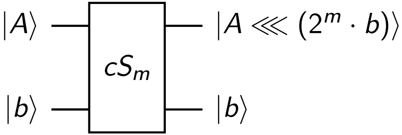


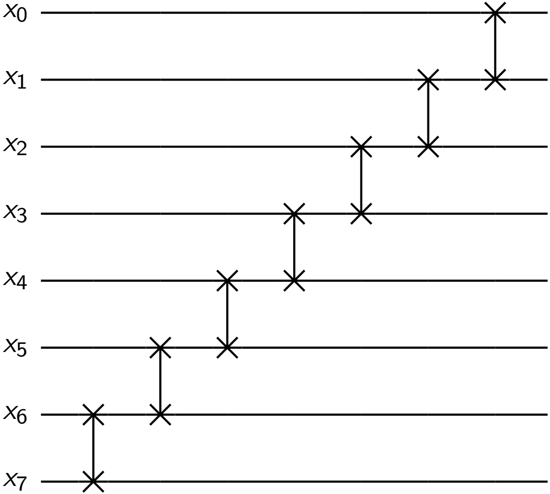
Figure: The quantum circuit cS_m ($A \in \{0,1\}^n$ and $b \in \{0,1\}$).

Even more simplified circuit

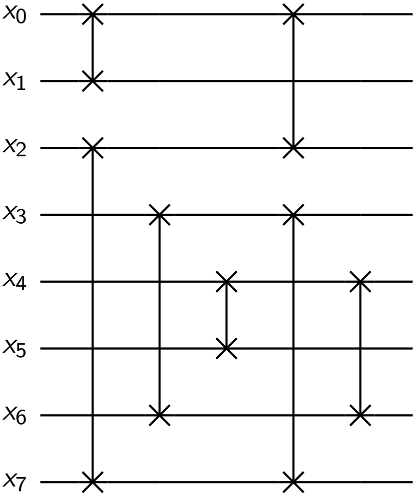
$$|A\rangle \; — \boxed{S_m} — \; |A \lll 2^m\rangle$$

Figure: The quantum circuit S_m ($A \in \{0,1\}^n$).

Example of 1-bit rotation



Example of Constant time 1-(qu)bit rotation



Constant time i-(qu)bit rotation

efficient i-(qu)bit rotation

The cyclic permutation on n qubits of parameter 2^i can be implemented :

- Depth : $\mathrm{O}(1)$
- Ancilae: none
- Volume : $\mathrm{O}(n)$

$$\sigma_0 = \left((1,0)\prod_{i=1}^{n/2-1}(i+1, n-i)\right)\cdot\left(\prod_{i=0}^{n/2-2}(i+1, n-i-1)\right)$$

From circuit to controlled circuit

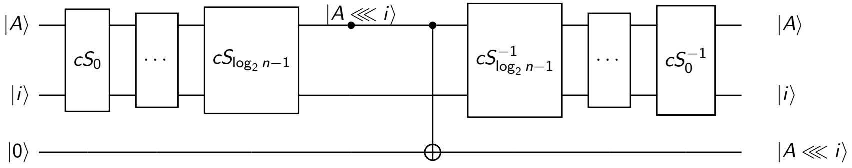
Controlled Version

Building a controlled version of S_j : cS_j can be done in with

- Depth : $\mathrm{O}(\log n)$
- Ancilae : $\mathrm{O}(n)$
- Volume : $\mathrm{O}(n)$

[0]Cristopher Moore and Martin Nilsson "Parallel quantum computation and quantum codes"

Overall circuit for multiplication by 2^x



- $i = \sum_i i_j 2^j$
- $A \lll i = A \lll i_0 2^0 \lll i_1 2^1 \lll \cdots \lll i_{\log_2(n-1)} 2^{log_2(n-1)}$
- The j-th bit of the decomposition of i in base 2 : i_j acts as the control bit of the circuit cS_j.

Complexity

Fine grain complexity analysis

The multiplication by a power of 2 can be inplemented on a circuit with:

- Depth : $12\lceil \log n \rceil$
- Ancilae: $n\lceil \log n \rceil$
- Volume : $12n\lceil \log n \rceil$

Results

Algorithm	Toffoli depth	♯ Ancilae	♯ Toffoli gates
Modular Addition	$O(\log n)$	$O\left(\frac{n}{\log n}\right)$	$O(n)$
Multiplication	$O\left(n^{1.143}\right)$	$O\left(n^{1.404}\right)$	$O\left(n^{1.55}\right)$
GCD	$O(n \log n)$	$O(n)$	$O\left(n^2\right)$
Bit rotation	$12 \log n$	$n \log n$	$12n \log n$
Logarithm	$4 \log n$	$4n$	$4n$
Binary quadratic form Reduction	$568n \log n + 896n$	$7n^2 + 26n$	$144n^2 \log n + 2834n^2$

Conclusion

We designed the first quantum circuit that

- generalizes GCD and is very important for number theory
- performs the core step of LLL Algorithm

We also derived a fine grain complexity analysis allowing a fine estimation for applications.
Article on eprint : https://eprint.iacr.org/2022/466.pdf

Conclusion

Thank you for your attention