Can we find a short path, without a map?
Can we parallelize maze-solving?

Romain Cosson

Based on joint works with Laurent Massoulié & Laurent Viennot

PhD Seminar

Keywords: collaborative exploration, trees, competitive analysis

20 slides / 20 mins
**Online framework:** the science of decision-making

*Online problem = Information arrives over time. Examples:*

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A simple example: **Linear Search**

Where should I look for my lost wallet?

- Introduced by Bellman in the 1950s
  - You start from $0 \in \mathbb{Z}$ in an infinite street, your Cost = #steps
  - Wallet is lost at $x \in \mathbb{Z}$ at distance $|x| = \text{OPT}$ (unknown)
  - Your strategy has competitive ratio $\alpha$ if you always find your wallet with at most

\[
\text{Cost} \leq \alpha \cdot \text{OPT}
\]

*The doubling strategy is 9-competitive*
Another example: **Learning with Expert advice**

*Who should I trust when there are many self-claimed « experts »?*

- Introduced independently in many fields in the late 20th century.
  - Each day $t \in \{1, \ldots, T\}$ there are $n$ experts forecasting rain ($y_{t,n} = 1$) or sun ($y_{t,n} = 0$)
  - $\text{OPT} =$ number of mistakes made by the most accurate forecaster
  - $\text{Cost} =$ the number of mistakes that you make
  - Your strategy has regret $R(n,T)$ if it satisfies

\[
\text{Cost} \leq \text{OPT} + R(n,T)
\]

*The multiplicative weights strategy has $R(n,T) = O(\sqrt{T \log n})$ regret*
Recap: on online problems

- **OPT**: the optimal possible cost, if you had all the information!
- **Cost**: the actual cost you pay, $\text{Cost} \geq \text{OPT}$
- A strategy acheives:
  - Competitive ratio $\alpha$ if $\text{Cost} \leq \alpha \times \text{OPT}$
  - Regret $R$ if $\text{Cost} \leq \text{OPT} + R$
Main dish!

• Layered Graph Traversal
  - Can you find the shortest path, when you don’t have a map?
    - Papadimitriou and Yannakakis, 1991 (online algorithms)

• Collective Tree Exploration
  - Is maze-solving parallelizable?
    - Fraigniaud, Gasieniec, Kowalski and Pelc, 2004 (distributed algorithms)
Layered Tree Traversal

Can you find the shortest path, Without a map?
Can you find the shortest path, without a map?

Notation: \( n = \#\text{nodes} = \#\text{edges}+1 \) and \( D = \text{depth} \) and \( w = \text{width} \)

- « the width »: max # nodes at given combinatorial depth
- « a layer »: set of nodes at the given combinatorial depth

**Online Problem**: Layers are revealed one after the other!

- \( \text{OPT} = D \) so an « \( \alpha(w) \)-competitive path » has length

\[
\text{Cost} \leq \alpha(w)D
\]

15 moves, i.e. 3-competitive

**Example with width w = 3, depth D = 5**

Layer 1
Layer 2
Layer 3
Layer 4
Layer 5
Is layered-feedback realistic?
The unweighted variant

✓ Observation: In unweighted tree $n = wD$
  ✓ Depth-First Search is thus $O(w)$-competitive

✓ Question: Can we do better than Depth-First Search?

Our work [CM]
✓ Yes! There is a $O(\sqrt{w})$-competitive strategy
  ✓ For a more general formulation of the problem
  ✓ Uses random choices!
Collective Tree Exploration

Is maze-solving parallelizable?
Is solving a (tree) maze parallelizable?

Notation: $n = \#\text{nodes} = \#\text{edges} + 1$ and $D = \text{depth}$

- With 1 agent: right-hand on wall (RHW, aka DFS) $\leq 2n$ moves
- With 2 agents: right+left-hand on wall (RHW+LHW) $\leq n$ moves each
- What about $k \geq 3$ agents? Moving synchronously at each round
- Exploration $\leftrightarrow$ Finding exit

Tic-tac-toe tree game

Silver coin of Knossos (Crete, 400 BC)

A computer-generated (tree) maze

Collective Tree Exploration
VOUS NE SORTIREZ JAMAIS D'ICI, ÉTRANGERS! CE TOMBEAU SERA VOTRE TOMBEAU!
Collective Tree Exploration

Goal: Traverse all edges of unknown tree $T = (V, E)$ with $n$ nodes and depth $D$

- With $k \in \mathbb{N}$ agents moving synchronously at each round

Communication Models:
- Centralized (full « complete » communication)
- Distributed (restricted « write-read » communication)

Main result [FGKP 2004]:
- Distributed algorithm SPLIT in which explorers split evenly at intersections
- Runtime $\text{SPLIT} \leq O\left(\frac{2n}{\log k} + D\right)$
The Competitive Ratio approach

What runtime could we hope for, had we know the tree in advance?

- **Offline variant**: $\text{OPT} = \text{Exploration time if tree were known in advance}
  
  $D \leq \text{OPT}$ and $\frac{n}{k} \leq \text{OPT}$ in fact, $\frac{n}{k} + D \approx \text{OPT}$

- **Consequence**: $\text{SPLIT}$ has a **competitive ratio** in $O\left(\frac{k}{\log k}\right)$

  $\text{Runtime}_{\text{SPLIT}} = O\left(\frac{n}{\log k} + D\right) \leq O\left(\frac{k}{\log k}\right)\text{OPT}$
Recent results

✓ Continuous analysis of online algorithms (online convex optimization)

✓ New idea: the explorers behave like electrons in an electric network

Latest results [C., Massoulié]

• Regret: $O(kD)$

• Competitive Ratio: $O(\sqrt{k})$
Open Questions
Open Question 1: Is there a competitive collective graph exploration algo
(competitive ratio) \[ c(k) \left( \frac{m}{k} + D \right) \]
or in \[ \frac{2m}{k} + f(k, D) \]
(regret)
where \( m \) is # of edges, \( D \) is graph diameter, and \( f(\cdot, \cdot) \) is some arbitrary function.

Open Question 2: Is there a (competitive) gap between distributed and centralized collective tree exploration?
Thank you!
Feel free to reach out
Office # C320
Why log \( k \) appears in SPLIT?

\[ \log_2 k \]

We need \( \frac{D}{\log_2(k)} \) such steps

This means \( \approx \frac{D^2}{2\log_2 k} \) rounds

At least \( \Omega \left( \frac{n}{\log_2 k} \right) \) rounds

\[ n \approx \frac{D^2}{2} \]

The « comb »

[HKLT 2014] This lower-bound applies to any « greedy » algorithm.

def. « Greedy » = a robot never goes upwards, if there is an unexplored edge below