# The Poisson problem: How can a simple equation be so complicated to solve efficiently ?

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Serena Team

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# Outline

# Serena Team

### Simulation for the Environment

- Environmental problems are modelled by Partial Differential Equations (PDEs).
- Often, such equations can not be solved exactly.

### Reliable and Efficient Numerical Algorithms

- How close is the approximate solution to the exact solution?
- How efficient is the method in terms of computational cost?

# Outline

# The Poisson problem

**We consider the boundary value problem**

Find $u : \Omega \mapsto \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The Laplacian $(\Delta \cdot)$ operator appears whenever there is diffusion.

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**Introducing the weak formulation is the starting point of FEM**

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega).$$

We do discretization from here.

# Discretization

**How to give numerical representation of a 2D continuous domain ?**

Mesh : a Finite union of Elements.

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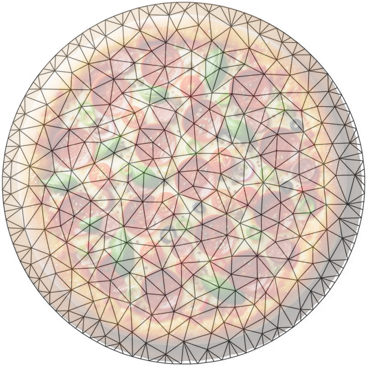
$\Omega_{\text{pizza}}$

# Discretization

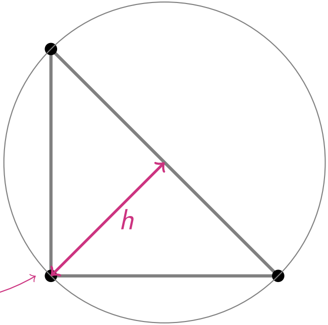
**How to give numerical representation of a 2D continuous domain ?**

Mesh : a Finite union of Elements.

$$\mathcal{T} = \bigcup_i K_i$$



$\mathcal{T}_{\text{pizza}}$




A mesh Element $K_i$

$h$ is the mesh size, it is related to the number of element $K_i$ in $\mathcal{T}$

# Approximation

We want the value of $u_h$ at specific points of each triangle
These function evaluations are called degrees of freedom (Dofs),
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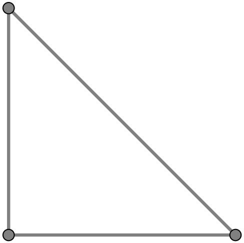

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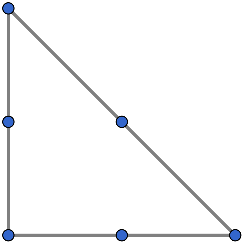

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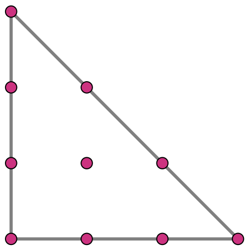
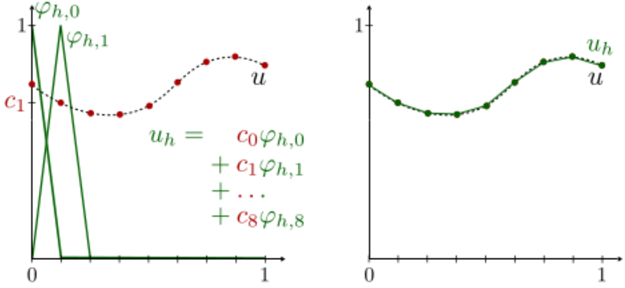


For $p = 1$, we have 3 Dofs on each element
For $p = 2$, we have 6 Dofs on each element
For $p = 3$, we have 10 Dofs on each element

Smaller $h$ : more elements
Higher $p$ : more Dofs per element

# Finite Element space

Chosing $h$ and $p$ allows us to build a discrete space $V_h^p$ where the basis functions are piecewise polynomials of degree $p$.
We look for $u_h$ in $V_h^p$ and not in $H_0^1(\Omega)$.




$V_h^p = \mathrm{Span}(\phi_{h,0}, ..., \phi_{h,n})$. We are looking for $c_0, c_1, ..., c_n$.

**What is commonly done**

$$\boxed{\text{Weak formulation}} \xrightarrow[\text{Choice of } p]{\text{Choice of } h} \boxed{\text{Finite Element Space}} \xrightarrow[\text{of a basis}]{\text{Choice}} \boxed{\text{Linear system}}$$

We have to solve a linear system for $c_0, c_1, \dots, c_n$,

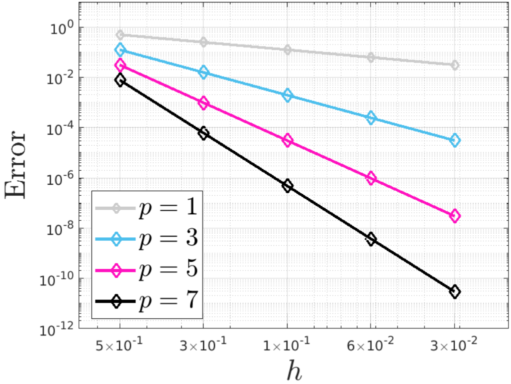
$$AU_h^p = F.$$

The size of $A$ depends on $h$ and $p$. The linear system has to be solved independently.

**How to chose $h$ and $p$ ?**

Exponential convergence of the $hp$ FEM

$$\|\nabla(u - u_h^p)\| \leq C h^{\min\{p,s-1\}} \text{ if } u \in H^s(\Omega), s \geq 1.$$



# High order FEM

**How to chose $h$ and $p$ ?**

Exponential convergence of the $hp$ FEM

$$\|\nabla(u - u_h^p)\| \leq C h^{\min\{p, s-1\}} \text{ if } u \in H^s(\Omega), s \geq 1.$$

Challenges

- The matrix becomes less sparse for high order $p$.
- The matrix becomes more ill-conditioned for high order $p$.

The system $AU_h^p = F$ is hard to solve for high $p$

Iterative solvers explode in terms of iterations

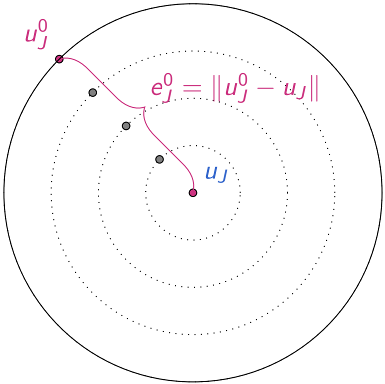
# Iterative solvers

Let $u_J$ be the exact solution of $Au_J = F$ and $u_J^i$ the iterates of the method.

Starting from $u_J^0$

$\downarrow$ $n$ iterations

Until reaching $u_J^n$
With $e_J^n = \|u_J^n - u_J\| \leq \epsilon$




Any iterative solver can be seen as a strategy to decrease the algebraic error

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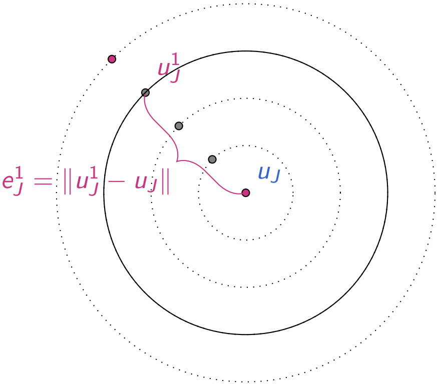


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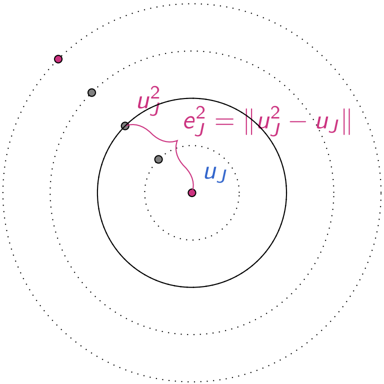


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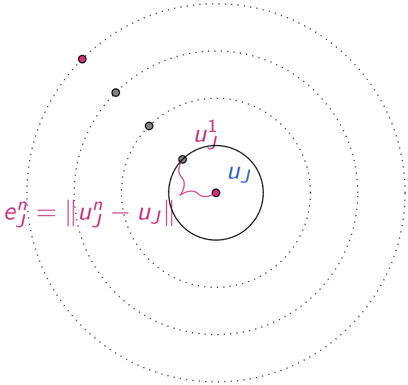
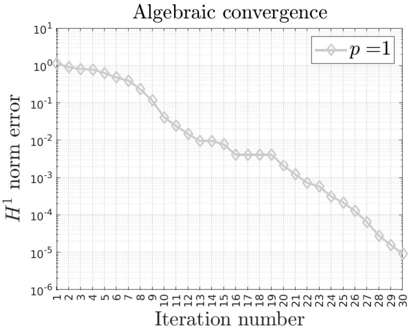
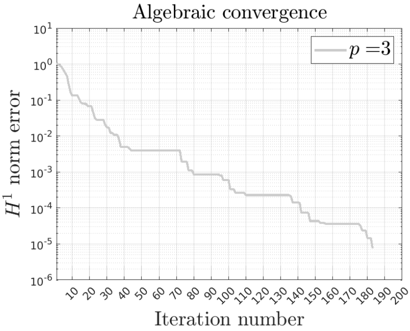



Any iterative solver can be seen as a strategy to decrease the algebraic error

# Algebraic convergence of CG

Conjugate gradient for the linear system corresponding to the Poisson equation




Tolerance: $\epsilon = 10^{-5}$

From 30 iterations with $p = 1$ to 180 iterations with $p = 3$.

**CG is not *p*-robust**

# Outline

## A multigrid FEM solver

Ingredients for the solver:

- A hierarchy of meshes
- Domain decomposition
- Orthogonal decomposition of the error

Leading to the following properties:

- The method is parallel by design
- The solver is memory efficient
- The solver is $p$-robust

Where $h$ and $p$ are arbitrary parameters

# How to decrease the algebraic error ?

The algebraic error is defined by $e_J^i = \|\nabla(u_J^i - u_J)\|$, with the $H^1$ semi norm.

How to decrease this quantity ?

We introduce $\rho_{J,\mathrm{alg}}^i$ a correction to go from $u_J^i$ to $u_J^{i+1}$

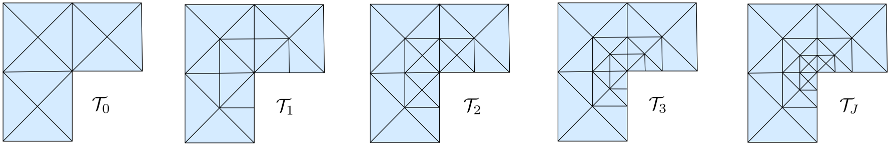
In the multigrid setting, $\rho_{J,\mathrm{alg}}^i$ is computed by going through a hierarchy of meshes.

$$\rho_{J,\mathrm{alg}}^i = \sum_{j=0}^{J} \rho_j^i$$

Where $\rho_j^i$ are the level-wise corrections.

## Multigrid : Multiple Meshes

We work with an initial mesh $\mathcal{T}_0$ and its refinements $\mathcal{T}_j$.




$$\rho^i_{J,\mathrm{alg}} = \rho^i_0 + \rho^i_1 + \rho^i_2 + \rho^i_3 + \ldots + \rho^i_J$$

The hierarchy of meshes does not have to be uniform

Low-cost information acquisition on coarser meshes

# How to compute the correction

1. Coarse solve on $\mathcal{T}_0$ for $\rho_0^i$

$$(\nabla\rho_0^i, \nabla v_0) = (f, v_0) - (\nabla u_J^i, \nabla v_0) \quad \forall v \in \mathcal{P}^0 \cap H_0^1(\mathcal{T}_0)$$

2. Level Solve for $\rho_j^i$

   **For** $\mathcal{T}_1$ to $\mathcal{T}_{\mathcal{J}}$ :

   - Decompose $\mathcal{T}_j$ into patches

     Local solve for $\rho_{j,a}^i$

     We obtain $\rho_j^i = \sum_a \rho_{j,a}^i$, the level-wise correction

   - Line Search : $\rho_j^i \longrightarrow \lambda_j^i$,

$$u_j^i = u_{j-1}^i + \lambda_j^i \rho_j$$

   **End For**

3. Update of the solution: $u_J^{i+1} = u_J^i$

# Domain decomposition

**Why are we doing domain decomposition?**

We need to compute $\rho_j^i$ on each mesh $\mathcal{T}_j$, to get the correction $\rho^i_{J,\mathrm{alg}}$.

We want to go from a single global problem to multiple local problems.

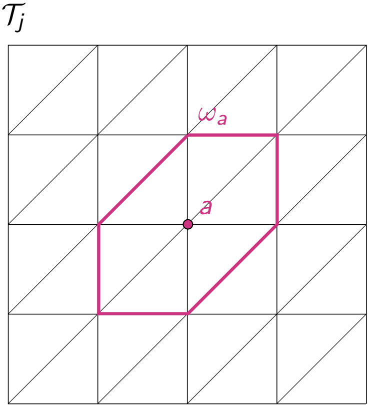
The reasons :

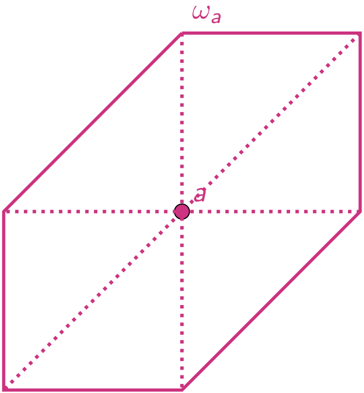
- More memory efficient
- More time efficient
- Parallelizable!

It is possible from theory using the so-called *Partition of unity*.

# But... What is a patch ?

Example of a patch at a mesh level $j$.




$a$ is the vertex patch

$\omega_a$ is the patch subdomain

# Global solve too costly

Solving for $\rho_j^i$ ?

$$\forall v \in V_j^p, \qquad \overbrace{(\nabla \rho_j^i, \nabla v)_{\mathcal{T}_j}}^{\text{Unknown}} = \overbrace{(f, v)_{\mathcal{T}_j} - (\nabla u_{j-1}^i, \nabla v)_{\mathcal{T}_j}}^{\text{Known}}$$

Choice of basis

$$K_j x_j = F_j^1 - F_j^2$$

The vector $x_j$ contains the coefficients of $\rho_j^i$ in the discrete space.

The matrix $K_j$ is assembled on the mesh $\mathcal{T}_j$... It is global!

Solving this linear system is really costly, especially for the mesh $\mathcal{T}_J$.

# Patch problems

We have the patch decomposition $\rho_j^i = \sum_a \rho_{j,a}^i$.

In order to get $\rho_{j,a}^i$, we solve :

$$\forall v \in V_a^p, \quad \overbrace{(\nabla \rho_{j,a}^i\ , \nabla v)_{\omega_a}}^{\text{Unknown}} = \overbrace{(f, v)_{\omega_a} - (\nabla u_{j-1}^i\ , \nabla v)_{\omega_a}}^{\text{Known}}$$

Choise of basis

$$K_a x_a \quad = \quad F_a^1 \quad - \quad F_a^2$$

The vector $x_a$ contains the coefficients of $\rho_a^i$ in the discrete space.

The matrix $K_a$ is assembled on the mesh $\omega_a$... It is local!

This results in

- $\texttt{size}(K_a) << \texttt{size}(K_j)$
- The computation of $\rho_{j,a}^i$can be parallelized

# Line Search

**How can we further improve the convergence ?**

At each level $\rho_j^i$ can be seen as the descent direction.

In order to get closer to $u_J$ at each level step $j$, we can introduce $\lambda_j^i$ solution of:

$$\lambda_j^i := \operatorname{argmin}_{\lambda \in \mathbb{R}} \|\nabla(u_J - (u_{J,j-1} + \lambda \rho_j^i))\|^2.$$

Minimization of $\lambda \mapsto f(\lambda)$ | $f'(\lambda) = 0$

$$\lambda_j^i = \frac{(f, \rho_j^i) - (\nabla u_{J,j-1}, \nabla \rho_j^i)}{\|\nabla \rho_j^i\|^2}$$

We update on level $j$ :

$$u_j = u_{j-1}^i + \lambda_j^i \rho_j^i$$

# Outline

# The Incredible V-cycle

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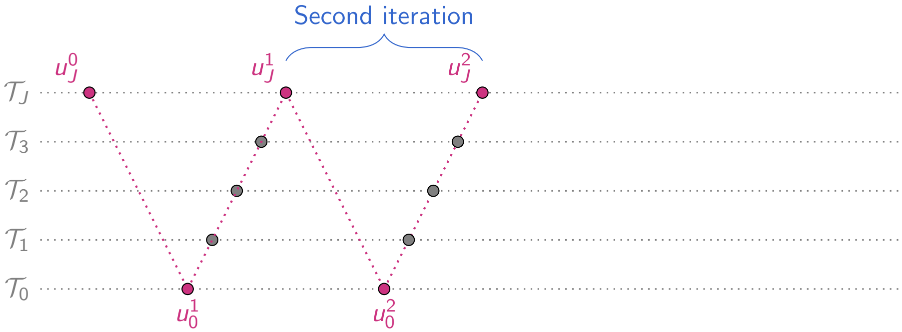
- On $\mathcal{T}_0$ : global coarse solve for $\rho_0^1$
- On $\mathcal{T}_1$ : Local patch solves to construct $\rho_1^1 = \sum_a \rho_{1,a}^1$

⋮

- On $\mathcal{T}_J$ : Local patch solves to construct $\rho_J^1 = \sum_a \rho_{J,a}^1$

If $e_J^1 \leq \epsilon$, we continue.

# The Incredible V-cycle



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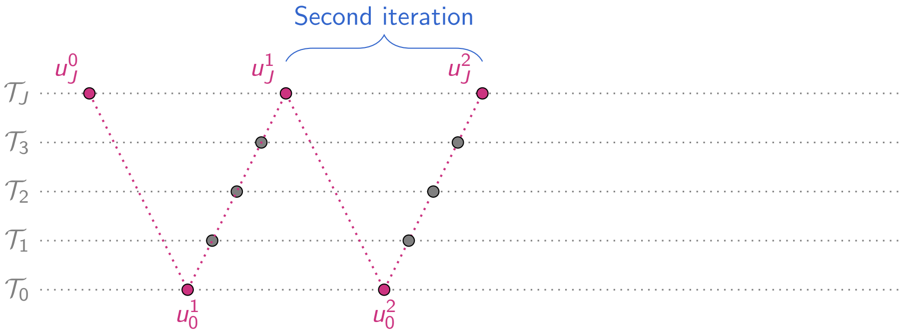


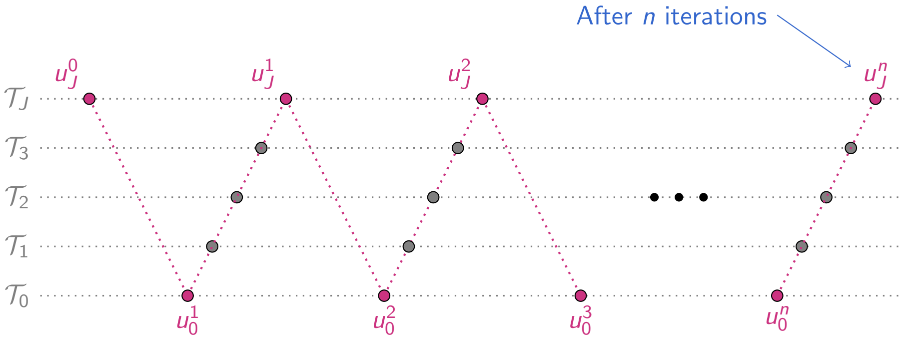

- On $\mathcal{T}_0$ : global coarse solve for $\rho_0^2$
- On $\mathcal{T}_1$ : Local patch solves to construct $\rho_1^2 = \sum_a \rho_{1,a}^2$

⋮

- On $\mathcal{T}_J$ : Local patch solves to construct $\rho_J^2 = \sum_a \rho_{J,a}^2$

If $e_J^2 \leq \epsilon$, we still continue...

# The Incredible V-cycle




We have performed $nJ$ correction on $u_J^0$

When the desired tolerance is reached, the solver stops.

We obtain $u_J^n$ the final approximation of $u_J$

# Properties

**We have the following orthogonal decomposition :**

$$\left\|\nabla\left(u_J - u_J^{i+1}\right)\right\|^2 = \left\|\nabla\left(u_J - u_J^{i}\right)\right\|^2 - \sum_{j=0}^{J}\left(\lambda_j^i\left\|\nabla\rho_j^i\right\|\right)^2$$

The error at iteration $i+1$ is the error at iteration $i$ minus a positive and computable quantity

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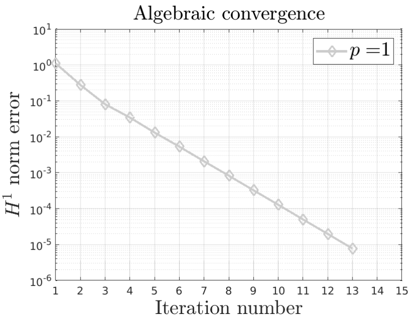
The error at iteration $i+1$ is the error at iteration $i$ minus a positive and computable quantity

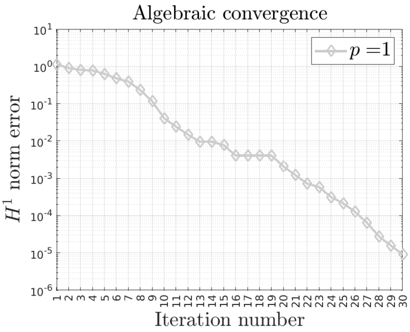
**And the $p-$robustness property**

$$\|\nabla(u_J - u_J^{i+1})\| \leq \alpha \|\nabla\left(u_J - u_J^{i}\right)\|$$

The contraction factor is given by $\alpha \in ]0,1[$. It doesn't depend on $p$

# $p$-robustness

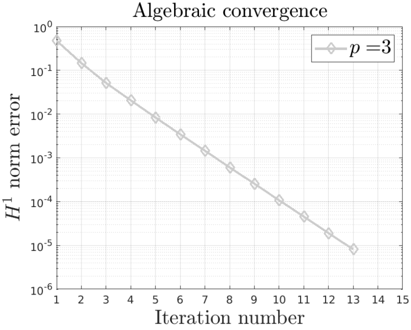


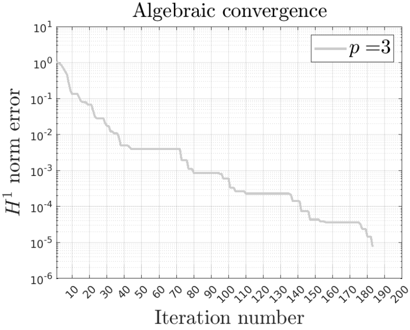




Multigrid : 13 iterations and **CG : 30 iterations**

Tolerance: $\epsilon = 10^{-5}$

# $p$-robustness

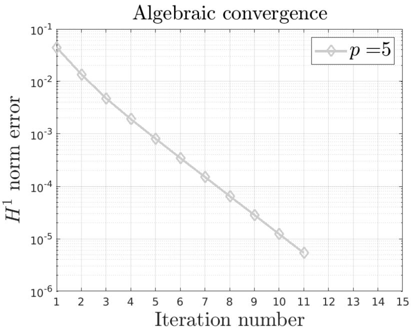


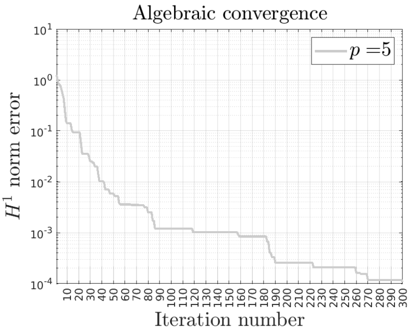




Multigrid : 13 iterations and **CG : 182 iterations**

Tolerance: $\epsilon = 10^{-5}$

# $p$-robustness

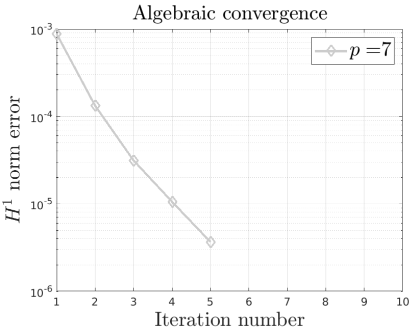


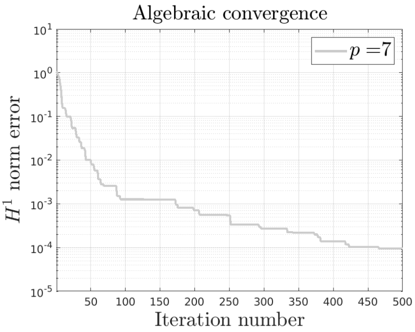




Multigrild : 11 iterations and **CG : + 300 iterations**

Tolerance: $\epsilon = 10^{-5}$

# $p$-robustness




Multrigrid : 5 iterations and **CG : + 500 iterations**

Tolerance: $\epsilon = 10^{-5}$

# Future work

While keeping the $p$-robustness property

- Localize even more (smaller problem than patch)
- Build a solver for other type of Finite Elements Methods
- Build a solver for other problems

# Conclusion

Thank you for your attention!