

A dynamic Markov chain model for pricing CDO

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1 The Model

We consider a portfolio of m firms, indexed by $i \in \{1, \dots, m\}$. The evolution of the default state of the portfolio is described by a default indicator process $Y = (Y_{t,1}, \dots, Y_{t,m})$, $t \geq 0$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We set $Y_{t,i} = 1$ if firm i has defaulted by time t and $Y_{t,i} = 0$ else, so that $Y_t \in S^Y := \{0, 1\}^m$. The corresponding default times are denoted by $\tau_i := \inf\{t \geq 0 : Y_{t,i} = 1\}$. We use the following notation for flipping the i th coordinate of a default state: given $y \in S^Y$ we define $y^i \in S^Y$ by

$$y_i^i := 1 - y_i \text{ and } y_j^i := y_j, j \in \{1, \dots, m\} \setminus \{i\}.$$

The default history is denoted by (\mathcal{H}_t) , i.e. $\mathcal{H}_t = \sigma(Y_s : s \leq t)$. An (\mathcal{H}_t) -adapted process $(\lambda_{t,i})$ is called the default intensity of default time τ_i (with respect to (\mathcal{H}_t)) if

$$Y_{t,i} - \int_0^{\tau_i \wedge t} \lambda_{s,i} ds \text{ is an } (\mathcal{H}_t)\text{-martingale.}$$

Intuitively, $\lambda_{t,i}$ gives the instantaneous chance of default of a non-defaulted firm i **given the default history up to time t** .

The default intensities $\lambda_i(t, Y_t)$ are crucial ingredients of the model. If the portfolio size N is large (such as in the pricing of typical synthetic CDO tranches) it is natural to assume that the portfolio has a homogeneous group structure. Denote the number of defaulted firms at time t by $M_t := \sum_{i=1}^N Y_{t,i}$. As discussed in [2], in a homogeneous model default intensities are necessarily of the form

$$\lambda_i(t, Y_t) = h(t, M_t) \text{ for some } h : [0, \infty) \times \{0, \dots, N\} \rightarrow \mathbb{R}_+.$$

Note that the assumption that default intensities depend on Y_t via the number of defaulted firms M_t makes sense also from an economic viewpoint, as unusually many defaults might have a negative impact on the liquidity of credit markets or on the business climate in general. In our context, we implement the following specific model

$$h(t, l) = \lambda_0 + \frac{\lambda_1}{\lambda_2} \left(\exp \left(\lambda_2 \frac{(l - \mu(t))_+}{m} \wedge 0.37 \right) - 1 \right), \quad \lambda_0, \lambda_2 > 0, \quad \lambda_1 \geq 0,$$

called *convex counterparty risk model*. The cap at 0.37 has been introduced in order to avoid an “explosion” of the intensity for high values of λ_2 . we take $\mu(t) := N(1 - \exp(-\text{Index Spread}/(1 - R)))$ as approximation for the expected number of defaulted firms, where R denotes the homogeneous recovery.

Note that λ_1 gives the slope of $h(t, l)$ at $\mu(t)$; intuitively this parameter models the strength of default interaction for “normal” realisations of M_t . The parameter λ_2 controls the degree of convexity of h and hence the tendency of the model to generate default cascades. The basic idea is simple: by increasing λ_2 we can generate occasional large clusters of defaults without affecting the left tail of the distribution of the loss process $L_t := \frac{1-R}{N} M_t$ too much; in this way we can reproduce the high spread of the CDO tranches in a way which is consistent with the observed spread of the equity tranche

2 Synthetic CDOs

Let B and A be the upper and lower attachment points of the tranche respectively. At each payment date, investors receive a coupon which is proportional to the notional of the tranche, net of the losses suffered

by the credit portfolio up to that point. Under above assumptions the process $(M_t)_{t \geq 0}$ is a markov chain taking value in $\{1, \dots, m\}$. The distribution of M_t can be determined via the following Kolmogorov forward equation

$$\frac{\partial P^M(t, s, l^1, l^2)}{\partial s} = \mathbf{1}_{\{l^2 > 0\}}(N - l^2 + 1)h(s, l^2 - 1)P^M(t, s, l^1, l^2 - 1) - (N - l^2)h(s, l^2)P^M(t, s, l^1, l^2) \quad (1)$$

with initial condition $P^M(t, t, l^1, l^2) = \mathbf{1}_{\{l^1\}}(l^2)$ for $0 \leq l^1, l^2 \leq N$. The function solving the above system and computing the quantity

$$\mathbb{E}(H_t^{A,B}) = \mathbb{E} \left[\left(\frac{1-R}{N} M_t - A \right)_+ - \left(\frac{1-R}{N} M_t - B \right)_+ \right]$$

is called **double EV_lu** in the Premia code.

2.1 Premium leg and default leg

The premium leg is equal to

$$pl^{A,B} = \sum_{j=1}^n (T_j - T_{j-1}) \mathbb{E} \left[e^{-rT_j} (B - A - H_{T_j}^{A,B}) \right],$$

where where n is the number of total payments occurring at dates T_1, \dots, T_n . The default leg is equal to

$$dl^{A,B} = \sum_{j=1}^n \mathbb{E} \left[e^{-rT_j} (H_{T_j}^{A,B} - H_{T_{j-1}}^{A,B}) \right].$$

The function computing $pl^{A,B}$, $dl^{A,B}$ and $CDO Spread^{A,B} := dl/pl$ is called **static int eber** in the Premia code.

2.2 Calibration

The model is calibrated on the to 6 months of observed 5 year tranche spreads on the iTraxx Europe in the period 23.9.2005-03.03.2006 business days from November 1st to November 6th 2006. The calibrated values used in Premia are taken from Table 4 of [2]. With the values $\lambda_0 = 0.004668$, $\lambda_1 = 0.1921$ and $\lambda_2 = 20.73$, we recover the data values of table 4 except for the first tranche since, in our implemented formulas, we did not take into account the upfront payment.

References

- [1] Eberlein, Ernst; Frey, Rüdiger; von Hammerstein, Ernst August. Advanced credit portfolio modeling and CDO pricing. Mathematics—key technology for the future, 253–279, Springer, Berlin, 2008.
- [2] Frey, Rüdiger; Backhaus, Jochen. Pricing and hedging of portfolio credit derivatives with interacting default intensities. Int. J. Theor. Appl. Finance 11 (2008), no. 6, 611–634. [1](#), [2](#)