

The Libor Market Model with Jumps

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Premia 22

Abstract

The aim of this note is to use a Lévy-driven model to describes the joint arbitrage-free dynamics of a set of forward Libor rates. Such model is called a Libor market model. This note is based on the paper of Tankov and Kohatsu-Higa (so for more details see [4]).

1 Preliminaries

We consider a d -dimensional Lévy process Z without diffusion componet. Thus $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, and ν is a Radon measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty.$$

By the Lévy-Itô decomposition, X can be written in the form

$$Z_t = \gamma t + \int_{|x|>1, s \in [0, t]} x J(dx \times ds) + \lim_{\delta \downarrow 0} \int_{\delta \leq |x| \leq 1, s \in [0, t]} x \tilde{J}(dx \times ds) \quad (1.1)$$

Here $\gamma \in \mathbb{R}^d$, J is a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity $\nu(dx)dt$, $\tilde{J}(dx \times ds) = J(dx \times ds) - \nu(dx)ds$ and ν is a Radon measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$. Given $\epsilon > 0$, we define the process R^ϵ by

$$R_t^\epsilon = \int_{0 \leq |x| \leq \epsilon, s \in [0, t]} x \tilde{J}(dx \times ds), \quad t \geq 0. \quad (1.2)$$

Note that we have

$$\mathbb{E} R_t^\epsilon = 0.$$

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On the other hand we denote by Σ^ϵ the covariance matrix of R_1^ϵ , and thus for any $i, j \in \{1, \dots, d\}$

$$\Sigma_{i,j}^\epsilon = \int_{|x| \leq \epsilon} x_i x_j \nu(dx).$$

Define the process Z^ϵ by

$$Z_t^\epsilon = \int_{|x| > \epsilon, s \in [0, t]} x J(dx \times ds), \quad t \geq 0.$$

Then we have

$$Z_t = \gamma_\epsilon t + Z_t^\epsilon + R_t^\epsilon, \quad t \geq 0, \quad (1.3)$$

where

$$\gamma_\epsilon = \gamma - \int_{\epsilon < |x| \leq 1} x \nu(dx). \quad (1.4)$$

We will call $(T_i^\epsilon)_{i \geq 1}$ the jump times of the process Z^ϵ .

2 Approximation of multidimensional SDE

Let X be a n -dimensional stochastic process, and the unique solution of the stochastic differential equation

$$dX_t = h(X_{t-}) dZ_t, \quad t \in [0, 1], \quad (2.5)$$

where h is a $n \times d$ matrix. A suitable approximation of X is \bar{X} defined by

$$d\bar{X}_t = h(\bar{X}_{t-}) (\gamma_\epsilon dt + dW_t^\epsilon + dZ_t^\epsilon), \quad (2.6)$$

where W^ϵ is a d -dimensional Brownian motion with covariance matrix Σ^ϵ . The choice of this approximation is explain in [4]. The process \bar{X} can be also written in this form

$$\begin{aligned} \bar{X}_t &= \bar{X}_{\eta_t} + \int_{\eta_t}^t h(\bar{X}_s) dW_s^\epsilon + \int_{\eta_t}^t h(\bar{X}_s) \gamma_\epsilon ds \\ \bar{X}_{T_i^\epsilon} &= \bar{X}_{T_i^{\epsilon-}} + h(\bar{X}_{T_i^{\epsilon-}}) \Delta Z_{T_i^\epsilon}, \end{aligned}$$

where $\eta_t = \sup T_i^\epsilon$, $T_i^\epsilon \leq t$. The idea of [4] is to approximate \bar{X} by

$$Y^0 + \frac{\partial}{\partial \alpha} Y^\alpha \Big|_{\alpha=0},$$

where the family of processes $(Y^\alpha)_{0 \leq \alpha \leq 1}$ is defined by

$$Y_t^\alpha = \bar{X}_{\eta_t} + \int_{\eta_t}^t h(Y_s^\alpha) dW_s^\epsilon + \int_{\eta_t}^t h(Y_s^\alpha) \gamma_\epsilon ds$$

Hence a new approximation of X , called \tilde{X} , is defined by

$$\begin{aligned}\tilde{X}_t &= Y_{0,t} + Y_{t,1}, \quad t > \eta_t \\ \tilde{X}_{T_i^\epsilon} &= \tilde{X}_{T_i^{\epsilon-}} + h\left(\tilde{X}_{T_i^{\epsilon-}}\right) \Delta Z_{T_i^\epsilon} \\ Y_{0,t} &= \tilde{X}_{\eta_t} + \int_{\eta_t}^t h(Y_{0,s}) \gamma_\epsilon ds \\ Y_{1,t} &= \int_{\eta_t}^t h(Y_{0,s}) dW_s^\epsilon + \sum_{i=1}^n \int_{\eta_t}^t \frac{\partial h}{\partial x_i}(Y_{0,s}) Y_{1,s}^i \gamma_\epsilon ds.\end{aligned}$$

The random vector $Y_{1,t}$ is Gaussian with mean zero and covariance matrix Ω_t satisfying

$$\Omega_t = \int_{\eta_t}^t \left(\Omega_s M_s + M_s^\perp \Omega_s^\perp + N_s \right) ds,$$

where M^\perp is the transpose of the matrix M and

$$M_t^{ij} = \frac{\partial h^{ij}(Y_{0,t})}{\partial x_j} \gamma_\epsilon^j, \quad N_t = h(Y_{0,t}) \Sigma^\epsilon h^\perp(Y_{0,t}).$$

3 Libor market model

Let $T_i = T_1 + (i-1)\delta$, $i = 1, \dots, n+1$ be a set dates, called tenor dates. The Libor rate L_t^i is the forward interest rate, defined at date t for the period $[T_i, T_{i+1}]$. The Libor rate can be expressed with respect to prices of zero-coupon bonds.

$$L_t^i = \frac{1}{\delta} \left(\frac{B_t(T_i)}{B_t(T_{i+1})} - 1 \right),$$

where $B_t(T)$ is the price at time t of a zero-coupon bond with maturity T . A arbitrage-free dynamics of L_t^1, \dots, L_t^n (see [3]) is

$$\frac{dL_t^i}{dL_{t-}^i} = \sigma_{i,t} dZ_t - \int_{\mathbb{R}^d} \sigma_{i,t} z \left[\prod_{j=i+1}^{n+1} \left(1 + \frac{\delta L_t^j \sigma_t^j z}{1 + \delta L_t^j} \right) - 1 \right] \nu(dz) dt, \quad (3.7)$$

where Z is a d -dimensional martingale pure jump Lévy process, with Lévy measure ν , and $\sigma_{i,t}$ are d -dimensional deterministic volatility functions. The dynamics are given under the so-called terminal measure. This means the last zero-coupon bond, $B_t(T_{n+1})$, is used as the numéraire. So the price at time t of an option with payoff $H = f(L_{T_1}^1, \dots, L_{T_1}^n)$ at time T_1 is given by

$$\pi_t(H) = \frac{B_t(T_1)}{\prod_{i=1}^n (1 + \delta L_t^i)} \mathbb{E} \left[f(L_{T_1}^1, \dots, L_{T_1}^n) \prod_{i=1}^n (1 + \delta L_{T_1}^i) / \mathcal{F}_t \right].$$

We introduce the process $(n+1)$ -dimensional X with $X_t^0 = t$ and $X_t^i = L_t^i$ (for $i = 1, \dots, n$), a $(d+1)$ -dimensional process $\tilde{Z} = (t, Z_t)^\perp$, and a $(n+1) \times (d+1)$ -dimensional function h with $h^{11} = 1$, $h^{1j} = 0$ for $2 \leq j \leq d+1$, $h^{i1} = f^i(x)$ and $h^{ij} = \sigma_{i,x_0}^{j-1}$ (for $2 \leq j \leq d+1$) with

$$f^i(x) = - \int_{\mathbb{R}^d} \sigma_{i,x_0} z \left[\prod_{j=i+1}^{n+1} \left(1 + \frac{\delta x_j \sigma_t^j z}{1 + \delta x_j} \right) - 1 \right] \nu(dz) dt,$$

so that the equation (3.7) takes the form

$$dX_t = h(X_{t-}) d\tilde{Z}_t.$$

For details about this model, see [4].

References

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References