

ap_ju_putamer

Output parameters:

- Price
- Delta

This method, described in [1], is based on the early exercise premium formula:

$$\begin{aligned}
 P_A = & P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) - K \int_0^T r e^{-rt} N(d_2(S, B_t, t)) dt \\
 & + S \int_0^T \delta e^{-\delta t} N(d_1(S, B_t, t)) dt
 \end{aligned} \tag{1}$$

where

$$\begin{aligned}
 d_1(x, y, t) &= \frac{\log(x/y) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}}, \\
 d_2(x, y, t) &= d_1(x, y, t) - \sigma\sqrt{t}
 \end{aligned}$$

P_E is the [Black and Scholes \(1973\)](#) price of the European put option, and B_t the exercise boundary at t .

As B_t appears only as $\log(S/B_t)$ in the definitions of d_1 and d_2 , a possible approximation for the exercise boundary would be one by exponential pieces. For instance, a two-exponential pieces' approximation consists in replacing B_t by $B_{21}e^{b_{21}t}$ for $t \in [T/2; T]$, $B_{22}e^{b_{22}t}$ for $t \in [0; T/2]$.

The advantage of this method is that the integrals $\int_{t_1}^{t_2} r e^{-rt} N(d_2(S, B e^{bt}, t)) dt$ and $\int_{t_1}^{t_2} \delta e^{-\delta t} N(d_1(S, B e^{bt}, t)) dt$, involved in equation (1), can be evaluated in closed form.

They become respectively $I(t_1, t_2, S, B, b, -1, r)$ and $I(t_1, t_2, S, B, b, 1, \delta)$ where I is defined by :

$$\begin{aligned}
I(t_1, t_2, S, B, b, \phi, \nu) = & e^{-\nu t_1} N(z_1 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) - e^{-\nu t_2} N(z_1 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) \\
& + \frac{1}{2} \left(\frac{z_1}{z_3} + 1 \right) e^{z_2(z_3 - z_1)} \left(N(z_3 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) \right) \\
& + \frac{1}{2} \left(\frac{z_1}{z_3} - 1 \right) e^{-z_2(z_3 + z_1)} \left(N(z_3 \sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - N(z_3 \sqrt{t_1} - \frac{z_2}{\sqrt{t_1}}) \right)
\end{aligned} \tag{2}$$

with

$$\begin{aligned}
z_1 &= \frac{r - \delta - b + \phi \sigma^2 / 2}{\sigma} \\
z_2 &= \frac{\log(S/B)}{\sigma} \\
z_3 &= \sqrt{z_1^2 + 2\nu}
\end{aligned}$$

By convention, when $t = 0$, $N(x\sqrt{t} + \frac{y}{\sqrt{t}}) = 0.5 \mathbf{1}_{\{y=0\}} + \mathbf{1}_{\{y>0\}}$

/*Mathematical functions*/
/*critical price*/

It calculates the critical price with [Mc Millan's method](#).

/*derivx*/

It computes the partial derivative of a function with respect to its first argument.

/*derivy*/

It computes the partial derivative of a function with respect to its second argument.

/*function d_1 */

/*function I */

It is defined in the equation (3).

/*function I_S */

It gives the partial derivative of I with respect to the spot S . We take the same convention for N as in function I .

$$\begin{aligned}
\frac{\partial I}{\partial S} &= I_S(t_1, t_2, S, B, b, \phi, \nu) \\
&= \left(\frac{e^{-\nu t_1}}{\sqrt{t_1}} n(z_1 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) 1_{\{t_1 \neq 0\}} - \frac{e^{-\nu t_2}}{\sqrt{t_2}} n(z_1 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) \right) \frac{1}{\sigma S} \\
&\quad + \frac{1}{2\sigma S} \left(\frac{z_1}{z_3} + 1 \right) (z_3 - z_1) e^{z_2(z_3 - z_1)} \left(N(z_3 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) \right) \\
&\quad + \frac{1}{2\sigma S} \left(\frac{z_1}{z_3} + 1 \right) e^{z_2(z_3 - z_1)} \left(\frac{1}{\sqrt{t_2}} n(z_3 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - \frac{1}{\sqrt{t_1}} n(z_3 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) 1_{\{t_1 \neq 0\}} \right) \\
&\quad - \frac{1}{2\sigma S} \left(\frac{z_1}{z_3} - 1 \right) (z_3 + z_1) e^{-z_2(z_3 + z_1)} \left(N(z_3 \sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - N(z_3 \sqrt{t_1} - \frac{z_2}{\sqrt{t_1}}) \right) \\
&\quad - \frac{1}{2\sigma S} \left(\frac{z_1}{z_3} - 1 \right) e^{-z_2(z_3 + z_1)} \left(\frac{1}{\sqrt{t_2}} n(z_3 \sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - \frac{1}{\sqrt{t_1}} n(z_3 \sqrt{t_1} - \frac{z_2}{\sqrt{t_1}}) 1_{\{t_1 \neq 0\}} \right) \quad (3)
\end{aligned}$$

/*det*/

It gives the determinant of the jacobian matrix for a couple of functions (f_1, f_2) .

/*coefficients of the inverse of the jacobian matrix*/
/*coefficient 00*/
/*coefficient 01*/
/*coefficient 10*/
/*coefficient 11*/
/*inverse of the jacobian matrix*/
/*Method of Newton-Raphson*/

The algorithm of Newton-Raphson for the system $\begin{cases} f(\mathbf{x}) = 0 \\ g(\mathbf{x}) = 0 \end{cases}$, where \mathbf{x}

is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$, is:

$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}} + \delta \mathbf{x},$$

where $\delta \mathbf{x}$ is solution of $\mathbf{J} \cdot \delta \mathbf{x} = -\mathbf{F}$.

\mathbf{F} represents the vector $\begin{pmatrix} f(x_{\text{old}}) \\ g(x_{\text{old}}) \end{pmatrix}$.

\mathbf{J} represents the jacobian matrix of the system.

The precision required to stop the algorithm is 10^{-7} .

The coefficients of the exponential pieces are obtained by solving the smooth fit system.

$$/*\text{APPROXIMATION BY ONE EXPONENTIAL}*/$$

$$B_{11}e^{b_{11}t}$$

The corresponding approximate price of the American put option is denoted by P_1 .

$$/*\text{APPROXIMATION BY TWO EXPONENTIAL PIECES}*/$$

$$B_{21}e^{b_{21}t} \text{ during } [T/2, T], \text{ and } B_{22}e^{b_{22}t} \text{ during } [0, T/2]$$

In this case, the system given by the condition of smooth fit is:

$$\begin{aligned} K - B_{21}e^{b_{21}T/2} &= P_E(B_{21}e^{b_{21}T/2}, K, T/2) + K(1 - e^{-rT/2}) \\ &\quad + B_{21}e^{b_{21}T/2}(1 - e^{-\delta T/2}) \\ &\quad - KI(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &\quad + B_{21}e^{b_{21}T/2}I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \end{aligned} \quad (4)$$

$$\begin{aligned} -1 &= -e^{-\delta T/2}N(-d_1(B_{21}e^{b_{21}T/2}, K, T/2)) - (1 - e^{-\delta T/2}) \\ &\quad - KIS(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &\quad + I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \\ &\quad + B_{21}e^{b_{21}T/2}IS(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \end{aligned} \quad (5)$$

and the couple (B_{22}, b_{22}) solution of:

$$\begin{aligned} K - B_{22} &= P_E(B_{22}, K, T) + K(1 - e^{-rT}) - B_{22}(1 - e^{-\delta T}) \\ &\quad - KI(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ &\quad + B_{22}I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &\quad - KI(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\ &\quad + B_{22}I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \end{aligned} \quad (6)$$

$$\begin{aligned} -1 &= -e^{-\delta T}N(-d_1(B_{22}, K, T)) - (1 - e^{-\delta T}) \\ &\quad - KIS(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ &\quad + I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &\quad + B_{22}IS(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &\quad - KIS(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\ &\quad + I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\ &\quad + B_{22}IS(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \end{aligned} \quad (7)$$

Ju suggests to use the Newton-Raphson algorithm to solve these systems. To initialize this algorithm, we take the critical price, calculated by [Mc Millan's method](#), and 0 as initial values for B_{21} and b_{21} , and the final values of B_{21} and b_{21} for the calculus of B_{22} and b_{22} .

The price P_2 of the put is given by:

$$P_2 = \begin{cases} P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) \\ -KI(0, T/2, S, B_{22}, b_{22}, -1, r) \\ +SI(0, T/2, S, B_{22}, b_{22}, 1, \delta) \\ -KI(T/2, T, S, B_{21}, b_{21}, -1, r) \\ +SI(T/2, T, S, B_{21}, b_{21}, 1, \delta) \text{ if } S > B_{22} \\ K - S \text{ if } S \leq B_{22} \end{cases} \quad (8)$$

/*APPROXIMATION BY THREE EXPONENTIAL PIECES*/
 $B_{31}e^{31* t}$ during $[2T/3; T]$, $B_{32}e^{32* t}$ during $[T/3; 2T/3]$
and $B_{33}e^{33* t}$ during $[0; T/3]$

The corresponding approximate price of the American put option is denoted by P_3 .

/*PRICING*/
/*Price*/

To improve the results of the method, we make a three-point Richardson extrapolation, so the price is given by: $\widehat{\mathbf{P}}_{\mathbf{A}} = 4.5\mathbf{P}_3 - 4\mathbf{P}_2 + 0.5\mathbf{P}_1$

/*Delta*/

To evaluate the delta, we compute: $\frac{\widehat{P}_A(S+h) - \widehat{P}_A(S)}{h}$ with the value 10^{-7} for h .

/*PROBLEMS*/

This method does not work, when the interest rate r equals 0.

References

- [1] N.JU. Pricing an american option by approximating its early exercise boundary as a multipiece exponential function. *The Review of Financial Studies*, 11, 3:627–646, 1998. [1](#)