

# Fourier-Cosine Method for Pricing Bermudan Options and American Options in Heston Model: Implementation in PREMIA

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### Abstract

Applying the pricing method based on Fourier-Cosine series expansion proposed in [4], we implement the algorithms for pricing Bermudan options under the Heston model. By increasing the exercise dates of the Bermudan options, we obtain an approximation for the price of the American options.

## 1 Introduction

The Fourier-Cosine pricing method (COS, in short) first proposed in [2] and further developed in [3] and [4], is based on the risk-neutral option valuation formula (discounted expected payoff approach). For an European option, the option value at time  $s$  given the state variable of the underlying asset taking value  $x$  is

$$v(x, s) = e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}} [v(X_t, t) | X_s = x] = e^{-r\Delta t} \int_{-\infty}^{\infty} v(y, t) f_{X_t|X_s}(y|x) dy, \quad (1)$$

where  $r$  is the interest rate,  $t$  and  $s$  are the expiration date and the initial date respectively and  $\Delta t := t - s$ ,  $X_t$  for  $t \geq 0$  is the state variable which can be any monotone functions of the underlying asset at time  $t$ . Function  $v(y, t)$ , which equals the payoff of the European option, is known, but the transitional density function,  $f_{X_t|X_s}(y|x)$ , typically is not. For a chosen sufficient wide domain  $[a, b]$ , we approximate the integration in the equation (1)

$$v(x, s) \approx e^{-r\Delta t} \int_a^b v(y, t) f_{X_t|X_s}(y|x) dy. \quad (2)$$

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Then the unknown conditional density function defined in  $[a, b]$  can be recovered from its characteristic function by a truncated Fourier-Cosine expansion as:

$$f_{X_t|X_s}(y|x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1} \text{Re} \left[ \varphi \left( \frac{k\pi}{b-a}; x \right) \exp \left( -i \frac{ak\pi}{b-a} \right) \right] \cos \left( k\pi \frac{y-a}{b-a} \right), \quad (3)$$

where  $\varphi(u; x)$  the characteristic function of  $f_{X_t|X_s}(y|x)$ , Re means taking the real part of the argument, and the prime at the sum symbol indicates the first term in the expansion is multiplied by one-half. The appropriate size of the integration interval can be determined with the help of the cumulants, as proposed in [2]:

$$[a, b] := \left[ c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}} \right], \quad (4)$$

with  $c_n$  denotes the  $n$ -th cumulant of  $X_t$  and  $L$  is a constant chosen to obtain sufficient precision,  $L = 12$  is large enough to obtain a satisfactory precision in Heston model.

Replacing  $f_{X_t|X_s}(y|x)$  by its approximation (3) in equation (2) and interchanging integration and summation gives the COS formula for computing the values of European options:

$$v(x, s) \approx e^{-r\Delta t} \sum_{k=0}^{N-1} \text{Re} \left[ \varphi \left( \frac{k\pi}{b-a}; x \right) \exp \left( -i \frac{ak\pi}{b-a} \right) \right] V_k(t), \quad (5)$$

where

$$V_k(t) = \frac{2}{b-a} \int_a^b v(y, t) \cos \left( k\pi \frac{y-a}{b-a} \right) dy, \quad k = 0, \dots, N-1$$

are the Fourier-Cosine coefficients of  $v(y, T)$ , available in closed form for several payoff functions.

Formula (5) also forms the basis for the pricing of Bermudan option, it can be further used to approximated the continuation value of the Bermudan options. We will show details in the next section.

This document is organized as follows: The Heston model will be introduced in Section 2; The pricing formula based on the Fourier-Cosine method will be derived in Section 3; Section 4 presents the manual of using the implementation in Premia.

## 2 Heston Model

In Heston model, the evolution of the scaled logarithm of the stock price (log-stock),  $X_t$  and its variance,  $U_t$  are described by the following stochastic differ-

ential equations (SDEs):

$$dX_t = \left( \mu - \frac{1}{2}U_t \right) dt + \rho\sqrt{U_t}dW_{1,t} + \sqrt{1-\rho^2}\sqrt{U_t}dW_{2,t} \quad (6)$$

$$dU_t = \lambda(\bar{u} - U_t)dt + \eta\sqrt{U_t}dW_{1,t}, \quad (7)$$

where the three non-negative parameters,  $\lambda, \bar{u}$  and  $\eta$ , represent the speed of mean reversion, the mean level of variance, and the volatility of the volatility process, respectively. The Brownian motions,  $W_{1,t}$  and  $W_{2,t}$  are independent and  $\rho$  is the correlation between the log-stock and the variance processes.

Since the problem of near-singular behavior in the variance direction of Heston model (we refer to [4] for more detail), we consider the log-variance process  $\sigma_t := \log(U_t), t > 0$  instead of the variance  $U_t$ . To price the path-dependent option we need to know the joint distribution of the log-stock price  $X_t$  and the log-variance processes  $\sigma_t$  at a future time  $t$ , given the information at the time  $s < t$ , i.e.  $p_{X,\sigma}(X_t, \sigma_t | X_s, \sigma_s)$ . Since the log-variance  $\sigma_t$  at time  $t$  is independent of the log-stock value at time  $s$ , i.e.  $p_\sigma(\sigma_t | X_s, \sigma_s) = p_\sigma(\sigma_t | \sigma_s)$ , then we have

$$p_{X,\sigma}(X_t, \sigma_t | X_s, \sigma_s) = p_{X|\sigma}(X_t | \sigma_t, X_s, \sigma_s) \cdot p_\sigma(\sigma_t | \sigma_s). \quad (8)$$

As shown in [1] and [5], the scaled variance process is governed by a non-central chi-square distribution, i.e.

$$2\zeta U_t \sim \chi^2(q, 2\zeta U_s e^{-\lambda(t-s)}), \quad \text{for } 0 < s < t,$$

where

$$q := 2\lambda\bar{u}/\eta^2 - 1, \quad \zeta := 2\lambda / \left( (1 - e^{-\lambda(t-s)})\eta^2 \right).$$

Then the probability density function of log-variance  $\sigma_t$  given  $\sigma_s$  is

$$p_\sigma(\sigma_t | \sigma_s) = \zeta e^{-\zeta(e^{-\sigma_s} e^{-\lambda(t-s)} + e^{\sigma_t})} \left( \frac{e^{\sigma_t}}{e^{\sigma_s} e^{-\lambda(t-s)}} \right)^{\frac{q}{2}} e^{\sigma_t} I_q \left( 2\zeta e^{-\frac{1}{2}\lambda(t-s)} \sqrt{e^{\sigma_s} + \sigma_t} \right), \quad (9)$$

where  $I_q(\cdot)$  is the modified Bessel function of the first kind with order  $q$ .

Therefore to derive the joint probability density function in (8), we only need to know  $p_{X|\sigma}(X_t | \sigma_t, X_s, \sigma_s)$ . But there is no closed-form expression for  $p_{X|\sigma}$ , instead we can derive its conditional characteristic function,  $\varphi(\omega; x_s, \sigma_t, \sigma_s)$ , based on the integration form of dynamics of  $X_t$  and  $U_t$ ,

$$\begin{aligned} \varphi(\omega; x_s, \sigma_t, \sigma_s) &:= \mathbb{E}_s \left[ e^{i\omega X_t | \sigma_t} \right] \\ &= \exp \left( i\omega \left[ x_s + \mu(t-s) + \frac{\rho}{\eta} (e^{\sigma_t} - e^{\sigma_s} - \lambda\bar{u}(t-s)) \right] \right) \\ &\quad \Phi \left( \omega \left( \frac{\lambda\rho}{\eta} - \frac{1}{2} \right) + \frac{1}{2}i\omega(1-\rho^2); e^{\sigma_t}, e^{\sigma_s} \right), \end{aligned} \quad (10)$$

where  $\Phi(\nu; U_t, U_s)$  is the characteristic function of the time integrated variance as given by

$$\begin{aligned}\Phi(\nu; U_t, U_s) &:= \mathbb{E} \left[ \exp \left( i\nu \int_s^t U_\tau d\tau \right) | U_t, U_s \right] \\ &= \frac{I_q \left[ \sqrt{U_t U_s} \frac{4\gamma(\nu) e^{-\frac{1}{2}\gamma(\nu)(t-s)}}{\eta^2 (1 - e^{-\gamma(\nu)(t-s)})} \right]}{I_q \left[ \sqrt{U_t U_s} \frac{4\lambda e^{-\frac{1}{2}\lambda(t-s)}}{\eta^2 (1 - e^{-\lambda(t-s)})} \right]} \cdot \frac{\gamma(\nu) e^{-\frac{1}{2}(\gamma(\nu) - \lambda)(t-s)} (1 - e^{-\lambda(t-s)})}{\lambda (1 - e^{-\gamma(\nu)(t-s)})} \\ &\quad \exp \left( \frac{U_t + U_s}{\eta^2} \left[ \frac{\lambda (1 + e^{-\lambda(t-s)})}{1 - e^{-\lambda(t-s)}} - \frac{\gamma(\nu) (1 + e^{-\gamma(\nu)(t-s)})}{1 - e^{-\gamma(\nu)(t-s)}} \right] \right),\end{aligned}$$

where  $\gamma(\nu)$  is defined as

$$\gamma(\nu) := \sqrt{\lambda^2 - 2i\eta^2\nu}. \quad (11)$$

### 3 Fourier-Cosine Method for Bermudan Options

From now on we use the scaled log-stock variable  $x_t := X_t - \log(K)$  to describe the state of the underlying asset, where  $K$  is the strike. The payoff function  $g(x, \tau)$  is given by

$$g(x, \tau) = [\alpha K (e^x - 1)]^+, \quad \alpha = \begin{cases} 1, & \text{for a call option} \\ -1, & \text{for a put option.} \end{cases} \quad (12)$$

Consider the Bermudan option has  $M$  exercise dates:  $t_0 < t_1 \leq t_2 \leq \dots t_M = T$  where  $t_0$  is the initial date and  $T$  is the expiration date. The Bermudan option pricing formula reads

$$v(x_{t_m}, \sigma_{t_m}, t_m) = \begin{cases} g(x_{t_m}, t_m), & m = M; \\ \max(g(x_{t_m}, t_m), c(x_{t_m}, \sigma_{t_m}, t_m)), & m = 1, 2, \dots, M-1; \\ c(x_{t_m}, \sigma_{t_m}, t_m), & m = 0, \end{cases} \quad (13)$$

with  $g(x_\tau, \tau)$  being the payoff at time  $\tau$ ,  $c(x_\tau, \sigma_\tau, \tau)$  and  $v(x_\tau, \sigma_\tau, \tau)$  the continuation value and the option value at time  $\tau$ , respectively, with scaled log-stock  $x_\tau$  and log-variance  $\sigma_\tau$ . To simplify we use  $x_m$  and  $\sigma_m$  for  $x_{t_m}$  and  $\sigma_{t_m}$  respectively.

By risk-neutral method, the continuation value is given by

$$\begin{aligned}& c(x_m, \sigma_m, t_m) \\ &:= e^{-r\Delta t} \mathbb{E}_{t_m}^Q [v(x_{m+1}, \sigma_{m+1}, t_{m+1})], \\ &= e^{-r\Delta t} \int_{\mathbb{R}} \int_{\mathbb{R}} v(x_{m+1}, \sigma_{m+1}, t_{m+1}) p_{X, \sigma}(x_{m+1}, \sigma_{m+1} | x_m, \sigma_m) d\sigma_{m+1} dx_{m+1} \\ &= e^{-r\Delta t} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} v(x_{m+1}, \sigma_{m+1}, t_{m+1}) p_{X|\sigma}(x_{m+1} | \sigma_{m+1}, x_m, \sigma_m) dx_{m+1} \right] \\ &\quad p_\sigma(\sigma_{m+1} | \sigma_m) d\sigma_{m+1},\end{aligned} \quad (14)$$

where the notation  $\mathbb{E}_{t_m}^{\mathbb{Q}}[\cdot]$  means that the expectation is taking under the risk-neutral measure and under the condition of the information till time  $t_m$  and the last equation derives from (8).

The outer integration in the above formula can be approximated by  $J$ -points quadrature integration rule (like Gauss-Legendre quadrature, composite Trapezoidal rule, etc.), which gives

$$\begin{aligned} & c(x_m, \sigma_m, t_m) \\ \approx & e^{-r\Delta t} \sum_{j=0}^{J-1} \omega_j p_{\sigma}(\zeta_j | \sigma_m) \cdot \left[ \int_{\mathbb{R}} v(x_{m+1}, \zeta_j, t_{m+1}) p_{X|\sigma}(x_{m+1} | \zeta_j, x_m, \sigma_m) dx_{m+1} \right]. \end{aligned} \quad (15)$$

The integration in the above formula corresponds to the pricing formula of European options defined between  $t_m$  and  $t_{m+1}$ , provided the the variance value at the future time is known, then the Fourier-Cosine expansion can be applied here, i.e. the conditional density function  $p_{X|\sigma}(x_{m+1} | \sigma_{m+1}, x_m, \sigma_m)$  can be approximated by its Fourier-Cosine series as in (3)

$$\begin{aligned} & p_{X|\sigma}(x_{m+1} | \sigma_{m+1}, x_m, \sigma_m) \\ \approx & \frac{2}{b-a} \sum_{k=0}^{N-1} \text{Re} \left[ \varphi \left( \frac{k\pi}{b-a}; x_m, \sigma_{m+1}, \sigma_m \right) \exp \left( -i \frac{ak\pi}{b-a} \right) \right] \cos \left( k\pi \frac{x_{m+1} - a}{b-a} \right) \\ = & \frac{2}{b-a} \sum_{k=0}^{N-1} \text{Re} \left[ \varphi \left( \frac{k\pi}{b-a}; 0, \sigma_{m+1}, \sigma_m \right) \exp \left( ik\pi \frac{x_m - a}{b-a} \right) \right] \cos \left( k\pi \frac{x_{m+1} - a}{b-a} \right), \end{aligned}$$

where  $\varphi(\omega; x_m, \sigma_{m+1}, \sigma_m)$  is the conditional characteristic function as defined in (10) and the last equation derives from the fact that

$$\varphi(\omega; x_m, \sigma_{m+1}, \sigma_m) = e^{i\omega x_m} \varphi(\omega; 0, \sigma_{m+1}, \sigma_m).$$

Substituting  $p_{X|\sigma}(x_{m+1} | \sigma_{m+1}, x_m, \sigma_m)$  by its Fourier-Cosine expansion and interchanging the summation over  $k$  with the integration over  $x_{m+1}$ , we have

$$c(x_m, \sigma_m, t_m) \approx e^{-r\Delta t} \sum_{j=0}^{J-1} \omega_j \sum_{k=0}^{N-1} V_{k,j}(t_{m+1}) \text{Re} \left\{ \tilde{\varphi} \left( \frac{k\pi}{b-a}, \zeta_j, \sigma_m \right) e^{ik\pi \frac{x_m - a}{b-a}} \right\}, \quad (16)$$

with

$$V_{k,j}(t_{m+1}) := \frac{2}{b-a} \int_b^a v(x_{m+1}, \zeta_j, t_{m+1}) \cos \left( k\pi \frac{x_{m+1} - a}{b-a} \right) dx_{m+1}, \quad (17)$$

and

$$\tilde{\varphi}(\omega, \zeta_j, \sigma_m) := p_{\sigma}(\zeta_j | \sigma_m) \varphi(\omega; 0, \zeta_j, \sigma_m). \quad (18)$$

By interchanging the summations in (16), we derive the discrete formula for the continuation value:

$$c(x_m, \sigma_m, t_m) \approx e^{-r\Delta t} \text{Re} \left\{ \sum_{k=0}^{N-1} \beta_k(\sigma_m, t_m) e^{ik\pi \frac{x_m - a}{b-a}} \right\}, \quad (19)$$

where

$$\beta_k(\sigma_m, t_m) := \sum_{j=0}^{J-1} \omega_j V_{k,j}(t_{m+1}) \tilde{\varphi} \left( \frac{k\pi}{b-a}, \zeta_j, \sigma_m \right). \quad (20)$$

By definition (18) of  $\tilde{\varphi} \left( \frac{k\pi}{b-a}, \zeta_j, \sigma_m \right)$ , it can be computed analytically by (9) and (10), then to obtain  $\beta_k(\sigma_m, t_m)$  we need to calculate  $V_{k,j}(t_{m+1})$ . Since the calculation of  $V_{k,j}(t_{m+1})$  in (17) involves the option value  $v(x_{m+1}, \zeta_j, t_{m+1})$ , by Bermudan option pricing formula (13), we will derive  $V_{k,j}(t_m)$  for  $m = M, M-1, \dots, 1, 0$  in succession.

For  $m = M$ ,

$$V_{k,j}(t_M) = \frac{2}{b-a} \int_b^a g(x_M, t_M) \cos \left( k\pi \frac{x_M - a}{b-a} \right) dx_M. \quad (21)$$

Then (21) can be computed analytically as

$$V_{k,j}(t_M) = \begin{cases} G_k(0, b), & \text{for call options} \\ G_k(a, 0), & \text{for put option,} \end{cases} \quad (22)$$

where

$$G_k(l, u) := \frac{2}{b-a} \int_a^b g(y, t_m) \cos \left( k\pi \frac{y-a}{b-a} \right) dy. \quad (23)$$

Since  $g(x, \tau)$  is given explicitly as in (12), the analytical formula of  $G_k(l, u)$  is shown in Appendix A1.

For  $m = M-1, \dots, 1$ , we will derive  $V_{k,j}(t_m)$  sequently, backwards in time, by repeating the following computational procedure. Suppose that we have  $V_{k,j}(t_{m+1})$ , we need to derive  $V_{k,j}(t_m)$ . By inserting  $V_{k,j}(t_{m+1})$  into (20), we obtain  $\beta_k(\zeta_p, t_m)$  for  $p = 0, 1, \dots, J-1$ . Then by (19), we have  $c(x_m, \zeta_p, t_m)$ . To derive  $V_{k,j}(t_m)$  we determine the *early-exercise point*,  $x_m^*(\zeta_p, t_m)$ , at time  $t_m$ , which is the point where the continuation value equals the payoff, i.e.,  $c(x_m^*(\zeta_p, t_m), \sigma_m, t_m) = g(x_m^*, t_m)$ . Many numerical methods can be used to search the early exercise point, Newton's method is used here. Based on  $x_m^*(\zeta_p, t_m)$ , we can split  $V_{k,j}(t_m)$  into two parts: one on the interval  $[a, x_m^*]$  and the other on  $(x_m^*, b]$ , i.e.,

$$V_{k,j}(t_m) = \begin{cases} C_{k,p}(a, x_m^*(\zeta_p, t_m), t_m) + G_k(x_m^*(\zeta_p, t_m), b) & \text{for a call} \\ C_{k,p}(x_m^*(\zeta_p, t_m), b, t_m) + G_k(a, x_m^*(\zeta_p, t_m)), & \text{for a put,} \end{cases} \quad (24)$$

where

$$C_{k,p}(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} c(x, \zeta_p, t_m) \cos \left( k\pi \frac{x-a}{b-a} \right) dx, \quad (25)$$

$G_k(x_1, x_2)$  as given in (23). To calculate  $V_{k,j}(t_m)$ , we need to calculate  $C_k(x_1, x_2, t_m)$ . Substituting the approximation of continuation value  $c(x, \zeta_p, t_m)$  of (19) into (25), we have the approximation of  $C_{k,p}(x_1, x_2, t_m)$ :

$$\begin{aligned} & \hat{C}_{k,p}(x_1, x_2, t_m) \\ &:= \frac{2}{b-a} \int_{x_1}^{x_2} e^{-r\Delta t} \text{Re} \left\{ \sum_{n=0}^{N-1} \beta_n(\zeta_p, t_m) e^{in\pi \frac{x-a}{b-a}} \right\} \cos \left( k\pi \frac{x-a}{b-a} \right) dx. \end{aligned} \quad (26)$$

Interchanging the order of integration and summation in the above formula, we have

$$\hat{C}_{k,p}(x_1, x_2, t_m) = e^{-r\Delta t} \sum_{n=0}^{N-1} \text{Re} [\mathcal{M}_{k,n}(x_1, x_2) \beta_n(\zeta_p, t_m)], \quad (27)$$

where

$$\mathcal{M}_{k,n}(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} e^{in\pi \frac{x-a}{b-a}} \cos\left(k\pi \frac{x-a}{b-a}\right) dx, \quad (28)$$

where it can be obtained analytically as in Appendix A2. The computation complexity of  $\hat{C}_{k,p}(x_1, x_2, t_m)$  for  $k = 0, \dots, N-1, p = 0, \dots, J-1, m = M-1, \dots, 1$  can be reduced by a property of the  $N \times N$  matrix  $\mathcal{M}(x_1, x_2)$  with elements  $\mathcal{M}_{k,n}(x_1, x_2), k, n = 0, \dots, N-1$ , which can be decomposed into a Hankel matrix  $\mathcal{M}^c(x_1, x_2)$  and a Toeplitz matrix  $\mathcal{M}^s(x_1, x_2)$ . Details about matrix  $\mathcal{M}_{k,n}(x_1, x_2)$  will be given in Appendix A2. Then by (27) and the analytical formula for  $G_k(l, u)$ , we compute  $V_{k,p}(t_m)$  from (24).

Repeating the same computational procedure, we can derive the  $V_{k,p}(t_m), k = 0, \dots, N-1, p = 0, \dots, J-1$  backward in time from  $t_{M-1}$  to  $t_1$ , finally we obtain a grid of option prices  $v(x_0, \zeta_p, t_0), p = 0, \dots, J-1$  by applying  $V_{k,p}(t_1)$  in (20) then (19). Then the initial price  $v(x_0, \sigma_0, t_0)$  can be obtained by a spline interpolation or a shift of the grid such that  $\sigma_0$  is exactly on the grid.

The backward recursion algorithm is summarized as follows:

#### Initialization

- Find  $a_v$  and  $b_v$  by Newton's method;
- Calculate  $V_{k,p}(t_M)$  with the analytic formula;
- By (18) calculate  $\tilde{\varphi}(\frac{k\pi}{b-a}, \zeta_j, \zeta_p)$ , for  $k = 0, \dots, N-1, p = 0, \dots, J-1, j = 0, 1, \dots, J-1$ .

#### Main loop to recover $V(t_m)$ from $V(t_{m+1})$ for $m = M-1$ to 1

- For  $k = 0, \dots, N-1, p = 0, \dots, J-1$ , with  $V_{k,p}(t_{m+1})$  and  $\tilde{\varphi}(\frac{k\pi}{b-a}, \zeta_j, \zeta_p)$  given for  $j = 0, \dots, J-1$ ,  $\beta_k(\zeta_p, t_m)$  is derived by (20).
- With  $\beta_k(\zeta_p, t_m)$  given for  $k = 0, \dots, N-1, p = 0, \dots, J-1$ , the continuation value  $c(x_m, \zeta_p, t_m)$  is calculated by (19).
- Determine early-exercise point  $x_m^*(\zeta_p, t_m)$  by solving the equation

$$c(x_m^*(\zeta_p, t_m), \sigma_m, t_m) = g(x_m^*, t_m)$$

using Newton's method;

- For  $(x_1, x_2) = (a, x_m^*(\zeta_p, t_m))$  and  $(x_1, x_2) = (x_m^*(\zeta_p, t_m), b)$ , calculate  $\mathcal{M}_{0,n}^s(x_1, x_2)$  for  $n = 0, \dots, N-1$ , and  $\mathcal{M}_{k,0}^c(x_1, x_2)$  for  $k = 0, \dots, N-1$  by (32) and (31), then derive  $\mathcal{M}_{k,n}^s(x_1, x_2)$  and  $\mathcal{M}_{k,n}^c(x_1, x_2)$  for  $k = 0, \dots, N-1, n = 0, \dots, N-1$  by the property of Hankel matrix and Toeplitz matrix;

- Calculate  $\hat{C}_{k,p}(a, x^*(\zeta_p, t_m), t_m)$  and  $\hat{C}_{k,p}(x^*(\zeta_p, t_m), b, t_m)$  with (27) for  $k = 0, \dots, N-1, p = 0, \dots, J-1$ , using the property of product of the matrix  $\mathcal{M}(x_1, x_2)$  with a vector  $\beta(\zeta_p, t_m) := (\beta_k(\zeta_p, t_m))_{k=0}^{N-1}$  stated in Appendix A2.
- Recover  $V_{k,p}(t_m)$  from (24).

#### Final step

- Calculate  $c(x_0, \zeta_p, t_0)$  by inserting  $V_{k,p}(t_1)$  into (20) and (19), then obtain  $v(x_0, \zeta_p, t_0)$ . Use spline interpolation to get  $v(x_0, \sigma_0, t_0)$ .

## 4 Program Manual

We implement the Bermudan options pricing by Fourier Cosine expansion. The program HAS TO work with the pnl library.

#### Model Parameters:

lambda: the speed of the mean reversion,  $\lambda$  in (6).

eta: the volatility of volatility,  $\eta$  in (6).

vbar: the mean level of variance,  $\bar{u}$  in model (6).

v0: the initial value of  $U_t$  in model (6).

rho: the correlation between log-stock and log-variance process,  $\rho$  in model (6).

#### Parameters of the product:

S0: the initial value of stock price.

strike: strike of the Bermudan option.

T: the maturity of the Bermudan option.

r: the discount interest rate.

divident: the payout dividend.

M: number of early-exercise dates, the American option price can be obtained by increasing  $M$  to sufficient large.

#### Flags for call option and put option:

flag:callput flag: 1 for call option, -1 for put option.

#### Parameters for Fourier-Cosine method:

N: number of Fourier-Cosine series,  $N$  in (16).

J: number of steps for quadrature method,  $J$  in (16).

## 5 Appendix

### 5.1 A1: $G_k(l, u)$

By definition of  $G_k(l, u)$  and payoff function  $g(x, t_m) = [\alpha \cdot K (e^x - 1)]^+$ , we have

$$G_k(l, u) = \frac{2}{b-a} \alpha K [\chi_k(l^*, u^*) - \psi_k(l^*, u^*)], \quad \alpha = \begin{cases} 1, & \text{for a call} \\ -1, & \text{for a put,} \end{cases} \quad (29)$$

with

$$l^* = \begin{cases} \max(l, 0), & \text{for a call} \\ \min(l, 0), & \text{for a put,} \end{cases} \quad u^* = \begin{cases} \max(u, 0), & \text{for a call} \\ \min(u, 0), & \text{for a put.} \end{cases} \quad (30)$$

and

$$\begin{aligned} \chi_k(l^*, u^*) &:= \int_{l^*}^{u^*} e^x \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \frac{1}{1 + \left(\frac{n\pi}{b-a}\right)^2} \left[ \cos\left(n\pi \frac{u-a}{b-a}\right) e^u - \cos\left(n\pi \frac{l-a}{b-a}\right) e^l \right. \\ &\quad \left. - \frac{n\pi}{b-a} \sin\left(n\pi \frac{u-a}{b-a}\right) e^u + \frac{n\pi}{b-a} \sin\left(n\pi \frac{l-a}{b-a}\right) e^l \right], \end{aligned}$$

$$\begin{aligned} \psi_k(l^*, u^*) &:= \int_{l^*}^{u^*} \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \begin{cases} \left[ \sin\left(n\pi \frac{u-a}{b-a}\right) - \sin\left(n\pi \frac{l-a}{b-a}\right) \right] \frac{b-a}{n\pi}, & n \neq 0 \\ u-l, & n = 0. \end{cases} \end{aligned}$$

## 5.2 A2: $\mathcal{M}_{k,n}(l, u)$

By definition of  $\mathcal{M}_{k,n}(l, u)$  in (28) and using  $e^{i\alpha} = \cos \alpha + i \sin \alpha$ , we have

$$\mathcal{M}_{k,n}(l, u) = -\frac{i}{\pi} (\mathcal{M}_{k,n}^c(l, u) + \mathcal{M}_{k,n}^s(l, u)),$$

where

$$\mathcal{M}_{k,n}^c(l, u) := \begin{cases} \frac{(u-l)\pi i}{b-a}, & k = n = 0, \\ \frac{\exp\left(i(n+k)\frac{(u-a)\pi}{b-a}\right) - \exp\left(i(n+k)\frac{(l-a)\pi}{b-a}\right)}{n+k}, & \text{otherwise} \end{cases} \quad (31)$$

$$\mathcal{M}_{k,n}^s(l, u) := \begin{cases} \frac{(u-l)\pi i}{b-a}, & k = n, \\ \frac{\exp\left(i(n-k)\frac{(u-a)\pi}{b-a}\right) - \exp\left(i(n-k)\frac{(l-a)\pi}{b-a}\right)}{n-k}, & k \neq n. \end{cases} \quad (32)$$

The matrices

$$\mathcal{M}^c(l, u) := \{\mathcal{M}_{k,n}^c(l, u)\}_{k,n=0}^{N-1}, \quad \mathcal{M}^s(l, u) := \{\mathcal{M}_{k,n}^s(l, u)\}_{k,n=0}^{N-1}$$

have special structure:  $\mathcal{M}^c(l, u)$  is a Hankel matrix and  $\mathcal{M}^s(l, u)$  is a Toeplitz matrix. Both of the Hankel matrix and the Toeplitz matrix yield the property that the products of a vector with Hankel matrix and the Toeplitz matrix respectively can be transformed into a circular convolution. The product of Toeplitz matrix  $\mathcal{M}^s(l, u)$  with the first column  $[m_0, m_{-1}, \dots, m_{1-N}]$  with a vector  $u = [u_0, u_1, \dots, u_{N-1}]$  is equal to the first  $N$  elements of  $\mathbf{m}_s \circledast \mathbf{u}_s$  with the  $2N$ -vectors:

$$\mathbf{m}_s = [m_0, m_{-1}, \dots, m_{1-N}, 0, m_{N-1}, m_{N-2}, \dots, m_1]^T,$$

and

$$\mathbf{u}_s = [u_0, u_1, \dots, u_{N-1}, 0, \dots, 0].$$

The product of Hankel matrix  $\mathcal{M}^c(l, u)$  with the first row  $[m_0, m_1, \dots, m_{N-1}]$  with a vector  $u = [u_0, u_1, \dots, u_{N-1}]$  is equal to the first  $N$  elements of  $\mathbf{m}_c \otimes \mathbf{u}_c$ , in reversed order, with the  $2N$ -vectors:

$$\mathbf{m}_c = [m_{2N-1}, m_{2N-2}, \dots, m_1, m_0]^T,$$

and

$$\mathbf{u}_c = [0, \dots, 0, u_0, u_1, \dots, u_{N-1}].$$

This property of the product of a vector with the Hankel or Toeplitz matrix reduces the computation complexity of Fourier-Cosine method. For more details about the Hankel matrix and the Toeplitz matrix, we refer to [3].

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[1](#), [2](#)  
[1](#), [10](#)