

Hull and White Two-factor model

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1 Hull and White two-factor model

Hull and White model is a short-rate model. One of its main characteristics is its ability to match the initial yield curve by using time-varying parameter. A one factor version of this model was first proposed in [1] (already implemented in Premia). In this project we consider the two-factor version proposed in [2].

Hull&White two-factor model is defined by an EDS which describes the evolution of the spot rate $r(t)$:

$$\begin{cases} dr(t) = [\theta(t) + u(t) - a r(t)] dt + \sigma_1 dW_1(t) \\ du(t) = -b u(t) dt + \sigma_2 dW_2(t), \quad u(0) = 0 \end{cases}$$

The two processes W_1 and W_2 are brownian motions with instantaneous correlation ρ , and θ is a deterministic function totally given by the market value of the zero coupon bonds.

Let us denote by $P_M(0, T)$ the market zero coupon bond value maturing at time T and $f_M(t) = -\frac{\partial \log(P_M(0, t))}{\partial t}$ the market present instantaneous forward rate, then with an appropriate choice for the function θ (see Hull&White 1994 for details), the model exactly fits the market bonds curve and we have several analytical formulas :

Zero coupon bond at time t knowing that $r(t) = r_t$ and $u(t) = u_t$:

$$P(t, T) = A(t, T)e^{-B(t, T)r_t - C(t, T)u_t}.$$

Explicite formulations for A , B and C can be found in [2].

The price at time t for a European Call on a ZC bond :

$$C_t = \mathbb{E}_t \left[e^{-\int_t^T r(s)ds} (P(T, S) - K)_+ \right] = P(t, S)\mathcal{N}(h) - KP(t, T)\mathcal{N}(h - \sigma_p).$$

Where \mathcal{N} is the cumulative function of the normal law,

$$h = \frac{1}{\sigma_p} \log \left(\frac{P(t, S)}{P(t, T)K} \right) + \frac{\sigma_p}{2}$$

and σ_p is given in [2].

This closed formula for european option on bond also leads to closed formula for cap and floor.

2 Trinomial Tree method

2.1 Tree for a one-factor process :

We first recall the procedure for the construction of a trinomial tree that approximates a process x of the form :

$$dx(t) = -ax(t)dt + \sigma dW(t), \quad x(0) = 0$$

Let $0 = t_0 < t_1 < \dots < t_n = T$ be a time scale for our tree in $[0, T]$, $\Delta t_i = t_{i+1} - t_i$, and $x_{i,j}$ the x node value at time t_i for the j^{th} space step of the tree (starting from the down). We need then :

$$\begin{cases} \mathbb{E}[x(t_{i+1})|x(t_i) = x_{i,j}] = M_{i,j} \\ \mathbb{V}ar[x(t_{i+1})|x(t_i) = x_{i,j}] = V_{i,j}^2 = V_i^2 \end{cases}$$

Knowing that x has is a gaussian process, $M_{i,j}$ and V_i can be computed :

$$\begin{cases} M_{i,j} = x_{i,j} e^{-a \Delta t_i} \\ V_i^2 = \frac{\sigma^2}{2a} [1 - e^{-a \Delta t_i}] \end{cases}$$

At time t_i , the nodes are equally spaced, so : $x_{i,j} = j \Delta x_i$, with $\Delta x_i = V_{i-1} \sqrt{3}$.

Starting at time t_i from node $x_{i,j}$, the process can move to three node at time t_{i+1} :

$$\begin{cases} \text{Up with probability } p_u(i, j) \text{ to the node } x_{i+1,k+1} \\ \text{Middle with probability } p_m(i, j) \text{ to the node } x_{i+1,k} \\ \text{Down with probability } p_d(i, j) \text{ to the node } x_{i+1,k-1} \end{cases}$$

The index k is chosen so that $x_{i+1,k}$ is as close as possible to the mean $M_{i,j}$, ie :

$$k = \text{round} \left(\frac{M_{i,j}}{\Delta x_{i+1}} \right) = \text{round} (j \beta_i)$$

$$, \text{ with } \beta_i = \frac{\Delta x_i}{\Delta x_{i+1}} e^{-a \Delta t_i}.$$

The probabilities $p_u(i, j)$, $p_m(i, j)$ and $p_d(i, j)$ are chosen so that that conditional mean and variance of the discrete process in the tree match those of the continuous process ($M_{i,j}$ and V_i). See [3] for the obtained formulas.

2.2 Tree for a two-factor process :

Now that we know how to construct a trinomial tree for a process of the kind $dx(t) = -a x(t)dt + \sigma(t) dW(t)$, we will use this technique for the two-factor process r .

First, we consider the process x verifying the same equation as r , with $\theta = 0$:

$$\begin{cases} dx(t) = [u(t) - a x(t)] dt + \sigma_1 dW_1(t), & x(0) = 0 \\ du(t) = -b u(t) dt + \sigma_2 dW_2(t), & u(0) = 0 \end{cases}$$

If we suppose that $a \neq b$, then the dependance of x on u can be eliminated by defining

$$y = x + \frac{u}{b-a}$$

so that

$$\begin{cases} dy(t) = -a y(t) dt + \sigma_3 dW_3(t), & y(0) = 0 \\ du(t) = -b u(t) dt + \sigma_2 dW_2(t), & u(0) = 0 \end{cases}$$

where

$$\sigma_3^2 = \sigma_3^2 + \frac{\sigma_2^2}{(b-a)^2} + \frac{2\rho\sigma_1\sigma_2}{b-a}$$

and W_3 is a brownian motion. The correlation between W_2 and W_3 is

$$\rho_{uy} = \frac{\rho\sigma_1 + \sigma_2/(b-a)}{\sigma_3}$$

The first step to construct a tree for x is to construct two trinomial trees, with the technique explained above, for the two processes u and y , then use the formula

$$x = y - \frac{u}{b-a}$$

The tree obtained for x will be a two-dimensional trinomial tree, where every node will have nine branches, result of the combination of the tree branches of u and y .

At time t_i , we define the nodes $y(i, h)$ and $u(i, l)$ so the node for x is $x(i, h, l)$. We define j the index of middle branche (in the tree of y) emanating from $y(i, h)$ and the corresponding probabilities pu , pm , pd , and define k the index of middle branche (in the tree of u) emanating from $u(i, l)$ and the corresponding probabilities qu , qm , qd . Then, starting from $x(i, h, l)$, the process move to nine branches $x(i, j + \epsilon_1, k + \epsilon_2)$, where ϵ_1 and ϵ_2 take the values 0, 1 or -1.

Finally we have to decide the probabilities associated with every node of the nine :

In case of zero correlation between u and y (ie $\rho_{uy} = 0$), the matrix of probabilities for the nine branches of x is simply :

		u-move		
		Down	Middle	Up
y-move	Down	$p_d q_d$	$p_d q_m$	$p_d q_u$
	Middle	$p_m q_d$	$p_m q_m$	$p_m q_u$
	Up	$p_u q_d$	$p_u q_m$	$p_u q_u$

In the case of correlated processes, the elements of the matrix above are shifted in such a way that the sum of shifts in each row and column is zero (to preserve the law of u and y) and to have the good correlation ρ_{uy} between u and y . See [2] for further details.

2.3 Calibration of the tree to the market yield curve :

After the construction of the tree for the process x , the process r can be defined as $r(t) = x(t) + \alpha(t)$ where α is a deterministic function. It is calculated using the Arrow-Debreu node prices and the market price of Zero Coupon Bonds.

We denote by $Q_{i+1,j,k}$ the present value of a security that pays 1 if the node $(i+1, j, k)$ is attained and zero otherwise. These quantities are calculated recursively, knowing α_i and $Q_{i,h,l}$ for all (h, l) , by :

$$Q_{i+1,j,k} = \sum_{h,l} Q_{i,h,l} q_i(h, l, j, k) \exp\{-(\alpha_i + x_{i,h,l}) \Delta t_i\}$$

where $q_i(h, l, j, k)$ is the probability of moving from (i, h, l) to $(i+1, j, k)$. Then, α_{i+1} is calculated by solving :

$$P_M(0, t_{i+2}) = \sum_{i,j} Q_{i+1,j,k} \exp\{-(\alpha_{i+1} + x_{i+1,j,k}) \Delta t_{i+1}\}$$

ie :

$$\alpha_{i+1} = \frac{1}{\Delta t_{i+1}} \ln \frac{\sum_{i,j} Q_{i+1,j,k} \exp\{-x_{i+1,j,k} \Delta t_{i+1}\}}{P_M(0, t_{i+2})}$$

The initial value for α and Q are : $Q_{0,0,0} = 1$ and $\alpha_0 = -\ln(P_M(0, t_1))/t_1$.

3 Pricing of a security using the tree :

Now that we have a trinomial tree of the spot rate $r_{i,j,k}$ with their transition probabilities we can compute the price $h(t, r(t), u(t))$ of any european option

with payoff $H(T, r(T), u(T))$ thanks to a backward induction, starting with $h(T, r(T), u(T)) = H(T, r(T), u(T))$:

$$h_{i,j,k} = e^{-r_{i,j,k}\Delta t_i} \sum_{\epsilon_1, \epsilon_2 \in \{-1, 0, 1\}} h_{i+1, j^* + \epsilon_2, k^* + \epsilon_2} q_i(j, k, j^* + \epsilon_1, k^* + \epsilon_2)$$

Where $(i+1, j^*, k^*)$ is the index of middle branche emanating from (i, j, k) and $q_i(j, k, j^* + \epsilon_1, k^* + \epsilon_2)$ is the probability of moving from (i, j, k) to $(i+1, j^* + \epsilon_1, k^* + \epsilon_2)$.

In the case of an american payoff, we compare the result of the backward induction with the payoff $H(t_i, r_{i,j,k}, u_{i,k})$:

$$h_{i,j,k} = \max \left(H(t_i, r_{i,j,k}, u_{i,k}), e^{-r_{i,j,k}\Delta t_i} \sum_{\epsilon_1, \epsilon_2 \in \{-1, 0, 1\}} h_{i+1, j^* + \epsilon_2, k^* + \epsilon_2} q_i(j, k, j^* + \epsilon_1, k^* + \epsilon_2) \right)$$

References

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References