

Importance Sampling and Statistical Romberg method for Lévy processes

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1 Input parameters

- First and second Monte Carlo samples N_1 and N_2 .
- Generator type
- Spot S_0 , the annual interest rate r , the dividend *divid*, ε and ε_{beta} .
- The CGMY parameters: $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$.
- Maturity T
- Strike K

2 Output parameters

- Price
- Length of the Price confidence interval
- Variance of the first and second Monte Carlo

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3 Description

3.1 CGMY model

The CGMY is a pure jump process with a Lévy density given by:

$$\nu(dx) = C \frac{e^{-Mx}}{x^{1+Y}} \mathbf{1}_{x>0} dx + \frac{C e^{-G|x|}}{|x|^{1+Y}} \mathbf{1}_{x<0} dx.$$

where $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$. The condition $Y < 2$ is induced by the requirement that Lévy density integrate x^2 in the neighborhood of 0.

The characteristic exponent is given by

$$\psi(u) = iu\gamma_c + \Gamma(-Y)M^Y C \left\{ \left(1 - \frac{iu}{M}\right)^Y - 1 + \frac{iuY}{M} \right\} + \Gamma(-Y)G^Y C \left\{ \left(1 + \frac{iu}{G}\right)^Y - 1 - \frac{iuY}{G} \right\}.$$

We use the Statistical Romberg method to compute the prices of European call options in an exponential Lévy model driven by a CGMY process. The stock price is given by:

$$S_T = S_0 e^{rT + L_T}.$$

where L_T is a Lévy process.

3.2 Importance sampling

The optimal θ is given by:

$$\theta_n^* = \arg \min_{\theta \in \mathbb{R}^2} \mathbb{E} \left[\left(\psi^2(L_T) + (\nabla \psi(L_T) \cdot W_T)^2 \right) e^{-\theta L_T + T\kappa(\theta)} \right].$$

To obtain the optimal θ , the idea is to make use a constrained Robbins Monro algorithm which is described as follows: Let $(L_{T,i})_{i \geq 1}$ be an i.i.d copies of the stochastic process (L_T) . Let B be a connected set in \mathbb{R}^d with $\{0\} \in B$. For $\theta_0 \in B$, we construct recursively the sequence of random variables $(\theta_i^\varepsilon)_{i \in \mathbb{N}}$ in \mathbb{R}^d defined by

$$\theta_{i+1}^\varepsilon = \Pi_B \left[\theta_i^\varepsilon - \gamma_{i+1} H(\theta_i^\varepsilon, L_{T,i+1}^\varepsilon) \right] \quad (1)$$

where Π_B is the Euclidean projection onto the constraint set B and the gain sequence $(\gamma_i)_{i \geq 1}$ is a decreasing sequence of positive real numbers satisfying

$$\sum_{i=1}^{\infty} \gamma_i = \infty \text{ and } \sum_{i=1}^{\infty} \gamma_i^2 < \infty \quad (2)$$

The function H is given by

$$H(\theta, L_T) = (T \nabla \kappa(\theta) - L_T) (\psi^2(L_T) + (\nabla \psi(L_T) \cdot W_T)^2) \exp(-\theta \cdot L_T + T\kappa(\theta)). \quad (3)$$

3.3 Calculus of the Statistical Romberg price

By rebalancing the optimal θ_n^* , we compute the price of the considered option using the Statistical Romberg method with importance sampling (SR + IS). Hence, we approximate our initial quantity of interest $\mathbb{E} \psi(L_T)$ ($N_1 = \frac{(2-Y)^2}{4C^2 \varepsilon^{4-2Y}}$ and $N_2 = \frac{(2-Y)}{2C} \varepsilon^{(2-Y)(\beta-2)}$)

$$V_\varepsilon^\theta = \frac{1}{N_1} \sum_{i=1}^{N_1} \psi(L_{T,i+1}^{\varepsilon^\beta, \theta_i^{\varepsilon^\beta}}) e^{-\theta_i^{\varepsilon^\beta} \cdot L_{T,i+1}^{\varepsilon^\beta, \theta_i^{\varepsilon^\beta}} + T\kappa(\theta_i^{\varepsilon^\beta})} + \frac{1}{N_2} \sum_{i=1}^{N_2} \left(\psi(L_{T,i+1}^{\varepsilon, \theta_i^\varepsilon}) - \psi(L_{T,i+1}^{\varepsilon^\beta, \theta_i^\varepsilon}) \right) e^{-\theta_i^\varepsilon \cdot L_{T,i+1}^{\varepsilon, \theta_i^\varepsilon} + T\kappa(\theta_i^\varepsilon)}$$

where with κ is the cummulant generating function of the marginal law at unit time of $\{L_t; t \geq 0\}$ under the probability measure \mathbb{P} , that is, $\kappa_t(\theta) := \ln \mathbb{E} [e^{\theta \cdot L_t}]$.

We compute the length of a 95% confidence interval and the CPU time of our method.

3.4 Complexity Analysis

The optimal average complexity of the Monte Carlo method is given by

$$\mathbb{E} [C_{MC}] = \mathbb{E} [N_T^\varepsilon] \times N$$

where $\mathbb{E} [N_T^\varepsilon] = \nu([\varepsilon, +\infty[) = O(\varepsilon^{-Y})$

Hence,

$$\mathbb{E} [C_{MC}] \simeq C \times \frac{1}{\varepsilon^{4-Y}}$$

The optimal average complexity of the Statistical Romberg method is given by

$$\mathbb{E} [C_{SR}] = \mathbb{E} [N_T^{\varepsilon^\beta}] \times N_1 + (\mathbb{E} [N_T^\varepsilon] + \mathbb{E} [N_T^{\varepsilon^\beta}]) \times N_2$$

where $\mathbb{E} [N_T^{\varepsilon^\beta}] = \nu([\varepsilon^\beta, +\infty[) = O(\varepsilon^{-\beta Y})$

Hence,

$$\mathbb{E} [C_{SR}] \simeq C \times \mathbb{E} [C_{MC}] \varepsilon^{Y(1-\beta)} \left(1 + \varepsilon^{(2-Y)\beta} + \varepsilon^{2\beta-Y} \right)$$

Since the only restriction on β is that $\beta \in (0, 1)$, we will choose the optimal β^* minimizing the average complexity of the Statistical Romberg algorithm. Numerically, we verify easily that for $\beta = \frac{Y}{2}$, the average complexity of the Statistical Romberg method will be minimal in comparison to the Monte Carlo one.