

# Premia - Optimal execution in a Hawkes-based price model

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### 1 The model

The optimal execution model presented here is described in detail in [1].

We consider a single asset and denote by  $P_t$  its price at time  $t$ . We assume

$$P_t = \underbrace{S_t}_{\text{fundamental price}} + \underbrace{D_t}_{\text{mesoscopic price deviation}}.$$

Typically, these quantities are related to the permanent and the transient impact of the market orders, respectively:

$$\begin{aligned} dS_t &= \frac{\nu}{q} \underbrace{dN_t}_{\text{market orders}} \\ dD_t &= \underbrace{-\rho D_t dt}_{\text{market resilience}} + \frac{1-\nu}{q} \underbrace{dN_t}_{\text{market orders}}, \end{aligned}$$

with  $\nu \in [0, 1]$  the proportion of permanent impact,  $\rho > 0$  the resilience speed of the price. The measure of liquidity  $q > 0$  corresponds to the quantity of shares that one should trade to shift the price of one unity (for instance, of one euro if  $P_t$  is the price of a European stock), if the order book were block-shaped.

We introduce the MIH (Mixed-market-Impact Hawkes) model for the process  $N$ :

$$N_t = N_t^+ - N_t^-,$$

where the process  $(N^+, N^-)$  is a symmetric two-dimensional Hawkes process of intensity  $(\kappa^+, \kappa^-)$ , with unpredictable jumps of average size  $m_1 > 0$ . We place ourselves in the Markovian settings

$$\begin{aligned} d\kappa_t^+ &= -\beta(\kappa_t^+ - \kappa_\infty) dt + \iota_s dJ_t^+ + \iota_c dJ_t^-, \\ d\kappa_t^- &= -\beta(\kappa_t^- - \kappa_\infty) dt + \iota_c dJ_t^+ + \iota_s dJ_t^-. \end{aligned} \tag{1}$$

where  $\beta > 0$  is the reversion speed of the intensity,  $\iota_s, \iota_c > 0$  are the self-excitation and cross-excitation parameters, respectively,  $\kappa_\infty > 0$  is the baseline intensity and  $J^\pm$  count the jumps of  $N^\pm$ .

Within this model, we consider a liquidating trader who has a position  $X_t$  in the considered asset at time  $t$ . The initial position  $X_0 = x_0$  is known and should be liquidated at time  $T > 0$ , thus  $X_{T+} = 0$  is imposed.

The liquidating trader impacts the price similarly to other traders, with a proportion  $\epsilon \in [0, 1]$  of permanent impact:

$$\begin{aligned} dS_t &= \frac{1}{q} (\nu dN_t + \epsilon dX_t), \\ dD_t &= -\rho D_t dt + \frac{1}{q} ((1 - \nu)dN_t + (1 - \epsilon)dX_t). \end{aligned} \quad (2)$$

but does not impact the intensities  $\kappa^\pm$ . We make the assumption of a block-shaped Limit Order Book: when the liquidating trader places at time  $t$  an order of size  $v \in \mathbb{R}$  ( $v > 0$  for a buy order and  $v < 0$  for a sell order), it has the cost

$$\pi_t(v) = \int_0^v \left[ P_t + \frac{1}{q} y \right] dy = \underbrace{P_t v}_{\text{cost at the current price}} + \underbrace{\frac{v^2}{2q}}_{\text{impact cost}}. \quad (3)$$

Therefore, the cost of the whole strategy  $X$  is given by

$$\begin{aligned} C(X) &= \int_{[0, T)} P_u dX_u + \frac{1}{2q} \sum_{\tau \in \mathcal{D}_X \cap [0, T)} (\Delta X_\tau)^2 - P_T X_T + \frac{1}{2q} X_T^2 \\ &= \int_{[0, T)} P_u dX_u^c + \sum_{\tau \in \mathcal{D}_X \cap [0, T)} P_\tau (\Delta X_\tau) + \frac{1}{2q} \sum_{\tau \in \mathcal{D}_X \cap [0, T)} (\Delta X_\tau)^2 - P_T X_T + \frac{1}{2q} X_T^2, \end{aligned}$$

since at time  $T$  all the remaining assets have to be liquidated. Here, the sum brings on the countable times of discontinuity  $\mathcal{D}_X$  of  $X$ , and the jumps  $\Delta X_\tau = X_{\tau+} - X_\tau \neq 0$  for  $\tau \in \mathcal{D}_X$ .

The optimal strategy for this model is obtained as a closed formula in [1]. We introduce the processes

$$\delta_t = \kappa_t^+ - \kappa_t^-, \quad \Sigma_t = \kappa_t^+ + \kappa_t^-, \quad (4)$$

along with the useful quantities

$$\alpha = \iota_s - \iota_c, \quad \eta = \beta - \alpha,$$

and the two continuously differentiable functions  $\zeta, \omega : \mathbb{R} \rightarrow \mathbb{R}^+$  defined by

$$\begin{aligned} \zeta(0) &= 1 \quad \text{and} \quad \forall y \neq 0, \quad \zeta(y) = \frac{1 - \exp(-y)}{y}, \\ \zeta'(0) &= -1/2 \quad \text{and} \quad \forall y \neq 0, \quad \zeta'(y) = \frac{(1 + y) \exp(-y) - 1}{y^2} = \frac{\exp(-y) - \zeta(y)}{y}, \\ \omega(0) &= 1/2 \quad \text{and} \quad \forall y \neq 0, \quad \omega(y) = \frac{\exp(-y) - 1 + y}{y^2} = \frac{1 - \zeta(y)}{y}, \\ \omega'(0) &= -1/6 \quad \text{and} \quad \forall y \neq 0, \quad \omega'(y) = \frac{2(1 - \exp(-y)) - y(1 + \exp(-y))}{y^3} = \frac{2\zeta(y) - 1 - \exp(-y)}{y^2}. \end{aligned}$$

The optimal strategy  $X^*$  is characterized by

$$(1 - \epsilon)X_t^* = -[1 + \rho(T - t)] D_t^* + \frac{m_1}{2\rho} \times [2 + \rho(T - t) \times \{1 + \zeta(\eta(T - t)) + \nu\rho(T - t) \omega(\eta(T - t))\}] \delta_t \quad (5)$$

for  $t \in [0, T)$ , and

$$X_{T+}^* - X_T^* = -X_T^*,$$

so that  $X_{T+}^* = 0$ . The expected cost of this strategy has the form

$$\begin{aligned}
q \times \mathcal{C}(T, x_0, D_0, S_0, \delta_0, \Sigma_0) &= -q(S_0 + D_0)x_0 + \left[ \frac{1-\epsilon}{2+\rho T} + \frac{\epsilon}{2} \right] x_0^2 + \frac{\rho T}{2+\rho T} \left[ qd - \mathcal{G}_\eta(T) \frac{\delta_0 m_1}{\rho} \right] x_0 \\
&\quad - \frac{1}{1-\epsilon} \times \frac{\rho T/2}{2+\rho T} \left[ qd - \mathcal{G}_\eta(T) \frac{\delta_0 m_1}{\rho} \right]^2 + \hat{c}_\eta(T) \left( \frac{\delta_0 m_1}{\rho} \right)^2 \\
&\quad + e(T) \Sigma_0 + g(T).
\end{aligned} \tag{6}$$

where for  $u \in [0, T]$ ,

$$\begin{aligned}
\mathcal{G}_\eta(u) &= \zeta(\eta u) + \nu \rho u \omega(\eta u), \\
\hat{c}_\eta(u) &= \frac{1}{1-\epsilon} \times (\eta - \nu \rho)^2 \frac{\rho u^3}{8} \omega'(\eta u) \zeta(\eta u).
\end{aligned}$$

The functions  $e$  and  $g$  are uniquely determined but have cumbersome expressions in general. However, in the case  $\iota_c = 0$ ,  $\beta = \alpha = \iota_s$ , one has the simpler formulas

$$\begin{aligned}
e(u) &= -\frac{(1-\nu)^2}{1-\epsilon} \times \left( m_2 - \frac{\beta m_1^2 (2\rho - \beta)}{\rho^2} \right) \times \left[ \frac{u}{2} - \frac{1}{\rho} \ln \left( 1 + \frac{\rho u}{2} \right) \right] \\
&\quad + \frac{\beta \nu (1-\nu) m_1^2}{4\rho^2 (1-\epsilon)} \times \left( 1 - \frac{\beta}{\rho} \right) \times \rho^2 u^2 - \frac{\beta^2 \nu^2 m_1^2}{8\rho^3 (1-\epsilon)} \times \left[ \rho^2 u^2 + \frac{1}{3} \rho^3 u^3 + \frac{1}{24} \rho^4 u^4 \right], \\
g(u) &= -2\beta \kappa_\infty \times \frac{(1-\nu)^2}{1-\epsilon} \times \left( m_2 - \frac{\beta m_1^2 (2\rho - \beta)}{\rho^2} \right) \left\{ \left( u + \frac{2}{\rho} \right) \left[ \frac{u}{2} - \frac{1}{\rho} \ln \left( 1 + \frac{\rho u}{2} \right) \right] - \frac{u^2}{4} \right\} \\
&\quad + \frac{\beta^2 \kappa_\infty \nu (1-\nu) m_1^2}{6\rho^3 (1-\epsilon)} \times \left( 1 - \frac{\beta}{\rho} \right) \times \rho^3 u^3 - \frac{\beta^3 \kappa_\infty \nu^2 m_1^2}{12\rho^4 (1-\epsilon)} \times \left[ \rho^3 u^3 + \frac{1}{4} \rho^4 u^4 + \frac{1}{40} \rho^5 u^5 \right].
\end{aligned}$$

## 2 Closed-formula cost

The ‘‘Closed-formula cost’’ functionality implemented in *Premia* takes the parameters of the model as input and returns

- The evaluation of formula (6) if the condition  $\iota_c = 0$ ,  $\beta = \alpha = \iota_s$  is satisfied.
- Zero as a default value otherwise.

## 3 Monte Carlo cost

The ‘‘Monte Carlo cost’’ functionality needs two additional input values: the number of simulations  $n \geq 1$  and the discretization step  $h > 0$ . It computes the cost of the optimal strategy averaged over all simulations, along with a confidence interval.

For each simulation, we first determine a realization of the bi-dimensional Hawkes process  $(N^+, N^-)$  on  $[0, T]$  using equation (1). We resort to the thinning method described by the following algorithm:

- Initialize  $t = 0$ ,  $\kappa^+ = \kappa_0^+$ ,  $\kappa^- = \kappa_0^-$ ,  $\Sigma = \kappa_0^+ + \kappa_0^-$ .

- While  $t < T$ ,
  1. Simulate an exponential variable  $\tau$  of rate  $\Sigma$ .
  2. Set  $t = t + \tau$  and  $\kappa^\pm = \kappa_\infty + (\kappa^\pm - \kappa_\infty) \exp(-\beta\tau)$ .
  3. Simulate a uniform variable  $U$  on  $[0, 1]$ .
  4. If  $U \leq \kappa^+/\Sigma$  and  $t \leq T$ , record  $t$  as a jump time of  $N^+$  and set  $\kappa^+ = \kappa^+ + \iota_s$ ,  $\kappa^- = \kappa^- + \iota_c$ .
  5. If  $\kappa^+/\Sigma < U \leq (\kappa^+ + \kappa^-)/\Sigma$  and  $t \leq T$ , record  $t$  as a jump time of  $N^-$  and set  $\kappa^+ = \kappa^+ + \iota_c$ ,  $\kappa^- = \kappa^- + \iota_s$ .
  6. Set  $\Sigma = \kappa^+ + \kappa^-$ .

Then, we choose a positive probability law  $\mu$  of expected value  $m_1$  for the amplitudes of the jumps. It should be easy to simulate (for instance, an exponential or a log-normal law are suitable). For each jump time of  $N^+$  and each jump time of  $N^-$ , we simulate a variable of law  $\mu$  independently of the rest and record it as the amplitude of the corresponding jump.

Once the trajectory of  $(N^+, N^-)$  is simulated, we determine the corresponding discretized optimal strategy. The discretization grid  $(t_i)_{i \geq 0}$  is built incrementally to have steps smaller than  $h$  and to include  $t = 0$ ,  $t = T$  and all the jumps of  $(N^+, N^-)$ :

- Set  $t_0 = 0$  and  $i = 0$ .
- While  $t_i < T - h$ ,
  1. Define  $\theta$  as the smallest jump time of  $(N^+, N^-)$  greater than  $t_i$ .
  2. If  $\theta < t_i + h$ , set  $t_{i+1} = \theta$ . Else, set  $t_{i+1} = t_i + h$ .
  3. Set  $i = i + 1$ .
- Set  $t_{i+1} = T$  and  $i = i + 1$ .

For each discretization time  $t_j$  (except the last one), we compute  $D_{t_j}$  and  $\delta_{t_j}$  using equations (1), (2) and (4), then we deduce the optimal order from equation (5)

$$\begin{aligned} \Delta X_{t_j}^* &= -\frac{x_0}{2 + \rho(T - t_j)} \\ &+ \frac{1}{1 - \epsilon} \times \frac{2 + \rho(T - t_j)[1 + \zeta(\eta(T - t_j)) + \nu\rho(T - t_j)\omega(\eta(T - t_j))]}{2 + \rho(T - t_j)} \times \frac{\delta_{t_j} m_1}{2\rho} \\ &- \frac{1}{1 - \epsilon} \times \frac{1 + \rho(T - t_j)}{2 + \rho(T - t_j)} \times D_{t_j}. \end{aligned}$$

At time  $T$ , one has  $\Delta X_T^* = -X_T^*$ . The cost of each order is given by equation (3), and we sum these costs to obtain the total cost of the strategy for the current simulation.

## References

- [1] Dynamic optimal execution in a mixed-market-impact Hawkes price model, Alfonsi, Aurélien and Blanc, Pierre *arXiv preprint arXiv:1404.0648v2, forthcoming in Finance and Stochastics*