

Premia 22

Libor market model with volatility skews

Let $P(t, T)$ denote the price at time $t \leq T$ of the zero-coupon bond maturing at time T . With the maturity structure $0 = T_0 < T_1 < \dots < T_{K+1}$, one associates the forward rates

$$F_k(t) = \frac{1}{T_{k+1} - T_k} \left(\frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right), \quad 0 \leq k \leq K.$$

Under the forward measure \mathbb{Q}^{k+1} induced by using $P(t, T_{k+1})$ as a numeraire, the process $(F_k(t))_{t \leq T_k}$ is a martingale. This martingale is assumed to be governed by a \mathbb{Q}^{k+1} -Brownian motion W_t^{k+1} and in order to generate volatility skews, Andersen and Andreasen [1] propose to generalize the usual lognormal dynamics $dF_k(t) = F_k(t)\lambda_k dW_t^{k+1}$ by using CEV processes

$$dF_k(t) = F_k^\alpha(t)\lambda_k dW_t^{k+1} \quad (1)$$

where $\alpha > 0$. In this model, the initial price $C_k(t)$ of the caplet paying $(T_{k+1} - T_k)(F_k(T_k) - H)^+$ at time T_{k+1} is given by the following formula generalizing Black's formula which holds for $\alpha = 1$:

$$\frac{C_k(t)}{(T_{k+1} - T_k)P(t, T_{k+1})} = \begin{cases} [F_k(t)(1 - \chi^2(a, b + 2, c)) - H\chi^2(c, b, a)] & \text{if } \alpha \in (0, 1), \\ [F_k(t)N(x^+) - HN(x^-)] & \text{if } \alpha = 1, \\ [F_k(t)(1 - \chi^2(c, -b, a)) - H\chi^2(a, 2 - b, c)] & \text{if } \alpha > 1, \end{cases}$$

where

$$\begin{cases} a = \frac{H^{2(1-\alpha)}}{(1-\alpha)^2 \lambda_k^2 (T_k - t)} \\ b = \frac{1}{1-\alpha} \\ c = \frac{F_k(t)^{2(1-\alpha)}}{(1-\alpha)^2 \lambda_k^2 (T_k - t)} \\ x_{\pm} = \frac{\ln(F_k(t)/H) \pm \frac{1}{2} \lambda_k^2 (T_k - t)}{\sqrt{\lambda_k^2 (T_k - t)}} \end{cases}.$$

and $N(\cdot)$ and $\chi^2(\cdot, \theta, \gamma)$ respectively denote the cumulative distribution functions of the normal law and of the non-central χ^2 distribution with non-centrality parameter γ and θ degrees of freedom.

Of course, in case the Brownian motion W_t^{k+1} is assumed to be m -dimensional and the constant real volatility coefficient λ_k is replaced by a \mathbb{R}^m -valued function $\lambda_k(\cdot)$ of the time variable, the above formulas are easily extended by replacing $\lambda_k^2(T_k - t)$ by $\int_t^{T_k} \|\lambda_k\|^2(u) du$.

Since

$$C_k(t) = (T_{k+1} - T_k)P(t, T_{k+1})\mathbb{E}^{\mathbb{Q}^{k+1}}((F_k(T_k) - H)^+ | \mathcal{F}_t)$$

Monte-Carlo valuation of caplets is easily implemented by means of a suitable time-discretization of the stochastic differential equation (1).

In order to price a more complex derivative security with payout V_T at time $T \leq T_{K+1}$ depending on the path of all forwards $F_k(u)$, $0 \leq k \leq K$ on the time interval $[t, T]$, one uses the spot measure \mathbb{Q} under which for $n(t) =$

$\sum_{k=1}^{K+1} k 1_{\{T_{k-1} \leq t < T_k\}}$, $W_t^{n(t)}$ is a Brownian motion . The numeraire of the spot measure is

$$B_t = P(t, T_{n(t)}) \prod_{j=0}^{n(t)-1} \frac{1}{P(T_j, T_{j+1})} = P(t, T_{n(t)}) \prod_{j=0}^{n(t)-1} (1 + (T_{j+1} - T_j)F_j(T_j)).$$

The brownian motions W_t^{k+1} and $W_t^{n(t)}$ respectively associated with the forward measure \mathbb{Q}_{k+1} and the spot measure \mathbb{Q} are related by

$$dW_t^{k+1} = dW_t^{n(t)} + \sum_{j=n(t)}^k \frac{(T_{j+1} - T_j)F_j^\alpha(t)\lambda_j}{1 + (T_{j+1} - T_j)F_j(t)}.$$

Since the price V_t at time $t \leq T$ of the derivative security is given by

$$V_t = B_t \mathbb{E}^{\mathbb{Q}} \left(\frac{V_T}{B_T} \middle| \mathcal{F}_t \right)$$

Monte-Carlo computation of the expectation is possible after discretization of the stochastic differential equations (1) under the spot measure \mathbb{Q} .

References

- [1] L. Andersen and J. Andreasen, *Volatility skews and extensions of the Libor market model*. Applied Mathematical Finance 7, 1-32, 2000 [1](#)