

Premia 22

FAST WIENER-HOPF FACTORIZATION METHOD FOR OPTION PRICING UNDER LÉVY PROCESSES

OLEG KUDRYAVTSEV

*Department of Informatics, Russian Customs Academy Rostov Branch,
Budennovskiy 20, Rostov-on-Don, 344004, Russia
E-mail: koe@donrta.ru*

ABSTRACT. We suggest new fast and accurate method, Fast Wiener-Hopf method (FWH-method), for pricing barrier and American options for a wide class of Lévy processes. The method uses the Wiener-Hopf factorization and Fast Fourier Transform algorithm. We demonstrate the accuracy and fast convergence of the method using Monte-Carlo simulations and an accurate finite difference scheme, compare our results with the results obtained by Cont-Voltchkova method, and explain the differences in prices near the barrier. The results are based on the ones obtained in Kudryavtsev, O.E., and S.Z. Levendorskiĭ (2007,2008).

1. INTRODUCTION

In this introductory section, we give a short overview of general results on pricing barrier options in exponential Lévy models. By now, there exist several large groups of relatively universal numerical methods for pricing of American and barrier options under exponential Lévy processes. The number of publications is huge, and, therefore, an exhaustive list is virtually impossible. We concentrate on the one-dimensional case. First, we will consider the model case of a down-and-out put option, without rebate. The down-and-out call options and up-and-out calls and puts can be reduced to the model case by suitable change of numeraire and/or the direction on the real axis. The method is applicable to American options as well, and, after straightforward modifications, to barrier options with a rebate (hence, to digitals as well). In Section 2, we introduce Fast Wiener-Hopf factorization method. In Section 3, we apply FWH-method to American options. Numerical examples are presented in Section 4. The background of the theory of Lévy processes and PDO and explicit numerical algorithms are relegated to appendices.

1.1. General set-up. Let T, K, H be the maturity, strike and barrier, and $S_t = e^{X_t}$ the stock price under a chosen risk-neutral measure. The riskless rate is assumed constant. Set $h = \ln H$. Then the payoff at maturity is $\mathbf{1}_{(h, +\infty)}(X_T)G(X_T)$, where $G(x) = (K - e^x)_+$, and the no-arbitrage price of the barrier option at time $t < T$ and $X_t = x > h$ is given by

$$(1.1) \quad V(t, x) = V(T, H; G; t, x) = E^{t, x} \left[e^{-r(T-t)} \mathbf{1}_{\underline{X}_T > h} G(X_T) \right],$$

1991 *Mathematics Subject Classification.* Primary-90A09, 60G44, 60G40, 60H30; secondary- 49K05.

Key words and phrases. Lévy processes, barrier options, American options, Wiener-Hopf factorization, numerical methods.

where $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ is the infimum process. In Subsection A.1, we remind the definitions of the Lévy density $F(dy)$ of the Lévy process X_t and characteristic exponent $\psi(\xi)$, and explicit formulas for ψ and L , the infinitesimal generator of X_t . We also list several classes of Lévy processes used in the main body of the paper.

We start with the reduction to the boundary problem for the option price, which is the first step for many methods.

1.2. Boundary problem for the price of the down-and-out barrier option and explicit formulas for the price. The boundary problem for $V(t, x)$ is of the form

$$(1.2) \quad (\partial_t + L - r)V(t, x) = 0, \quad x > h, t < T;$$

$$(1.3) \quad V(t, x) = 0, \quad x \leq h, t \leq T;$$

$$(1.4) \quad V(T, x) = G(x), \quad x > h.$$

Boyarchenko and Levendorskii (1999, 2002b, c) derived the generalization of the Black-Scholes equation (1.2) under a weak regularity condition: the process (t, X_t) in 2D satisfies (ACP)-condition (for the definition, see, e.g., Sato (1999)). Note that (ACP) condition is satisfied if the process X_t has the transition density. Equation (1.2) is understood in the sense of the theory of generalized functions: for any infinitely smooth function u with compact support $\text{supp } u \subset (-\infty, T) \times (h, +\infty)$,

$$(1.5) \quad (V, (-\partial_t + \tilde{L} - r)u)_{L_2} = 0,$$

where \tilde{L} is the infinitesimal generator of the dual process. One cannot state that the boundary problem (1.2)–(1.3) has a unique bounded solution for an arbitrary Lévy process. However, if the characteristic exponent ψ is sufficiently regular, which is the case for Lévy processes used in empirical studies of financial markets, then the general technique of the theory of PDO can be applied to show that a bounded solution, which is continuous on $\text{supp } u \subset (-\infty, T) \times (h, +\infty)$, is unique – see, e.g., Kudryavtsev and Levendorskii (2006).

Later, Cont and Voltchkova (2005) (see also Cont and Tankov (2004)) derived the same equation (1.2) under much more stringent conditions. The latter version is called *partial integro-differential equation* (PIDE) and uses the viscosity solution technique, which is better known in probability than the language and technique of the theory of pseudo-differential operators (PDO) used by Boyarchenko and Levendorskii (1999, 2002b, c). Notice, however, that the definition of a solution in the sense of generalized functions is standard in analysis for half a century, at least, and it is more natural for linear problems than the language of viscosity solutions invented to tackle much more difficult non-linear problems. Moreover, the PDO technique based on the Fourier transform is much more powerful than the technique based on the study of the kernel of the PIDE. This was the reason the theory of PDO was invented in the first place – see, e.g., Eskin (1973) and Hörmaner (1985).

1.3. Carr’s randomization and Wiener-Hopf factorization.

1.3.1. Carr’s randomization. The methods constructed in this paper start with time discretization (the method of lines). This method was introduced to finance by Carr and Faguet (1994); Carr (1998) suggested a new important probability interpretation of the method, which we call Carr’s randomization. One discretizes time $(0 =)t_0 < t_1 < \dots <$

$t_N(=T)$ but not the space variable. Set $V_N(x) = G(x)$. For $s = N-1, N-2, \dots, 0$, set $\Delta_s = t_{s+1} - t_s$, $q^s = r + (\Delta_s)^{-1}$, denote by $V_s(x)$ the Carr's randomized approximation to $V(t_s, x)$, and replace the time derivative in (1.2) with the finite difference. The result is a sequence of boundary problems on the line, which can be written as follows: for $s = N-1, N-2, \dots$, find a (unique) bounded measurable V^s , which satisfies

$$(1.6) \quad (q^s)^{-1}(q^s - L)V_s(x) = \frac{1}{q^s \Delta_s} V_{s+1}(x), \quad x > h,$$

$$(1.7) \quad V_s(x) = 0, \quad x \leq h.$$

The probabilistic version is: for $s = N-1, N-2, \dots$, calculate

$$(1.8) \quad V_s(x) = E^x \left[\int_0^\tau e^{-q^s t} (\Delta_s)^{-1} V_{s+1}(X_t) dt \right],$$

where τ is the hitting time of $(-\infty, h]$. In the paper, we will use the uniform spacing, therefore, q^s and Δ_s will be independent of s and denoted q and Δt , respectively.

The convergence of the Carr's randomization procedure for barrier options is proved by M.Boyarchenko (2008). Each problem in the sequence can be solved explicitly using the operator form of the Wiener-Hopf factorization developed in Boyarchenko and Levendorskii (2002a-c, 2005, 2007, 2008) to price barrier and American options. The operator form of the Wiener-Hopf factorization is a standard analytical tool for solution of boundary problems for pseudo-differential equations; the new element is the interpretation of the factors denoted \mathcal{E}^\pm as the expected present value operators (EPV-operators) – operators which calculate the (normalized discounted) expected present values of streams of payoffs under supremum and infimum processes.

1.3.2. The Wiener-Hopf factorization. Let q be a positive real number. The Wiener-Hopf factorization formula used in probability reads:

$$(1.9) \quad E[e^{i\xi X_T}] = E[e^{i\xi \bar{X}_T}] E[e^{i\xi \underline{X}_T}], \quad \forall \xi \in \mathbb{R},$$

where $T \sim \text{Exp } q$, and $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ are the supremum and infimum processes. Introducing the notation

$$(1.10) \quad \phi_q^+(\xi) = qE \left[\int_0^\infty e^{-qt} e^{i\xi \bar{X}_t} dt \right] = E \left[e^{i\xi \bar{X}_T} \right],$$

$$(1.11) \quad \phi_q^-(\xi) = qE \left[\int_0^\infty e^{-qt} e^{i\xi \underline{X}_t} dt \right] = E \left[e^{i\xi \underline{X}_T} \right]$$

we can write (1.9) as

$$(1.12) \quad \frac{q}{q + \psi(\xi)} = \phi_q^+(\xi) \phi_q^-(\xi).$$

Equation (1.12) is a special case of the Wiener-Hopf factorization of the symbol of a PDO (see Subsection A.2 for the definition of PDO). In applications to Lévy processes, the symbol is $q/(q + \psi(\xi))$, and the PDO is $\mathcal{E} = q/(q - L) = q(q + \psi(D))^{-1}$: the normalized resolvent of the process X_t or, using the terminology of Boyarchenko and Levendorskii

(2005, 2007, 2008), the expected present value operator (EPV-operator) of the process X_t . The name is due to the observation that, for a stream $g(X_t)$,

$$\mathcal{E}g(x) = E \left[\int_0^{+\infty} qe^{-qt} g(X_t) dt \mid X_0 = x \right].$$

The factors $\phi_q^\pm(\xi)$ also admit interpretation as the symbols of the EPV-operators $\mathcal{E}^\pm = \phi_q^\pm(D)$ under supremum and infimum processes

$$\begin{aligned} \mathcal{E}^+g(x) : &= qE \left[\int_0^\infty e^{-qt} g(\bar{X}_t) dt \mid X_0 = x \right] \\ \mathcal{E}^-g(x) : &= qE \left[\int_0^\infty e^{-qt} g(\underline{X}_t) dt \mid X_0 = x \right]. \end{aligned}$$

One of the basic observations in the theory of PDO is that the product of symbols corresponds to the product of operators. In our case, it follows from (1.12) that

$$(1.13) \quad \mathcal{E} = \mathcal{E}^+ \mathcal{E}^- = \mathcal{E}^- \mathcal{E}^+$$

as operators in appropriate function spaces.

The general results in this paper are based on simple properties of the EPV operators, which are immediate from the interpretation of \mathcal{E}^\pm as expectation operators. For details, see Boyarchenko and Levendorskii (2005).

Proposition 1.1. *EPV-operators \mathcal{E}^\pm enjoy the following properties*

- (a) *If $g(x) = 0 \ \forall x \geq h$, then $\forall x \geq h$, $(\mathcal{E}^+g)(x) = 0$ and $((\mathcal{E}^+)^{-1}g)(x) = 0$.*
- (b) *If $g(x) = 0 \ \forall x \leq h$, then $\forall x \leq h$, $(\mathcal{E}^-g)(x) = 0$ and $((\mathcal{E}^-)^{-1}g)(x) = 0$.*
- (c) *If $g(x) \geq 0 \ \forall x$, then $(\mathcal{E}^+g)(x) \geq 0, \ \forall x$. If, in addition, there exists x_0 such that $g(x) > 0 \ \forall x > x_0$, then $(\mathcal{E}^+g)(x) > 0 \ \forall x$.*
- (d) *If $g(x) \geq 0 \ \forall x$, then $(\mathcal{E}^-g)(x) \geq 0, \ \forall x$. If, in addition, there exists x_0 such that $g(x) > 0 \ \forall x < x_0$, then $(\mathcal{E}^-g)(x) > 0 \ \forall x$.*
- (e) *If g is monotone, then \mathcal{E}^+g and \mathcal{E}^-g are also monotone.*
- (f) *If g is continuous and satisfies*

$$(1.14) \quad |g(x)| \leq C(e^{\sigma_-x} + e^{\sigma_+x}), \quad \forall x \in \mathbb{R},$$

where $\sigma_- \leq 0 \leq \sigma_+$ and C are independent of x , then \mathcal{E}^+g and \mathcal{E}^-g are continuous.

1.3.3. *Explicit solution of problem (1.6)–(1.7).* In Boyarchenko and Levendorskii (2002a, b), it was shown that the unique bounded solution is given by

$$(1.15) \quad V^s = \frac{1}{q\Delta t} \mathcal{E}^- \mathbf{1}_{[h, +\infty)} \mathcal{E}^+ V^{s+1},$$

where $\mathbf{1}_{[h, +\infty)}$ is the indicator function of $[h, +\infty)$. In Boyarchenko and Levendorskii (2002a-c, 2005, 2007), no efficient numerical realization of the action of EPV-operators was suggested. Levendorskii (2004) constructed a very accurate and fast realization of the pricing procedure for the American put in Kou's model and more general HEJD model. For these models, efficient procedures are possible because the EPV-operators are linear combinations of convolution operators with exponential kernels. If the analytic

expressions for the Wiener-Hopf factors $\phi_q^\pm(\xi)$ are available, one can calculate V^s using Fourier transform F and its inverse F^{-1} :

$$(1.16) \quad V^s(x) = \frac{1}{q\Delta t} F_{\xi \rightarrow x}^{-1} \phi_q^-(\xi) F_{x \rightarrow \xi} \mathbf{1}_{[h, +\infty)}(x) F_{\xi \rightarrow x}^{-1} \phi_q^+(\xi) F_{x \rightarrow \xi} V^{s+1}(x).$$

1.4. New variants of Carr's randomization–WHF method. The advantage of the straightforward WHF-method and of the two variations constructed in the paper is that the price of each option in the sequence has the same order of the asymptotics near the barrier as the initial barrier option with finite time horizon, therefore, one can expect that WHF-approach is more accurate than approaches, which approximate small jumps by an additional diffusion component.

The main new element of the Fast Wiener-Hopf (FWH) method constructed in Section 2 is an efficient approximations to the EPV-operators, which leads to a very fast and accurate pricing procedure. FWH-method is fully implicit because we approximate the operators in the exact formula for the solution of the boundary problem.

2. FAST WIENER-HOPF FACTORIZATION METHOD (FWH-METHOD)

2.1. General formulas. We start with the exact formula (1.16). Once sufficiently accurate approximations to the factors ϕ_q^\pm are constructed, the calculation of the option price becomes accurate and fast. The FWH-method enjoys an additional appealing feature: for arbitrary number of time steps, approximate formulas for ϕ_q^\pm are needed at the first and last steps in the cycle in s only. At all intermediate steps, the exact analytic expression $q/(q + \psi(\xi))$ is used, and, therefore, almost all errors are errors of FFT and iFFT only. Indeed, for $s = N - 1, N - 2, \dots, 0$, define

$$(2.1) \quad W^s = \mathbf{1}_{[h; +\infty)} \mathcal{E}^+ V^{s+1}.$$

Then

$$(2.2) \quad V^s = (q\Delta t)^{-1} \mathcal{E}^- W^s(x).$$

Using the Wiener-Hopf factorization formula (1.13), we obtain that

$$(2.3) \quad W^s = (q\Delta t)^{-1} \mathbf{1}_{[h; +\infty)} \mathcal{E} W^{s+1},$$

or, equivalently,

$$(2.4) \quad \hat{W}^s(\xi) = (q\Delta t)^{-1} F_{x \rightarrow \xi} \mathbf{1}_{[h; +\infty)} F_{\xi \rightarrow x}^{-1} \frac{q}{q + \psi(\xi)} \hat{W}^{s+1}(\xi).$$

Thus, if we are interested in the option value V^0 at the last step only, the algorithm becomes very simple. However, the crucial problem of efficient calculation of the Wiener-Hopf factors remains.

2.2. Reduction to symbols of order 0. The new ingredient is the reduction of the factorization problems to symbols of order 0, which stabilize at infinity to 1. We explain the reduction for a wide class of Lévy processes which consists of Variance Gamma processes and RLPE of order $\nu \in (0; 2]$ (see basic facts and definitions on Lévy processes in Appendix A).

Introduce functions

$$(2.5) \quad \Lambda_-(\xi) = \lambda_+^{\nu_+/2} (\lambda_+ + i\xi)^{-\nu_+/2};$$

$$(2.6) \quad \Lambda_+(\xi) = (-\lambda_-)^{\nu_-/2} (-\lambda_- - i\xi)^{-\nu_-/2};$$

$$(2.7) \quad \Phi(\xi) = q \left((q + \psi(\xi)) \Lambda_+(\xi) \Lambda_-(\xi) \right)^{-1}.$$

Choices of ν_+ and ν_- depend on properties of ψ , hence on order ν (see (A.16)–(A.17)) and drift μ . We have to consider the following cases.¹

- (1) If X_t – RLPE of order $\nu \in (1, 2)$, we set $\nu_+ = \nu_- = \nu/2$.
- (2) If X_t – RLPE of order $\nu \in (0, 1]$ and drift $\mu = 0$, we set $\nu_+ = \nu_- = \nu/2$.
- (3) If X_t – RLPE of order $\nu \in (0, 1)$ and drift $\mu > 0$, we set $\nu_+ = 0, \nu_- = 1$.
- (4) If X_t – RLPE of order $\nu \in (0, 1)$ and drift $\mu < 0$, we set $\nu_+ = 1, \nu_- = 0$.
- (5) If X_t – VGP and drift $\mu > 0$, we set $\nu_+ = 0, \nu_- = 1$.
- (6) If X_t – VGP and drift $\mu < 0$, we set $\nu_+ = 1, \nu_- = 0$.

Functions $\Lambda_{\pm}(\xi)$ are analytic and do not vanish in the half-plane $\pm \operatorname{Im} \xi > 0$, continuous up to the boundary. In addition, $\Lambda_{\pm}(\xi)$ and its reciprocal grow not faster than a polynomial. Therefore, it remains to factorize

$$(2.8) \quad \Phi(\xi) = \Phi^+(\xi) \Phi^-(\xi),$$

and then set

$$(2.9) \quad \phi_q^{\pm}(\xi) = \Lambda_{\pm}(\xi) \Phi^{\pm}(\xi).$$

2.3. Approximation of symbols $\Phi^{\pm}(\xi)$. In this subsection, we, first, approximate Φ by a periodic function Φ_d with a large period $2\pi/d$, which is the length of the truncated region in ξ -state, then approximate the latter by a partial sum of the Fourier series, and, finally, use the factorization of the latter instead of the exact one.

We can apply this realization both after the reduction to symbols of order 0 has been made, and without this reduction. In the latter case, we obtain a Poisson type approximation.

It is well-known (see e.g. Lukacs, E., 1960) that the limit of a sequence of the Poisson type characteristic functions is infinitely divisible characteristic function. The converse is also true. Every infinitely divisible characteristic function can be written as the limit of a sequence of finite products of Poisson type characteristic functions. Since $\psi(\xi)$ is the characteristic exponent of Lévy process, then the function $q/(q + \psi(\xi))$ is infinitely divisible characteristic function.

The second step is straightforward. We impose an additional condition

$$(2.10) \quad |\Phi'(\xi)| \leq C(1 + |\xi|)^{-\rho},$$

where $\rho > 0$; this condition is satisfied by all RLPEs and VGPs (and can be relaxed), which makes the following lemma applicable.

¹We can reduce the case $\nu \in (0, 1]$ and $\mu \neq 0$ to (2) after the elimination of the drift. For driftless VGPs we reduce to the cases (5) or (6) by suitable change of numeraire

Consider a function $f_d \in C^1[-\pi/d, \pi/d]$ depending on a small parameter d , and, for $m > 0$, construct the partial sum

$$S_m(f_d) = \frac{d}{2\pi} \sum_{|k| \leq m} \hat{f}_{d,k} e^{ikx}$$

of the Fourier series for f_d .

Lemma 2.1. *Let $f_d(-\pi/d) = f_d(\pi/d)$, and let there exists $C > 0$ such that for all $x \in [-\pi/d, \pi/d]$ and all $d \in (0, 1]$, $|\partial_x f_d(x)| \leq C$.*

Then there exists a function $d \mapsto m_0(d)$ such that $\forall m \geq m_0(d)$, $x \in [-\pi/d, \pi/d]$ and $d \in (0, 1]$,

$$(2.11) \quad |f_d(x) - S_m(f_d)(x)| \leq \epsilon.$$

Proof. Changing the variables $x = x'/d$, we see that it suffices to prove lemma for a $g_d \in C^1[-\pi, \pi]$, such that $g_d(-\pi) = g_d(\pi)$ and for all $x \in [-\pi, \pi]$ and all $d \in (0, 1]$, $|\partial_x g_d(x)| \leq C/d$. Set

$$S_m(g_d) = \frac{1}{2\pi} \sum_{|k| \leq m} \hat{g}_{d,k} e^{ikx}.$$

We need to prove that there exists a function $d \mapsto m_0(d)$ such that $\forall m \geq m_0(d)$, $x \in [-\pi, \pi]$ and $d \in (0, 1]$,

$$(2.12) \quad |g_d(x) - S_m(g_d)(x)| \leq \epsilon.$$

The proof is a modification of the classical proof of the uniform convergence of the Fourier series due to U.Dini. We use

$$S_n(g_d)(x) - g_d(x) = \frac{1}{\pi} \int_0^\pi h_d(x, t) \frac{\sin((n+1/2)t)}{2 \sin(t/2)} dt,$$

where $h_d(x, t) = g_d(x+t) + g_d(x-t) - 2g_d(x)$, which presumes that g_d is extended to a periodic function. For any $\epsilon > 0$, set $\delta = \epsilon d / (2C)$. Then

$$(2.13) \quad \left| \frac{1}{\pi} \int_0^\delta h_d(x, t) \frac{\sin((n+1/2)t)}{2 \sin(t/2)} dt \right| \leq \frac{1}{2} \int_0^\delta \frac{|h_d(x, t)|}{t} dt \leq \delta C / d \leq \epsilon / 2.$$

Next, integrating by parts,

$$\begin{aligned} \left| \frac{1}{\pi} \int_\delta^\pi h_d(x, t) \frac{\sin((n+1/2)t)}{2 \sin(t/2)} dt \right| &\leq \frac{1}{2\delta(n+1/2)} \left| \cos((n+1/2)t) h_d(x, t) \Big|_{t=\delta}^{t=\pi} \right. \\ &\quad \left. - \int_\delta^\pi \cos((n+1/2)t) \partial_t h_d(x, t) dt \right| \leq \frac{A(d)}{n+1/2}, \end{aligned}$$

where $A(d)$ depends on d and C but not on n . Choosing $m_0 > 2A(d)/\epsilon$ and taking (2.13) into account, we obtain (2.12). \square

For the first and third steps, we fix a small positive d and large even M (on the strength of Lemma 2.1, M should be much larger than $1/d$), set

$$(2.14) \quad b_k^d = \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \ln \Phi(\xi) e^{-i\xi kd} d\xi, \quad k \neq 0,$$

$$(2.15) \quad b_{d,M}(\xi) = \sum_{k=-M/2+1}^{M/2} b_k^d (\exp(i\xi kd) - 1),$$

$$(2.16) \quad b_{d,M}^+(\xi) = \sum_{k=1}^{M/2} b_k^d (\exp(i\xi kd) - 1),$$

$$(2.17) \quad b_{d,M}^-(\xi) = \sum_{k=-M/2+1}^{-1} b_k^d (\exp(i\xi kd) - 1);$$

$$(2.18) \quad \Phi_{d,M}(\xi) = \exp(b_{d,M}(\xi)),$$

$$(2.19) \quad \Phi_{d,M}^\pm(\xi) = \exp(b_{d,M}^\pm(\xi)).$$

2.4. Approximation of $\phi_q^\pm(D)$ using Fast Fourier Transform. Let d be the step in x -space, ζ —the step in ξ -space, and $M = 2^m$ the number of the points on the grid; decreasing d and increasing (even faster) M , we obtain a sequence of approximations to the option price. Approximants for EPV-operators can be efficiently computed by using the Fast Fourier Transform (FFT). Consider the algorithm (the discrete Fourier transform (DFT)) defined by

$$(2.20) \quad G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi i kl/M}, \quad l = 0, \dots, M-1.$$

(It differs in sign in front of i from the algorithm `fft` in MATLAB). The DFT maps m complex numbers (the g_k 's) into m complex numbers (the G_l 's). The formula for the inverse DFT which recovers the set of g_k 's exactly from G_l 's is:

$$(2.21) \quad g_k = iDFT[G](k) = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{-2\pi i kl/M}, \quad k = 0, \dots, M-1.$$

In our case, the data consist of a real-valued array $\{g_k\}_{k=0}^M$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. Since this complex-valued array has real values G_0 and $G_{M/2}$, and $M/2 - 1$ other independent complex values $G_1, \dots, G_{M/2-1}$, then it has the same “degrees of freedom” as the original real data set. In this case, it is inefficient to use full complex FFT algorithm. The main idea of FFT of real functions is to pack the real input array cleverly, without extra zeros, into a complex array of half of length. Then a complex FFT can be applied to this shorter length; the trick is then to get the required values from this result (see Press, W. et al (1992) for technical details). To distinguish DFT of real functions we will use notation RDFT.

Depending on the type of the option under consideration, we choose real ω , and apply the Fourier transform $F_{x \rightarrow \xi}$ and the inverse Fourier transform $F_{\xi \rightarrow x}^{-1}$ with x living in \mathbb{R} and ξ living in $\mathbb{R} + i\omega$. Thus, a grid for ξ is the grid $\xi_j = \eta_j + i\omega$ on the line $\text{Im } \xi = \omega$.

Then $\psi_1(\eta) = \psi(\eta + i\omega) - \psi(i\omega)$ be also characteristic exponent of infinitely divisible distribution. Set $q_1 = q + \psi(i\omega)$, and we will apply FWH-method for factorization of $q_1/(q_1 + \psi_1(\eta))$ with q_1 and $\psi_1(\eta)$ instead q and $\psi(\eta)$, respectively. In the case of down-and-out options, for typical parameters values, we will choose $\omega = -2$; for up-and-out options, $\omega = 1$ is a good choice.

Fix the space step $d > 0$ and number of the space points $M = 2^m$. Define the partitions of normalized log-price domain $[-\frac{Md}{2}; \frac{Md}{2})$ by points $x_k = -\frac{Md}{2} + kd$, $k = 0, \dots, m-1$, and frequency domain $[-\frac{\pi}{d} + i\omega; \frac{\pi}{d} + i\omega]$ by points $\xi_l = \frac{2\pi l}{dM} + i\omega$, $l = -M/2, \dots, M/2$. Then the Fourier transform of a function g on the real line can be approximated as follows:

$$\hat{g}(\xi_l) \approx \int_{-Md/2}^{Md/2} e^{-ix\xi_l} g(x) dx \approx \sum_{k=0}^{M-1} g(x_k) e^{-ix_k \xi_l} d = d e^{i\pi l} \sum_{k=0}^{M-1} g(x_k) e^{\omega x_k} e^{-2\pi i k l / M},$$

and finally,

$$(2.22) \quad \hat{g}(\xi_l) \approx d e^{i\pi l} \overline{RDFT[ge^{\omega \cdot}]}(l), \quad l = 0, \dots, M/2.$$

Here \bar{z} denotes the complex conjugate of z . Now, we approximate \mathcal{E} . Using the notation $p(\xi) = q(q + \psi(\xi))^{-1}$, we can approximate

$$(\mathcal{E}g)(x_k) = \frac{1}{2\pi} \int_{-\infty + i\omega}^{+\infty + i\omega} e^{ix_k \xi} p(\xi) \hat{g}(\xi) d\xi$$

by

$$\begin{aligned} e^{-\omega x_k} & \left(\frac{2}{M} \operatorname{Re} \sum_{l=1}^{M/2-1} e^{-2\pi i k l / M} p(-\xi_l) RDFT[ge^{\omega \cdot}]_l \right. \\ & \left. + \frac{1}{M} (RDFT[ge^{\omega \cdot}]_0 + \operatorname{Re} p(-\xi_{M/2}) RDFT[ge^{\omega \cdot}]_{M/2}) \right). \end{aligned}$$

Finally,

$$(2.23) \quad (\mathcal{E}g)(x_k) \approx e^{-\omega x_k} i RDFT[\bar{p} * RDFT[ge^{\omega \cdot}]](k), \quad k = 0, \dots, M-1.$$

To approximate operators $\mathcal{E}^\pm = \phi_q^\pm(D)$, we find an approximation of function $\ln \Phi$ by the Fourier series using the formula (2.15). The coefficients b_k^d in (2.15) are defined by (2.14) and can be efficiently computed by using iRDFT. We have:

$$(2.24) \quad b_k^d = \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \ln \Phi(\xi) e^{-i\xi k d} d\xi \approx i RDFT[\ln \Phi](k).$$

Next, we calculate $b_{d,M}^\pm$ in (2.16)–(2.17), and then, calculate the approximations to the Wiener-Hopf factors

$$(2.25) \quad p^\pm(\xi_l) = \Lambda_{-\nu/2}^\pm(\xi_l) \exp(b_{d,M}^\pm(\xi_l)), \quad l = -M/2, \dots, 0.$$

The action of the EPV-operator \mathcal{E}^\pm is approximated as follows:

$$\begin{aligned} (\mathcal{E}^\pm g)(x_k) &= \frac{1}{2\pi} \int_{-\infty+i\omega}^{\infty+i\omega} e^{ix_k \xi} \phi^\pm(\xi) \hat{g}(\xi) d\xi \approx \frac{1}{2\pi} \int_{-\pi/d+i\omega}^{\pi/d+i\omega} e^{-ix_k \xi} p^\pm(-\xi) \overline{\hat{g}(\xi)} d\xi \\ &\approx \frac{1}{2\pi} \sum_{l=-M/2+1}^{M/2} e^{-ix_k \xi_l} p^\pm(-\xi_l) \overline{\hat{g}(\xi_l)} \frac{2\pi}{dM}, \end{aligned}$$

and, finally,

$$(2.26) \quad (\mathcal{E}^\pm g)(x_k) = e^{-\omega x_k} iRDFT[\overline{p^\pm} \cdot RDFT[ge^{\omega \cdot}]](k), \quad k = 0, \dots, M-1.$$

The reader can find the detailed algorithm of FWH method in Subsection C.1.

3. FWH-METHOD FOR PRICING AMERICAN OPTIONS

3.1. Free boundary problem for the price of the American put option. We consider the American put on a stock which pays no dividends; the generalization to the case of a dividend-paying stock and the American call is straightforward. (Moreover, as it is well-known, changing the direction on the line, the unknown function, the riskless rate and the process, one can reduce the pricing problem for the American call to the pricing problem for the American put).

Let $V(t, S_t)$ be the price of American put with the strike price K and the terminal date T . Set $x = \ln(S/K)$, $g(x) = K(1 - e^x)$ and $v(t, x) = V(t, Ke^x)$. Assume that the optimal stopping time is of the form $\tau'_B \wedge T$, where τ'_B is the hitting time of a closed set $B \subset \mathbf{R} \times (-\infty, T]$ by the two-dimensional process $\hat{X}_t = (X_t, t)$. Set $\mathcal{C} = \mathbf{R} \times [0, T] \setminus B$ (this is the continuation region, where the option remains alive), and consider the following boundary value problem

$$\begin{aligned} (3.1) \quad & (\partial_t + L - r)v(t, x) = 0, \quad (t, x) \in \mathcal{C}; \\ (3.2) \quad & v(t, x) = g(x), \quad (t, x) \in B \text{ or } t = T; \\ (3.3) \quad & v(t, x) \geq g(x)^+, \quad t \leq T, x \in \mathbf{R}; \\ (3.4) \quad & (\partial_t + L - r)v(t, x) \leq 0, \quad t < T, (t, x) \notin \bar{\mathcal{C}}, \end{aligned}$$

where $g(x)^+ := \max\{g(x), 0\}$.

Under certain regularity conditions (see Theorem 6.1 in Boyarchenko and Levendorskiĭ (2002b)), the continuous bounded solution to the free boundary problem (3.1)-(3.4) gives the optimal early exercise region, B , and the rational option price, v .

3.2. General formulas. Again we apply the Lévy analog of Carr's randomization procedure developed in Section 6.2.2 of Boyarchenko and Levendorskiĭ (2002b) for the American put. Normalize the strike price to 1, divide $[0, T]$ into n subperiods by points $t_j = j\Delta t$, $j = 0, 1, \dots, n$, where $\Delta t = T/n$, and denote by $v_j(x)$ the approximation to $v(x, t_j)$; h_j denotes the approximation to the early exercise boundary at time t_j . Then $v_n(x) = K(1 - e^x)^+$, and by discretizing the derivative ∂_t in (3.1), we obtain, for $j = n-1, n-2, \dots, 0$,

$$(3.5) \quad \frac{v_{j+1}(x) - v_j(x)}{\Delta t} - (r - L)v_j(x) = 0, \quad x > h_j.$$

Equation (3.2) assumes the form

$$(3.6) \quad v_j(x) = g(x), \quad x \leq h_j.$$

The approximation h_j to the early exercise boundary is found so that the v_j be maximal.

Introduce $\tilde{v}_j(x) = v_j(x) - g(x)$ and substitute $v_j(x) = \tilde{v}_j(x) + g(x)$ into (3.5)–(3.6):

$$(3.7) \quad \frac{\tilde{v}_{j+1}(x) - \tilde{v}_j(x)}{\Delta t} - (r - L)\tilde{v}_j(x) = (r - L)g, \quad x > h_j,$$

$$(3.8) \quad \tilde{v}_j(x) = 0, \quad x \leq h_j.$$

Set $q = \Delta t^{-1} + r$ and $G_j = \tilde{v}_{j+1} - \Delta t(r - L)g$, then equation (3.7) can be rewritten as follows.

$$(3.9) \quad q^{-1}(q - L)\tilde{v}_j(x) = (q\Delta t)^{-1}G_j(x), \quad x > h_j.$$

Note that continuous function $g(x)$ is decreasing admits bound (1.14) with $\sigma_+ = 1$, $\sigma_- = 0$, and satisfies condition $g(-\infty) > 0 > g(+\infty) = -\infty$. Taking into account that $(r - L)g = Kr$ one can show that the following statements hold (for details see Boyarchenko S.I. and S.Z. Levendorskiĭ (2005)).

For $j = n - 1, n - 2, \dots, 0$:

a) function G_j is a non-decreasing continuous function satisfying bound (1.14) with $\sigma_+ = 1$, $\sigma_- = 0$; in addition,

$$(3.10) \quad G_j(-\infty) < 0 < G_j(+\infty) = +\infty;$$

b) function

$$(3.11) \quad \tilde{w}_j := \mathcal{E}^+ G_j$$

is continuous; it increases and satisfies (3.10);

c) equation

$$(3.12) \quad \tilde{w}_j(h) = 0$$

has a unique solution, denote it h_j ;

d) the hitting time of $(-\infty, h_j]$, $\tau(h_j)$, is a unique optimal stopping time;

e) (Carr's approximation to) the option value at the moment j is given by

$$(3.13) \quad v_j = (q\Delta t)^{-1}\mathcal{E}^- \mathbf{1}_{(h_j, +\infty)} \tilde{w}_j + g;$$

equivalently,

$$(3.14) \quad \tilde{v}_j = (q\Delta t)^{-1}\mathcal{E}^- \mathbf{1}_{(h_j, +\infty)} \tilde{w}_j;$$

f) $\tilde{v}_j = v_j - g$ is a positive non-decreasing function that admits bound (1.14) with $\sigma_+ = 1$, $\sigma_- = 0$, and satisfies $\tilde{v}_j(+\infty) = +\infty$; it vanishes below h_j and increases on $[h_j, +\infty)$.

It follows from Wiener-Hopf formula that functions \tilde{w}_j and \tilde{w}_{j+1} are connected by the formula

$$(3.15) \quad \tilde{w}_j = (q\Delta t)^{-1}\mathcal{E} \tilde{w}_{j+1} - Kr\Delta t.$$

Thus, for calculations of v_0 and early exercise boundaries h_j we need to apply \mathcal{E}^+ and \mathcal{E}^- only on the first and last steps, respectively.

To improve the convergence of the method we will choose $\omega = -2$ and $\omega = 1$ (see (2.23) and (2.26)), in the case of American put and American call, respectively.

4. NUMERICAL EXAMPLES

In this section, we compare the performance of the two methods: Cont and Voltchkova (2005) method and FWH-method.

In numerical examples, we implement the algorithm of FWH-method described in Subsection C.1.

We use Monte-Carlo method (MC-method) and the accurate finite-difference scheme of FDS-method as the benchmarks. It is well known that the convergence of Monte Carlo estimators of quantities involving first passage is very slow. Hence, a large number of paths was needed to obtain convergence. For the Monte Carlo calculations we used 500, 000 paths with time step = 0.00005 for $\nu < 1$ and time step = 0.00001 for $\nu > 1$. For simulating trajectories of the tempered stable (KoBoL) process we implemented the code of J. Poirrot and P. Tankov (www.math.jussieu.fr/~tankov/). The program uses the algorithm in Madan and Yor (2005), see also Poirrot and Tankov (2006).

For sequentially generating VG sample paths on $[0, T]$, we used the algorithm of simulating Variance Gamma as Gamma time-changed Brownian motion (see Madan and Yor (2006)).

We consider the down-and-out put option with strike K , barrier H and time to expiry T . The option prices were calculated on a PC with characteristics Intel Core(TM)2 Due CPU, 1.8GHz, RAM 1024Mb, under Windows Vista. For computation of the prices by CV-method we used Premia 8 routine (www.premia.fr).

We consider two types of processes, two times to maturity, and two ratios H/K . Certainly, it would be interesting to study more variants but a detailed study of several case will require dozens of tables. The examples, which we analyze in detail below, are fairly representative. The option price depends on several parameters of the chosen scheme. For a fixed number of time steps, N , and step in x -space, d , we will vary the other parameters of the scheme to ensure that the price does not change significantly (the details will be explained below). This explains why we can denote the price by $V_{d,N}$ (the scheme will be indicated separately). It should be noted that computation of the prices under Premia 8 increases the CPU time by a factor of 3.

First, we take KoBoL model of order $\nu \in (0, 1)$, with parameters $\sigma = 0$, $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$. We choose instantaneous interest rate $r = 0.072310$, time to expiry $T = 0.5$ year, strike price $K = 100$ and the barrier $H = 90$. In this case, the drift parameter μ is approximately zero. The localization domain is $(x_{\min}; x_{\max})$ with $x_{\min} = -\ln 2$ and $x_{\max} = \ln 2$; we check separately that if we increase the domain two-fold, and the number of points 4-fold, the prices change by less than 0.0001.

Table B.1, Panel A, reports prices for down-and-out put options calculated using Monte-Carlo simulation and FDS, FWH and CV methods, with very fine grids. The options are priced at five spot levels. ExtCV labels option prices obtained by linear extrapolation of prices $V_{d,N}$ with $d = 0.000005$ and $d = 0.000002$. In Panel B, the sample mean values are compared with the prices computed by FDS, FWH and CV-methods. The results show a general agreement between the Monte Carlo simulation results and those computed by FDS and FWH methods. FWH-prices converge very fast and agree with MC-prices and FDS-prices very well (relative error less than 1% even in the out-of-the money region) after 0.5 sec whereas CV-method produces relative

errors larger than 3% after dozens of hours of calculation. In Table B.2, we consider the same model for smaller time to expiry $T = 0.1$ year. Notice that, near the barrier, the CV-prices are lower than WHF-prices and MC-prices, which is to be expected for processes of order $\nu < 1$ with zero drift. Indeed, by construction, CV prices are of class C^2 up to the barrier, whereas the theoretical prices have the asymptotics (B.1). The same systematic error is observed for processes of order $\nu > 1$ (Table B.3).

In the case of Variance Gamma processes, we consider two sets of parameters. In Table B.4, the drift is negative, and in Table B.5 - positive.

APPENDIX A. BASIC FACTS

A.1. Lévy processes: a short reminder. A Lévy process is a process with stationary independent increments (for details, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. We denote it by $F(dy)$. A Lévy process can be completely specified by its characteristic exponent, ψ , definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case). The characteristic exponent is given by the Lévy-Khintchine formula:

$$(A.1) \quad \psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y| \leq 1}) F(dy),$$

where σ^2 is the variance of the Gaussian component, and $F(dy)$ satisfies

$$(A.2) \quad \int_{\mathbf{R} \setminus \{0\}} \min\{1, y^2\} F(dy) < +\infty.$$

If the jump component is a process of finite variation, equivalently, if

$$(A.3) \quad \int_{\mathbf{R} \setminus \{0\}} \min\{1, |y|\} F(dy) < +\infty,$$

then (A.1) can be simplified

$$(A.4) \quad \psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y}) F(dy),$$

with a different μ , and the new μ is the drift of the gaussian component.

Assume that under a risk-neutral measure chosen by the market, the stock has the dynamics $S_t = e^{X_t}$. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\text{Im } \xi \in (-1, 0)$ and continuous continuation into the closed strip $\text{Im } \xi \in [-1, 0]$. Further, if the riskless rate, r , is constant, and the stock does not pay dividends, then the discounted price process must be a martingale. Equivalently, the following condition must hold

$$(A.5) \quad r + \psi(-i) = 0,$$

which can be used to express μ via the other parameters of the Lévy process:

$$(A.6) \quad \mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y \mathbf{1}_{|y| \leq 1}) F(dy).$$

Example 2.1. [Tempered stable Lévy processes] The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$(A.7) \quad \psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu],$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (A.7) is derived in Boyarchenko and Levendorskii (1999, 2000, 2002b) from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$(A.8) \quad F_\mp(dy) = ce^{\lambda_\pm y}|y|^{-\nu-1}dy;$$

in the first two papers, the name extended Koponen family was used. Later, the same class of processes was used in Carr et al. (2002) under the name CGMY-model. Note that Boyarchenko and Levendorskii (2000, 2002b) consider a more general version with c_\pm instead of c , as well as the case $\nu = 1$ and cases of different exponents ν_\pm .

Example 2.2. [Normal Inverse Gaussian processes] A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see Barndorff-Nielsen (1998))

$$(A.9) \quad \psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}],$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example 2.3. [Variance Gamma processes] The Lévy density of a Variance Gamma process is of the form (A.8) with $\nu = 0$, and the characteristic exponent is given by (see Madan et al. (1998))

$$(A.10) \quad \psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln \lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)],$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2.4. [Kou model] If $F_\mp(dy)$ are given by exponential functions on negative and positive axis, respectively:

$$F_\mp(dy) = c_\pm(\pm\lambda_\pm)e^{\lambda_\pm y},$$

where $c_\pm \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain Kou model. The characteristic exponent of the process is of the form

$$(A.11) \quad \psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}.$$

The version with one-sided jumps is due to Das and Foresi (1996), the two-sided version was introduced in Duffie, Pan and Singleton (2000), see also S.G.Kou (2002).

The infinitesimal generator of X , denote it L , is an integro-differential operator, which acts as follows:

$$(A.12) \quad Lu(x) = \frac{\sigma^2}{2}u''(x) + \mu u'(x) + \int_{-\infty}^{+\infty} (u(x+y) - u(x) - y\mathbf{1}_{|y| \leq 1}u'(x))F(dy).$$

A.2. The infinitesimal generator of a Lévy process as a PDO. The infinitesimal generator L can be represented as a pseudo-differential operator $-\psi(D)$. Recall that a PDO $A = a(D)$ with the symbol $a(\xi)$ acts as follows:

$$(A.13) \quad Au(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi) \hat{u}(\xi) d\xi,$$

where \hat{u} is the Fourier transform of a function u :

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x) dx.$$

Note that the inverse Fourier transform in (A.13) is defined in the classical sense only if the symbol $a(\xi)$ and function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform, $F_{x \rightarrow \xi}$ (the subscript means that a function defined on the x -space becomes a function defined on the dual ξ -space), and the inverse Fourier transform, $F_{\xi \rightarrow x}^{-1}$ by duality; in many cases, in particular, in the context of pricing of put and call options, one can use the classical definition of the integral but integrate in (A.13) not over the real line but along an appropriate line or contour in the complex plane. See, e.g., Boyarchenko and Levendorskiĭ (2002a-c) for details and examples. Formally, the action of a PDO A with the constant symbol $a(\xi)$ can be described as the composition

$$(A.14) \quad Au(x) = F_{\xi \rightarrow x}^{-1} a(\xi) F_{x \rightarrow \xi} u(x)$$

If the functions u and a are represented as arrays suitable for application of the Fast Fourier Transform and inverse Fast Fourier Transform algorithms (FFT and iFFT), then (A.14) can be programmed as $Au = iFFT(a * (FFT(u)))$.

A.3. Regular Lévy processes of exponential type. In order that the PDO technique were not complicated and solutions to boundary problems be regular, the symbols should be sufficiently nice. For PDO with constant symbols (meaning: symbols independent of x), a convenient condition is: the symbol admits a representation

$$(A.15) \quad a(\xi) = a_m(\xi) + O(|\xi|^{m_1}), \quad \text{as } \xi \rightarrow \infty,$$

where a_m is positively homogeneous of degree m , and $m_1 < m$ (for PDO with “variable” symbols $a(x, \xi)$, additional conditions on the derivatives of the symbol are needed). The m is called the order of PDO $a(D)$, and a_m is called the principal symbol of $a(D)$. If it is necessary to consider the action of a PDO in spaces with exponential weights, then the representation (A.15) must be valid in an appropriate strip of the complex plane (in the multi-dimensional case, in a tube domain in \mathbf{C}^n) – see e.g. Eskin (1973), Barndorff-Nielsen and Levendorskiĭ (2001) and Boyarchenko and Levendorskiĭ (2002b). Essentially, these two properties (the characteristic exponent is analytic in a strip, and (A.15) is valid in the strip) are used in Boyarchenko and Levendorskiĭ (1999, 2000, 2002a-c) to introduce the class of RLPE in terms of the characteristic exponent; the other definition starts with the Lévy density.

Loosely speaking, a Lévy process X is called a *Regular Lévy Process of Exponential type* (RLPE) if its Lévy density has a polynomial singularity at the origin and decays exponentially at infinity (see Boyarchenko and Levendorskiĭ (1999, 2000, 2002a-c). An

almost equivalent definition is: the characteristic exponent is analytic in a strip $\text{Im } \xi \in (\lambda_-, \lambda_+)$, continuous up to the boundary of the strip, and admits the representation

$$(A.16) \quad \psi(\xi) = -i\mu\xi + \phi(\xi),$$

where $\phi(\xi)$ stabilizes to a positively homogeneous function at the infinity:

$$(A.17) \quad \phi(\xi) \sim c_{\pm}|\xi|^{\nu}, \quad \text{as } \text{Re } \xi \rightarrow \pm\infty, \quad \text{in the strip } \text{Im } \xi \in (\lambda_-, \lambda_+),$$

where $c_{\pm} > 0$. "Almost" means that the majority of classes of Lévy processes used in empirical studies of financial markets satisfy conditions of both definitions. These classes are: Brownian motion, Kou's model (Kou (2002)), Hyperbolic processes (Eberlein and Keller (1995), Eberlein et al (1998)), Normal Inverse Gaussian processes and their generalization (Barndorff-Nielsen (1998) and Barndorff-Nielsen and Levendorskiĭ (2001)), and extended Koponen's family (a.k.a. KoBoL model and CGMY model). Koponen (1995) introduced a symmetric version; Boyarchenko and Levendorskiĭ (1999, 2000), gave a non-symmetric generalization; later a subclass of this model appeared under the name CGMY-model in Carr et al. (2002), and Boyarchenko and Levendorskiĭ (2002a-c) used the name KoBoL family. The two important exceptions are Variance Gamma Processes (VGP; see, e.g., Madan et al. (1998)) and stable Lévy processes. The characteristic exponent of a stable Lévy process does not admit the analytic continuation into a strip adjacent to the real line, therefore, call options cannot be priced. VGP satisfy the conditions of the first definition of RLPE but not the second one, since the characteristic exponent behaves like $\text{const} \cdot \ln |\xi|$, as $\xi \rightarrow \infty$.

If $\nu \geq 1$ or $\mu = 0$, then the order of the KoBoL process equals to the order of the infinitesimal generator as PDO, but if $\nu < 1$ and $\mu \neq 0$, then the order of the process is ν , and the order of the PDO $-L = \psi(D)$ is 1. Hyperbolic and normal inverse Gaussian processes are RLPEs of order 1, and the generalization of normal inverse processes constructed in Barndorff-Nielsen and Lévendorskiĭ (2001) contains processes of any order $\nu \in (0, 2)$. If there is a diffusion component, the order of the process is 2.

The difference between the definition in terms of the Lévy density and the one in terms of the characteristic exponent is apparent in the case of a VGP without the diffusion component: the Lévy density is given by (A.8) with $\nu = 0$ but the characteristic exponent involves a logarithmic function. Hence, the infinitesimal generator of a VGP is not of order 0.

APPENDIX B. BEHAVIOR OF THE PRICE OF THE BARRIER OPTION NEAR THE BARRIER

It is extremely difficult if at all possible to study the behavior of the price of a barrier option near the barrier for an arbitrary Lévy process X_t . We concentrate on several classes of processes used in empirical studies of financial markets. M.Boyarchenko (2009) proved for wide classes processes of finite variation with positive drift and non-vanishing transition densities (this class contains RLPEs), and payoff functions, which are not identically zero above the barrier, that the price of a down-and-out option is discontinuous at the boundary. Below, we prove that the delta of the down-and-out barrier option is unbounded in the vicinity of the barrier assuming that X_t is an RLPE of order $\nu \geq 1$ or of order $\nu \in (0, 1)$ with $\mu = 0$.

Let $V_{\text{eur}}(t, x)$ be the price of the European option with the payoff $G(X_T)$, and set $V_1 = V_{\text{eur}} - V$. This is the price of the option which pays $G_1(t, X_t) = V_{\text{eur}}(t, X_t)$ the

first time X_t reaches or crosses the barrier, and expires; at $t > T$, the option expires worthless. For an RLPE, the jump density is positive on $\mathbb{R} \setminus 0$, therefore, for all $t < T$ and $x \leq h$, $G_1(t, x) = V_{\text{eur}}(t, x) > 0$. Suppose for the moment that $G_1 \equiv 1$; then V_1 is the price of the first touch digital option. Boyarchenko and Levendorskiĭ (2002c, Theorem 7.4 and Eq. (7.56)) proved that if $\nu \geq 1$, then there exist $\kappa_- \in (0, 1)$, $\delta > 0$ and, for each $t < T$, $c(t) > 0$ such that, as $x \downarrow h$,

$$(B.1) \quad V_1(t, x) = 1 - c(t)(x - h)^{\kappa_-} + O((x - h)^{\kappa_- + \delta}).$$

Equivalently, the price of the digital down-and-out option has the asymptotics

$$(B.2) \quad V(t, x) = c(t)(x - h)^{\kappa_-} + O((x - h)^{\kappa_- + \delta}).$$

In the case $G_1 = V_{\text{eur}}$, one can either prove (B.1) modifying in the straightforward fashion the proof of Theorem 7.4 in Boyarchenko and Levendorskiĭ (2002c). The first step of the proof is the explicit formula for the solution, which we describe in the case $h = 0$. In Eq. (7.13) in Boyarchenko and Levendorskiĭ (2002c), one should replace $1/(\xi\eta)$, which is the Fourier transform of $\mathbf{1}_{x < 0}\mathbf{1}_{\tau > 0}$, with the Fourier transform of $G_1(T - \tau, x)\mathbf{1}_{x < 0}\mathbf{1}_{\tau > 0}$. After that, one has to modify the calculations in Boyarchenko and Levendorskiĭ (2002c). The calculations are typical for calculations of asymptotic expansions and rather tedious (cf., e.g., Eskin (1973), Chapter 9).

APPENDIX C. ALGORITHMS

C.1. Algorithm of the FWH-method. We assume that the “drift” $\mu = 0$ (except the VGP case); non-zero μ can be eliminated by the change of variable $x = x' + \mu t$.

I. PRELIMINARY STEPS

- Step 1. Input r , ω and parameters of the characteristic exponent $\psi(\xi)$ (see (A.16)-(A.17)). Redefine $\psi(\xi) = \psi(\xi + i\omega) - \psi(i\omega)$.
- Step 2. Input space step d .
- Step 3. Input the scale of logprice range L . Define the localization domain $(x_{\min}; x_{\max})$ for the space variable x . Set $x_{\min} = L \cdot \ln(0.5)$ and $x_{\max} = L \cdot \ln(2.0)$. The choice $L = 1$ is optimal for typical parameter values.
- Step 4. Define the number of space points M as follows. Find a positive integer m such that $2^{m-1} < \frac{x_{\max} - x_{\min}}{2d} \leq 2^m$, and set $M = 2^m$.
- Step 5. Input time to maturity T , the number of time steps N , and set $\Delta t = T/n$, $q = (\Delta t)^{-1} + r + \psi(i\omega)$.
- Step 6. Set $\xi_k = \frac{\pi k}{Md}$, $k = -M+1, \dots, M$, and find $p_k = q(q + \psi(\xi_k))^{-1}$, $k = -M+1, \dots, -1$, and $p_0 = 1$, $p_{-M} = q \operatorname{Re}(q + \psi(-\xi_M))^{-1}$. We need array p_k , $k = -M, \dots, 0$ for approximation of \mathcal{E} (see (2.23)).
- Step 7. Find $(\ln \Phi)_k = \ln \Phi(\xi_k)$, $k = 1, \dots, M-1$, and $(\ln \Phi)_0 = 0$, $(\ln \Phi)_M = \operatorname{Re} \ln \Phi(\xi_M)$. We need array $(\ln \Phi)_k$, $k = -M, \dots, 0$ for calculation of coefficients b_k^d (see (2.15) and (2.24)).
- Step 8. Using inverse FFT for real-valued functions, we find (see (2.24))

$$b_k = b_k^d = iRFFT[\ln \Phi](k), \quad k = -M+1, \dots, M$$

- Step 9. Set $b_0^- = -\sum_{l=-M+1}^{l=-1} b_l$, $b_k^- = b_{-k}$, $k = 1, \dots, M$, $b_k^- = 0$, $k = -M+1, \dots, -1$. We need the array b_k^- , $k = -M+1, \dots, M$ for calculation of $b^-(-\xi_l)$ (see (2.17), (2.25) and (2.26))
- Step 10. Find $b^-(-\xi_l) = RDFT[b^-](l)$, $l = 0, \dots, M$.
- Step 11. Set $b_0^+ = b_0 - b_0^-$, $b_k^+ = b_{-k}$, $k = -M+1, \dots, -1$, $b_k^+ = 0$, $k = 1, \dots, M$. We need array b_k^+ , $k = -M+1, \dots, M$ for calculation of $b^+(-\xi_l)$ (see (2.16), (2.25) and (2.26))
- Step 12. Calculate $b^+(-\xi_l) = RDFT[b^+](l)$, $l = 0, \dots, M$.
- Step 13. We find $p_l^\pm = \Lambda_\pm(\xi_l) \exp(b^\pm(\xi_l))$, $l = -M, \dots, 0$ (see (2.25)). We need arrays p_l^\pm , $l = -M, \dots, 0$ for the approximation of \mathcal{E}^\pm (see (2.26)).

II.A PRICING BARRIER OPTION

Denote $v_j^k = v_j(x_k)$ and $w_j^k = w_j(x_k)$, $k = -M+1, \dots, M$, $j = N, N-1, \dots, 0$. Let $G(S)$ be the payoff function.

- Step 1. Input barrier H and strike K . Set $\gamma = (q\Delta t)^{-1}$.
- Step 2. Calculate values of payoff function: $v_N^k = e^{\omega x_k} G(He^{x_k})$, $k = -M+1, \dots, M$.
- Step 3. Calculate (see (2.1) and (2.26))

$$w_{N-1}^k = iRDFT[\overline{p^+} \cdot RDFT[v_N]](k), \quad k = -M+1, \dots, M.$$

- Step 4. Calculation in cycle $s = N-1, N-2, \dots, 1$.

- Set $w_s^k = 0$, $k \leq 0$.
- Applying direct and inverse FFT, we calculate (see (2.3) and (2.23))

$$w_{s-1}^k = \gamma iRDFT[\overline{p} \cdot RDFT[w_s]](k), \quad k = -M+1, \dots, M$$

- Step 5. Set $w_0^k = 0$, $k \leq 0$.

- Step 6. Applying direct and inverse FFT, we calculate (see (2.2) and (2.26))

$$v_0^k = \gamma iRDFT[\overline{p^-} \cdot RDFT[w_0]](k), \quad k = -M+1, \dots, M$$

- Step 7. Output: spot price: $S_k = He^{x_k}$; option price: $V_k = e^{-\omega x_k} v_0^k$, $k = 1, \dots, M$.

II.B PRICING AMERICAN PUT OPTION

Denote $v_j^k = v_j(x_k)$ and $w_j^k = w_j(x_k)$, $k = -M+1, \dots, M$, $j = N, N-1, \dots, 0$.

- Step 1. Input strike K . Set $\gamma = (q\Delta t)^{-1}$.
- Step 2. Calculate values of payoff function: $v_N^k = 0$, $k = -M+1, \dots, 0$ and $v_N^k = e^{\omega x_k} (Ke^{x_k} - K)$, $k = 1, \dots, M$.
- Step 3. Calculate (see (3.11) and (2.26))

$$w_{N-1}^k = iRDFT[\overline{p^+} \cdot RDFT[v_N]](k), \quad k = -M+1, \dots, M.$$

- Step 4. Calculation in cycle $s = N-1, N-2, \dots, 1$.

- Set $w_s^k = w_s^k - e^{\omega x_k} K \Delta t r$. If $w_s^k < 0$, then set $w_s^k = 0$.
- Applying direct and inverse FFT, we calculate (see (3.15) and (2.23))

$$w_{s-1}^k = \gamma iRDFT[\overline{p} \cdot RDFT[w_s]](k), \quad k = -M+1, \dots, M$$

- Step 5. Set $w_0^k = w_0^k - e^{\omega x_k} K \Delta t r$. If $w_0^k < 0$, then set $w_0^k = 0$.

Step 6. Applying direct and inverse FFT, we calculate (see (2.2) and (2.26))

$$v_0^k = \gamma i R D F T[\overline{p^-} \cdot R D F T[w_0]](k), \quad k = -M + 1, \dots, M$$

Step 7. Output: spot price: $S_k = K e^{x_k}$; option price: $V_k = e^{-\omega x_k} v_0^k - K e^{x_k} + K$, $k = 1, \dots, M$.

Remark:

a) Approximation of the Fourier transform (resp., inverse Fourier transform) using FFT (resp., iFFT) involves two types of errors: truncation error and discretization error. For an RLPE, the truncation error for FFT can be made small very easily because the put option price decays exponentially as $x \rightarrow +\infty$ (we may consider the put options only because of the call-put parity) and, after that, the discretization error can be controlled by decreasing the step d of the grid in x -space, equivalently, increasing the number of points $M = 2^m$ of the grid. Now, consider the inverse Fourier transform. Assuming that the truncated region on the line $\text{Im } \xi = \omega$ is of the form $[-\Lambda + i\omega, \Lambda + i\omega]$, and denoting the step of the uniform grid by ζ , we have $d\zeta/(2\pi) = 1/M$, hence, $\Lambda = 2^{m-1}\zeta = \pi/d$. It follows that if we keep the truncated region in x -space fixed and decrease d (equivalently, increase m) then we can control the truncation error of iFFT but not the discretization error. To control the latter, we need to increase the truncated region in x -space and, in addition, increase M by a larger factor. In the numerical examples in Section 4, we use the truncated region in x -space sufficiently large so that the doubling the region and increasing the number of points 4-fold changes the option price by 0.01% or less.

Note that real-FFT is two times faster than FFT.

b) Note that in the program implemented to Premia 11 one can manage by three parameters of the algorithm: the space step d , the scale of logprice range L and the number of time steps N . Parameter L controls the size of the truncated region in x -space (see remark a)). The typical values of the parameter are $L = 1$, $L = 2$ and $L = 4$. To improve the results one should decrease d and/or increase N , when L is fixed. It should be noted that FWH-method converges sufficiently fast.

REFERENCES

- [1] Avram, F., T. Chan, and M. Usabel, 2002, "On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts", *Stoch. Proc. Appl.*, 100, 75–107.
- [2] Barndorff-Nielsen, O. E., 1998, "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2, 41–68.
- [3] Barndorff-Nielsen, O. E., and S. Levendorskiĭ, 2001, "Feller Processes of Normal Inverse Gaussian type", *Quantitative Finance*, 1, 318–331.
- [4] Boyarchenko, M., "Carr's Randomization for Finite-Lived Barrier Options: Proof of Convergence" (September 29, 2008). Available at SSRN: <http://ssrn.com/abstract=1275666>
- [5] Boyarchenko, M., "Discontinuity of value functions of certain options with barriers" (working paper, January 1, 2009). Available at SSRN: <http://ssrn.com/abstract=1322285>
- [6] Boyarchenko, N., and S. Levendorskiĭ, 2007, "On errors and bias of Fourier transform methods in quadratic term structure models", *International Journal of Theoretical and Applied Finance*, 10, No. 2, 273–306
- [7] Boyarchenko, S. I., and S. Z. Levendorskiĭ, 1999, "Generalizations of the Black-Scholes equation for truncated Lévy processes", Working Paper, University of Pennsylvania, Philadelphia.

- [8] Boyarchenko, S. I., and S. Z. Levendorskiĭ, 2000, "Option pricing for truncated Lévy processes", *International Journal of Theoretical and Applied Finance*, 3, 549–552.
- [9] Boyarchenko, S. I., and S. Z. Levendorskiĭ, 2002a, "Perpetual American options under Lévy processes", *SIAM Journal of Control and Optimization*, 40, 1663–1696.
- [10] Boyarchenko, S. I., and S. Z. Levendorskiĭ, 2002b, *Non-Gaussian Merton-Black-Scholes theory*, World Scientific, New Jersey, London, Singapore, Hong Kong.
- [11] Boyarchenko, S. I., and S. Z. Levendorskiĭ, 2002c, Barrier options and touch-and-out options under regular Levy processes of exponential type, *Annals of Applied Probability*, 12:4, 1261–1298.
- [12] Boyarchenko S.I. and S.Z. Levendorskiĭ 2005, "American options: the EPV pricing model", *Annals of Finance*, 1:3, 267–292.
- [13] Boyarchenko S.I. and S.Z. Levendorskiĭ 2007: "American options in Lévy models with stochastic interest rates", to appear in *Computational Finance*
- [14] Boyarchenko, S.I. and S.Z. Levendorskiĭ 2008, "Pricing American Options in Regime-Switching Models: FFT Realization" . Available at SSRN: <http://ssrn.com/abstract=1127562>
- [15] Broadie, M., and J. Detemple, 2004, "Option pricing: valuation models and applications," *Management Science* 50, 1145–1177
- [16] Carr, P., 1998, "Randomization and the American put", *Review of Financial Studies*, 11, 597–626.
- [17] Carr, P., and D. Faguet, 1994, "Fast accurate valuation of American options", Working paper, Cornell University, Ithaca.
- [18] Carr, P., H. Geman, D.B. Madan, and M. Yor, 2002, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75, 305–332.
- [19] Carr, P., and A. Hirsa, 2003, "Why be backward?", *Risk* January 2003, 26, 103–107.
- [20] Cont, R., and P. Tankov, 2004, *Financial modelling with jump processes*, Chapman & Hall/CRC Press.
- [21] Cont, R., and E. Voltchkova, 2005, "A finite difference scheme for option pricing in jump diffusion and exponential Lévy models.", *SIAM Journal on Numerical Analysis*, 43, No. 4, 1596–1626.
- [22] Duffie, D., J. Pan, and K. Singleton, 2000, "Transform analysis and options pricing for affine jump diffusions", *Econometrica*, 68, 1343–1376.
- [23] Eberlein, E., and U. Keller, 1995, "Hyperbolic distributions in finance", *Bernoulli*, 1, 281–299.
- [24] Eberlein, E., U. Keller and K. Prause, 1998, "New insights into smile, mispricing and value at risk: The hyperbolic model", *Journal of Business*, 71, 371–406.
- [25] Eskin, G.I., 1973, *Boundary Problems for Elliptic Pseudo-Differential Equations*. Nauka, Moscow (Transl. of Mathematical Monographs 52 Providence. Rhode Island: Amer. Math. Soc. 1980).
- [26] Hirsa, A., and D.B. Madan, 2003, "Pricing American options under Variance Gamma", *Journal of Computational Finance*, 7:2.
- [27] Hörmaner, L., 1985, *Analysis of Partial Differential Operators. Vol. III*, Springer-Verlag, Berlin Heidelberg New-York.
- [28] I. Koponen, 1995, "Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process", *Physics Review E*, 52, 1197–1199.
- [29] S.G. Kou, 2002, "A jump-diffusion model for option pricing", *Management Science*, 48, 1086–1101
- [30] Kudryavtsev, O.E., and S.Z. Levendorskiĭ, 2006, "Pricing of first touch digitals under normal inverse Gaussian processes", *International Journal of Theoretical and Applied Finance*, Vol. 9, No. 6, 915–949.
- [31] Kudryavtsev, O.E., and S.Z. Levendorskiĭ, 2007, "Fast and accurate pricing of barrier options under Lévy processes" (November 30, 2007). Available at SSRN: <http://ssrn.com/abstract=1040061>
- [32] Kudryavtsev, O.E., and S.Z. Levendorskiĭ, 2008, "Fast and accurate pricing of barrier options under Lévy processes", submitted to *Finance and Stochastics*

- [33] KYPRIANOU, A.E., AND M.R. PISTORIUS, 2003, "Perpetual options and Canadization through fluctuation theory", *Annals of Applied Probability*, 13, 1077–1098
- [34] Levendorskiĭ, S.Z., 2004, "Pricing of the American put under Lévy processes", *International Journal of Theoretical and Applied Finance*, 7, 303–335.
- [35] Levendorskii, S., Kudryavtsev, O., and V.Zherder, 2006 "The relative efficiency of numerical methods for pricing American options under Lévy processes", *Journal of Computational Finance*, Vol. 9. No 2, Winter 2005/06.
- [36] Lukacs, E., 1960, *Characteristic functions*. Charles Griffin & Company limited, London.
- [37] Madan, D.B., Carr, P., and E. C. Chang, 1998, "The variance Gamma process and option pricing", *European Finance Review*, 2, 79–105.
- [38] Madan, D.B., and M. Yor, 2005, "CGMY and Meixner subordinators are absolutely continuous with respect to one sided stable subordinators", *Prépublication du Laboratoire de Probabilités et Modèles Aléatoires*
- [39] Madan, D.B., and M. Yor, 2006, "Representing the CGMY and Meixner processes as time changed Brownian motions", to appear in *Computational Finance*
- [40] Poirrot, J. and P. Tankov, 2006, "Monte Carlo option pricing for tempered stable (CGMY) processes", *Asia Pacific Financial Markets*, Vol. 13, No 4, 327–344.
- [41] Press, W., Flannery, B., Teukolsky, S. and W. Vetterling, 1992, *Numerical recipes in C: The Art of Scientific Computing*, Cambridge Univ. Press, available at www.nr.com.
- [42] Sato, K., 1999, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge.

TABLE C.1. Down-and-out put prices in KoBoL model, $\nu = 0.5$ **A**

	MC	FDS	FWH			CV		
Spot price	Sample mean	$d = 0.0001$ $N = 1600$	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.000005$ $N = 10000$	$d = 0.000002$ $N = 18000$	ExtCV
$S = 91$	0.236500	0.235866	0.237139	0.236491	0.236720	0.218617	0.223599	0.226920
$S = 101$	0.569974	0.566907	0.568303	0.567766	0.568134	0.552174	0.556747	0.559795
$S = 111$	0.383990	0.384982	0.385349	0.385557	0.385287	0.377460	0.379963	0.381632
$S = 121$	0.209492	0.208093	0.208223	0.208406	0.208196	0.204459	0.205700	0.206528
$S = 131$	0.108359	0.107262	0.107439	0.107516	0.107403	0.105499	0.106115	0.106526
CPU-time (sec)	25,000	97,300	0.22	0.45	0.46	26,000	116,000	

B

	MC	FDS	FWH			CV		
Spot price	MC error	$d = 0.0001$ $N = 1600$	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.000005$ $N = 10000$	$d = 0.000002$ $N = 18000$	ExtCV
$S = 91$	1.3%	-0.0027	0.0027	0.0000	0.0009	-0.0756	-0.0546	-0.0405
$S = 101$	0.8%	-0.0054	-0.0029	-0.0039	-0.0032	-0.0312	-0.0232	-0.0179
$S = 111$	1.0%	0.0026	0.0035	0.0041	0.0034	-0.0170	-0.0105	-0.0061
$S = 121$	1.4%	-0.0067	-0.0061	-0.0052	-0.0062	-0.0240	-0.0181	-0.0141
$S = 131$	1.9%	-0.0101	-0.0085	-0.0078	-0.0088	-0.0264	-0.0207	-0.0169

C

	FWH			CV		
Spot price	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.000005$ $N = 10000$	$d = 0.000002$ $N = 18000$	ExtCV
$S = 91$	0.00540	0.00265	0.00362	-0.07313	-0.05201	-0.03793
$S = 101$	0.00246	0.00152	0.00216	-0.02599	-0.01792	-0.01254
$S = 111$	0.00095	0.00149	0.00079	-0.01954	-0.01304	-0.00870
$S = 121$	0.00062	0.00150	0.00049	-0.01747	-0.01150	-0.00752
$S = 131$	0.00165	0.00237	0.00131	-0.01644	-0.01069	-0.00686

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$, $\mu \approx 0$.

Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.

Algorithm parameters: d – space step, N – number of time steps, S – spot price.

Panel A: Option prices calculated by using MC, FDS, FWH and CV methods.

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

Panel C: Relative errors w.r.t. FDS.

TABLE C.2. Down-and-out put prices in KoBoL model, $\nu = 0.5$, $T = 0.1$

A

	MC	FDS	FWH				CV	
Spot price	Sample mean	$d = 0.0001$ $N = 1600$	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.000002$ $N = 4000$	$d = 0.000001$ $N = 8000$	ExtCV
$S = 91$	2.34771	2.349327	2.35472	2.35229	2.35060	2.247170	2.271554	2.295938
$S = 101$	1.00787	1.009248	1.01028	1.01081	1.00977	1.003694	1.005384	1.007074
$S = 111$	0.17782	0.177806	0.17798	0.17801	0.17802	0.176402	0.176776	0.177150
$S = 121$	0.04990	0.04938	0.04933	0.04932	0.04938	0.048932	0.049045	0.049158
$S = 131$	0.01700	0.017067	0.01726	0.01725	0.01708	0.016900	0.016941	0.016982
CPU-time (sec)	5,000	99,000	0.22	0.45	0.47	18,300	96,400	

B

	MC	FDS	FWH				CV	
Spot	MC error (error)	$d = 0.0001$ $N = 1600$	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.000002$ $N = 4000$	$d = 0.000001$ $N = 8000$	ExtCV
$S = 91$	0.4%	0.0007	0.0030	0.0020	0.0012	-0.0428	-0.0324	-0.0221
$S = 101$	0.6%	0.0014	0.0024	0.0029	0.0019	-0.0041	-0.0025	-0.0008
$S = 111$	1.5%	-0.0001	0.0009	0.0011	0.0011	-0.0080	-0.0059	-0.0038
$S = 121$	2.9%	-0.0104	-0.0115	-0.0116	-0.0105	-0.0194	-0.0172	-0.0149
$S = 131$	5.1%	0.0041	0.0153	0.0151	0.0049	-0.0057	-0.0033	-0.0009

C

	FWH				CV		
Spot price	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.000002$ $N = 4000$	$d = 0.000001$ $N = 8000$	ExtCV	
$S = 91$	0.00230	0.00126	0.00054	-0.04348	-0.03310	-0.02273	
$S = 101$	0.00103	0.00155	0.00052	-0.00550	-0.00383	-0.00215	
$S = 111$	0.00100	0.00117	0.00118	-0.00790	-0.00579	-0.00369	
$S = 121$	-0.00109	-0.00115	-0.00002	-0.00908	-0.00679	-0.00450	
$S = 131$	0.01119	0.01096	0.00076	-0.00978	-0.00740	-0.00501	

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$, $\mu \approx 0$.

Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.1$.

Algorithm parameters: d – space step, N – number of time steps, S – spot price.

Panel A: Option prices calculated by using MC, FDS, FWH and CV methods.

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

Panel C: Relative errors w.r.t. FDS.

TABLE C.3. Down-and-out put prices in KoBoL model, $\nu = 1.2$ **A**

	MC	FWH			CV		
Spot price	Sample mean	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.00005$ $N = 1600$	$d = 0.000025$ $N = 6400$	ExtCV
$S = 81$	0.52594	0.51967	0.51910	0.52006	0.481031	0.492062	0.503093
$S = 91$	2.43882	2.44192	2.44293	2.44245	2.407295	2.417447	2.427599
$S = 101$	2.38503	2.37340	2.37687	2.37367	2.357705	2.363777	2.369849
$S = 111$	1.59189	1.57551	1.57833	1.57566	1.570379	1.573140	1.575901
$S = 121$	0.87757	0.87454	0.87569	0.87463	0.872098	0.873271	0.874444
CPU-time (sec)	50,000	0.22	0.45	0.47	121	1949	

B

	MC	FWH			CV		
Spot price	MC error	$d = 0.001$ $N = 400$	$d = 0.001$ $N = 800$	$d = 0.0005$ $N = 400$	$d = 0.00005$ $N = 1600$	$d = 0.000025$ $N = 6400$	ExtCV
$S = 81$	1.3%	-0.0119	-0.0130	-0.0112	-0.0854	-0.0644	-0.0434
$S = 91$	0.5%	0.0013	0.0017	0.0015	-0.0129	-0.0088	-0.0046
$S = 101$	0.5%	-0.0049	-0.0034	-0.0048	-0.0115	-0.0089	-0.0064
$S = 111$	0.7%	-0.0103	-0.0085	-0.0102	-0.0135	-0.0118	-0.0100
$S = 121$	0.9%	-0.0035	-0.0021	-0.0033	-0.0062	-0.0049	-0.0036

KoBoL parameters: $\nu = 1.2$, $\lambda_+ = 8.8$, $\lambda_- = -14.5$, $c = 1$, $\mu = 0.824313$.

Option parameters: $K = 100$, $H = 80$, $r = 0.04879$, $T = 0.1$.

Algorithm parameters: d – space step, N – number of time steps, S – spot price.

Panel A: Option prices calculated using MC, FWH and CV methods.

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

TABLE C.4. Down-and-out put prices in VG model, negative drift

A

	MC	FDS	FWH				CV	
Spot price	Sample mean	$d = 0.0001$ $N = 800$	$d = 0.005$ $N = 200$	$d = 0.005$ $N = 400$	$d = 0.0025$ $N = 200$	$d = 0.0001$ $N = 800$	$d = 0.00005$ $N = 1600$	ExtCV
$S = 81$	3.390775	3.398799	3.389713	3.390362	3.396733	3.240458	3.308974	3.37749
$S = 91$	7.383015	7.38668	7.394133	7.390715	7.394453	7.38255	7.384723	7.386896
$S = 101$	1.40095	1.403506	1.405228	1.403736	1.405659	1.405159	1.403894	1.402629
$S = 111$	0.041495	0.042795	0.042338	0.042406	0.042369	0.042577	0.042523	0.042469
$S = 121$	0.000768	0.000724	0.000736	0.000734	0.000713	0.000705	0.000706	0.000707
CPU-time (sec)	25,000	128,000	0.0079	0.0158	0.0158	9.5	40	

B

	MC	FDS	FWH				CV	
Spot price	MC error	$d = 0.0001$ $N = 800$	$d = 0.005$ $N = 200$	$d = 0.005$ $N = 400$	$d = 0.0025$ $N = 200$	$d = 0.0001$ $N = 800$	$d = 0.00005$ $N = 1600$	ExtCV
$S = 81$	0.5%	0.0024	-0.0003	-0.0001	0.0018	-0.0443	-0.0241	-0.0039
$S = 91$	0.2%	0.0005	0.0015	0.0010	0.0015	-0.0001	0.0002	0.0005
$S = 101$	0.3%	0.0018	0.0031	0.0020	0.0034	0.0030	0.0021	0.0012
$S = 111$	2.7%	0.0313	0.0203	0.0220	0.0211	0.0261	0.0248	0.0235
$S = 121$	19.3%	-0.0573	-0.0417	-0.0443	-0.0716	-0.0820	-0.0807	-0.0794

C

	FWH				CV		
Spot price	$d = 0.005$ $N = 200$	$d = 0.005$ $N = 400$	$d = 0.0025$ $N = 200$	$d = 0.0001$ $N = 800$	$d = 0.00005$ $N = 1600$	ExtCV	
$S = 81$	-0.00267	-0.00248	-0.00061	-0.04659	-0.02643	-0.00627	
$S = 91$	0.00101	0.00055	0.00105	-0.00056	-0.00026	0.00003	
$S = 101$	0.00123	0.00016	0.00153	0.00118	0.00028	-0.00062	
$S = 111$	-0.01068	-0.00909	-0.00995	-0.00509	-0.00636	-0.00762	
$S = 121$	0.01657	0.01381	-0.01519	-0.02624	-0.02486	-0.02348	

VG parameters: $\lambda_+ = 56.4414$, $\lambda_- = -21.8735$, $c = 5$, $\mu = -0.0973754$.

Option parameters: $K = 100$, $H = 80$, $r = 0.04879$, $T = 0.5$.

Algorithm parameters: d – space step, N – number of time steps, S – spot price.

Panel A: Option prices calculated by using MC, FDS, FWH and CV methods.

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

Panel C: Relative errors w.r.t. FDS.

TABLE C.5. Down-and-out put prices in VG model, positive drift

A

	MC	FDS	FWH				CV	
Spot price	Sample mean	$d = 0.00025$ $N = 600$	$d = 0.005$ $N = 400$	$d = 0.005$ $N = 800$	$d = 0.0025$ $N = 400$	$d = 0.001$ $N = 200$	$d = 0.0005$ $N = 400$	ExtCV
$S = 81$	2.99941	2.993117	2.977941	2.996258	2.974461	2.982662	2.993702	3.004742
$S = 91$	3.56916	3.577056	3.554800	3.568326	3.554509	3.537752	3.549693	3.561634
$S = 101$	1.87517	1.870955	1.863893	1.867966	1.863789	1.864022	1.864893	1.865764
$S = 111$	0.833637	0.836202	0.835492	0.835848	0.835473	0.835016	0.834656	0.834296
$S = 121$	0.358665	0.361734	0.362460	0.361995	0.362432	0.361219	0.361063	0.360907
CPU-time (sec)	25,000	17,300	0.0140	0.0280	0.0280	0.21	0.94	

B

	MC	FDS	FWH				CV	
Spot price	MC error	$d = 0.00025$ $N = 600$	$d = 0.005$ $N = 400$	$d = 0.005$ $N = 800$	$d = 0.0025$ $N = 400$	$d = 0.001$ $N = 200$	$d = 0.0005$ $N = 400$	ExtCV
$S = 81$	0.5%	-0.0021	-0.0072	-0.0011	-0.0083	-0.0056	-0.0019	0.0018
$S = 91$	0.4%	0.0022	-0.0040	-0.0002	-0.0041	-0.0088	-0.0055	-0.0021
$S = 101$	0.6%	-0.0022	-0.0060	-0.0038	-0.0061	-0.0059	-0.0055	-0.0050
$S = 111$	0.9%	0.0031	0.0022	0.0027	0.0022	0.0017	0.0012	0.0008
$S = 121$	1.4%	0.0086	0.0106	0.0093	0.0105	0.0071	0.0067	0.0063

C

	FWH				CV		
Spot price	$d = 0.005$ $N = 400$	$d = 0.005$ $N = 800$	$d = 0.0025$ $N = 400$	$d = 0.001$ $N = 200$	$d = 0.0005$ $N = 400$	ExtCV	
$S = 81$	-0.00507	0.00105	-0.00623	-0.00349	0.00020	0.00388	
$S = 91$	-0.00622	-0.00244	-0.00630	-0.01099	-0.00765	-0.00431	
$S = 101$	-0.00377	-0.00160	-0.00383	-0.00371	-0.00324	-0.00277	
$S = 111$	-0.00085	-0.00042	-0.00087	-0.00142	-0.00185	-0.00228	
$S = 121$	0.00201	0.00072	0.00193	-0.00142	-0.00185	-0.00229	

VG parameters: $\lambda_+ = 14.4093$, $\lambda_- = -60.2427$, $c = 6.25$, $\mu = 0.363531$.

Option parameters: $K = 100$, $H = 80$, $r = 0.04879$, $T = 0.5$.

Algorithm parameters: d – space step, N – number of time steps, S – spot price.

Panel A: Option prices calculated by using MC, FDS, FWH and CV methods.

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

Panel C: Relative errors w.r.t. FDS.