

Exponential Moments of the Discrete Maximum of a Lévy process

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Abstract

To estimate the lookback option price we will use a method elaborated by Feng and Linetsky (see [1]). This method consist of estimating the exponential moments of the discrete maximum by using Hilbert transform.

1 Preliminaries

Assume that the spot S_t follows an exponential Levy process (i.e $S_t = e^{X_t}$) where X_t is a Levy process defined by the characteristic triplet (μ, σ, ν) . Hence its characteristic is obtained by:

$$\mathbb{E}(e^{i\xi X_t}) = e^{-t\psi(\xi)}$$

where

$$\psi(\xi) = \frac{\sigma^2 \xi^2}{2} - i\mu\xi - \int_{\mathbb{R}} e^{i\xi y} - 1 - i\xi y 1_{|y|<1} \nu(dy)$$

One denotes M_N as the discretely observed maximum of the Levy process is given by X_t

$$M_N = \max_{j \leq N} X_j$$

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Since the interest is focused on the estimation of $\mathbb{E}(e^{sM_N})$, let Z_t be the process defined by

$$Z_t = e^{sX_t + t\psi(-is)}$$

then Z_t is a positive martingal with an expectation equal to 1 which is used to define an equivalent measure probability \mathbb{P}^* as following, for $t \leq T$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t$$

Under this new measure X_t remains a Levy process with a characteristic function $\phi_t^*(\xi) = \frac{\phi_t(\xi - is)}{\phi_t(-is)}$. This measure transform (called the Esscher transform) allows one to write

$$\begin{aligned} \mathbb{E}(e^{sM_N}) &= \mathbb{E}(e^{sM_N - sX_N} e^{sX_N}) \\ &= e^{-T\psi(-is)} \mathbb{E}(Z_T e^{s(M_N - X_N)}) \\ &= e^{-T\psi(-is)} \mathbb{E}^*(e^{s(M_N - X_N)}) \end{aligned}$$

It is easier to consider $M_N - X_N$. Indeed, one can compute its distribution recursively.

$$\begin{aligned} M_j - X_j &= \max(M_{j-1} - X_j, 0) \\ &= \max((M_{j-1} - X_{j-1}) - (X_j - X_{j-1}), 0) \end{aligned}$$

\tilde{f}_j is the distribution density of $M_j - X_j$ and p_Δ^* the transition probability density of the process X_Δ under \mathbb{P}^* . One notices clearly that $(M_{j-1} - X_{j-1})$ and $(X_j - X_{j-1})$ are independent according to the proprieties of the Levy process. Due to this independence, a recurrence relation can be easily established between $(M_{j-1} - X_{j-1})$ and $(X_j - X_{j-1})$ in order to compute by convolution \tilde{f}_{j+1} .

\tilde{f}_j can be therefore decomposed into 2 parts, $\tilde{f}_j(x) = 1_{]0, \infty[}(x) \cdot f_j(x) + C_j \cdot \delta_0(x)$. Thus, $\tilde{f}_0(x) = 1_{]0, \infty[}(x) \cdot f_0(x) + C_0 \cdot \delta_0(x)$ with $f_0(x) = 0$ and $C_0 = 1$

$$f_0 = 0, C_0 = 1$$

$$f_j(x) = C_{j-1} p_\Delta^*(-x) + \int f_{j-1}(y) \cdot 1_{]0, \infty[}(y) p_\Delta^*(y - x) dy$$

$$C_j = 1 - \int f_j(x) \cdot 1_{]0, \infty[}(x) dx$$

After N computations, one will be able to determine

$$\begin{aligned} \mathbb{E}(e^{sM_N}) &= \phi_T(-is) \mathbb{E}^*(e^{s(M_N - X_N)}) \\ &= \phi_T(-is) (C_N + \int (e^{sx}) 1_{]0, \infty[}(x) \cdot f_N(x) dx) \end{aligned}$$

2 Computing the exponential moment in the Fourier space

By passing to the Fourier space, one avoids using the transition probability density p_Δ^* which is often difficult to approach and will use instead the characteristic function ϕ^* . But first, it is important to denote that

$$g_j(x) = 1_{]0, \infty[}(x)(C_{j-1}e^{sx}p_\Delta^*(-x) + \int f_{j-1}(y)e^{sx-sy}p_\Delta^*(y-x)dy)$$

which leads to obtain

$$\phi_T(-is)\mathbb{E}^*(e^{s(M_N - X_N)}) = \phi_T(-is)(C_N + \int g_N(x)dx)$$

The Hilbert Transformation is also introduced and defined by

$$\mathcal{H}(f)(x) = \frac{1}{\pi}P.V. \int_{\mathbb{R}} \frac{f(y)}{x-y}dy$$

where $P.V.$ is the principal value. Among its properties, there is:

$$\mathcal{F}(\text{sign}(x)f(x))(\xi) = i\mathcal{H}(\hat{f})(\xi)$$

which if considered from a different perspective, gives almost everywhere

$$1_{]0, \infty[} = \frac{1}{2}(\text{sign}(x) + 1)$$

Thus, one obtains immediately

$$\mathcal{F}(f.1_{]0, \infty[})(\xi) = \frac{1}{2}\hat{f}(\xi) + \frac{i}{2}\mathcal{H}(\hat{f})(\xi)$$

And then, the recurrence becomes

$$c_0 = 1, \hat{f}_0 = 0$$

$$\hat{f}_j(\xi) = \frac{1}{2}\phi_\Delta^*(-\xi)(c_{j-1} + \hat{f}_{j-1}(\xi)) + \frac{i}{2}\mathcal{H}(\phi_\Delta^*(-\eta)(c_{j-1} + \hat{f}_{j-1}(\eta)))(\xi)$$

$$\hat{g}_j(\xi) = \frac{1}{2}\phi_\Delta^*(-\xi + is)(c_{j-1} + \hat{g}_{j-1}(\xi)) + \frac{i}{2}\mathcal{H}(\phi_\Delta^*(-\eta + is)(c_{j-1} + \hat{g}_{j-1}(\eta)))(\xi)$$

$$c_j = 1 - \hat{f}_j(0)$$

This will give finally

$$\mathbb{E}(e^{sM_N}) = \phi_T(-is)(c_N + \hat{g}_N(0))$$

3 Approximation of the Hilbert transform

For our numerical computation, one will estimate the Hilbert transform by

$$\mathcal{H}(f)(x) = \sum_{m=-M}^M f(mh) \frac{1 - \cos\left[\pi \frac{(x-mh)}{h}\right]}{\pi \frac{(x-mh)}{h}}$$

It is important to note that if $x = m_0 h$ then

$$\mathcal{H}(f)(x) = \sum_{m=-M, m \neq m_0}^M f(mh) \frac{1 - (-1)^{m_0-m}}{\pi(m_0 - m)}$$

So one will approximate the $(\hat{f}_j)_{j \leq N}$ and $(\hat{g}_j)_{j \leq N}$ on the grid $kh; -M \leq k \leq M$ and the final algorithm which will be implemented becomes

$$\begin{aligned} \hat{f}_{j,h,M}(hk) &= \frac{1}{2} \phi_{\Delta}^*(-kh)(c_{j-1} + \hat{f}_{j-1}(kh)) \\ &+ \frac{i}{2\pi} \sum_{m=-M, m \neq k}^M \phi_{\Delta}^*(-mh)(c_{j-1} + \hat{f}_{j-1}(mh)) \frac{1 - (-1)^{k-m}}{\pi(k-m)} \\ \hat{g}_{j,h,M}(hk) &= \frac{1}{2} \phi_{\Delta}^*(-kh + is)(c_{j-1} + \hat{g}_{j-1}(kh)) \\ &+ \frac{i}{2\pi} \sum_{m=-M, m \neq k}^M \phi_{\Delta}^*(-mh + is)(c_{j-1} + \hat{g}_{j-1}(mh)) \frac{1 - (-1)^{k-m}}{\pi(k-m)} \end{aligned}$$

Let's consider

$$\begin{aligned} H_M &= \frac{1}{2} \begin{pmatrix} 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & 0 & \frac{2i}{5\pi} & \dots & 0 & \frac{2i}{2M+1\pi} \\ -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & 0 & \ddots & \ddots & \vdots \\ 0 & \frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & \ddots & \ddots & \vdots \\ -\frac{2i}{3\pi} & 0 & \frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \ddots & \ddots & \vdots \\ 0 & \frac{2i}{3\pi} & 0 & \frac{2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots & \vdots \\ -\frac{2i}{5\pi} & 0 & \frac{2i}{3\pi} & 0 & \frac{2i}{\pi} & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \end{pmatrix} \\ \Phi_{\Delta,F}^* &= \begin{pmatrix} \phi_{\Delta}^*(Mh) & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \phi_{\Delta}^*(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \phi_{\Delta}^*(-Mh) \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
\Phi_{\Delta,G}^* &= \begin{pmatrix} \phi_{\Delta}^*(Mh + is) & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \phi_{\Delta}^*(0 + is) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \phi_{\Delta}^*(-Mh + is) \end{pmatrix} \\
F_j &= \begin{pmatrix} \hat{f}_{j,h,M}(-Mh) \\ \vdots \\ \hat{f}_{j,h,M}(kh) \\ \vdots \\ \hat{f}_{j,h,M}(Mk) \end{pmatrix} \\
G_j &= \begin{pmatrix} \hat{g}_{j,h,M}(-Mh) \\ \vdots \\ \hat{g}_{j,h,M}(kh) \\ \vdots \\ \hat{g}_{j,h,M}(Mk) \end{pmatrix} \\
C_j &= c_j \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}
\end{aligned}$$

Then the algorithm becomes

$$\begin{aligned}
F_j &= H_M[\Phi_{\Delta,F}^*(C_{j-1} + F_{j-1})] \\
G_j &= H_M[\Phi_{\Delta,G}^*(C_{j-1} + G_{j-1})] \\
c_j &= 1 - (F_j)_{M+1,1} = 1 - \hat{f}_{j,h,M}(0)
\end{aligned}$$

H_M is a Toeplitz matrix, it is therefore possible to transform it into a circulant

Matrix K_{4M+1}

$$K_{4M+1}\left(\frac{V}{2}\right) = \frac{1}{2} \left(\begin{bmatrix} 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & \cdots & \frac{2i}{2M+1\pi} \\ -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \ddots & \ddots \\ 0 & -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\ -\frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} & 1 & \ddots & \ddots \\ 0 & \frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} & \ddots & \ddots \\ -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots & 1 \end{bmatrix} \begin{bmatrix} -\frac{2i}{2M+1\pi} & \cdots & -\frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} \\ \frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \frac{2i}{2M+1\pi} & -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & -\frac{2i}{\pi} \\ 0 & \frac{2i}{2M+1\pi} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{2i}{2M+1\pi} \\ \frac{2i}{\pi} & \cdots & \cdots & 0 & \frac{2i}{2M+1\pi} \end{bmatrix} \begin{bmatrix} 1 & \frac{2i}{\pi} & \ddots & \ddots & \ddots \\ -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \frac{2i}{\pi} \\ -\frac{2i}{2M+1\pi} & 0 & \cdots & -\frac{2i}{\pi} & 1 \end{bmatrix} \right)$$

where

$$V = \begin{pmatrix} 1 \\ -\frac{2i}{\pi} \\ 0 \\ -\frac{2i}{3\pi} \\ \vdots \\ -\frac{2i}{2M+1\pi} \\ \frac{2i}{2M+1\pi} \\ \vdots \\ \frac{2i}{3\pi} \\ 0 \\ \frac{2i}{\pi} \end{pmatrix}$$

Thanks to the Fast Fourier Transform (FFT), the product matrix-vector can be easily computed in $O(M \log_2(M))$ operations instead of $O(M^2)$. Indeed,

$$FFT^{-1}(FFT(\frac{V}{2}) \circ FFT(X)) = K_{4M+1}(\frac{V}{2})X$$

So to calculate $H_M X$, one computes

$$\frac{1}{2} \left(\begin{bmatrix} 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & \cdots & \frac{2i}{2M+1\pi} \\ -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \ddots & \ddots \\ 0 & -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\ -\frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} & 1 & \ddots & \ddots \\ 0 & \frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} & \ddots & \ddots \\ -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots & 1 \\ \frac{2i}{2M+1\pi} & -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & -\frac{2i}{\pi} \\ 0 & \frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{2i}{2M+1\pi} & \frac{2i}{\pi} \\ \frac{2i}{\pi} & \cdots & \cdots & 0 & \frac{2i}{2M+1\pi} & \frac{2i}{\pi} \end{bmatrix} \begin{bmatrix} -\frac{2i}{2M+1\pi} & \cdots & -\frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} \\ \frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & \frac{2i}{\pi} & \ddots & \ddots & \ddots \\ -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \frac{2i}{\pi} \\ -\frac{2i}{2M+1\pi} & 0 & \cdots & -\frac{2i}{\pi} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{2M+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

References

- [1] FENG, L. ET V. LINETSKY. Computing Exponential Moments of the Discrete Maximum of a Levy process and Look-back Options. Finance and Stochastics 13(4), 501-529 (2009). [1](#)