

# Evaluating fair premiums of equity-linked policies with surrender option in a bivariate model\*

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## Abstract

We tackle the problem of computing fair periodical premiums of an equity-linked policy with a maturity guarantee and an embedded surrender option. We consider the policy as a Bermudan-style contingent claim that can be exercised at the premium payment dates. The evaluation framework is based on a discretization of a bivariate model that considers the joint evolution of the equity value with stochastic interest rates. To deeply reduce the computational complexity of the pricing problem we use the singular points framework that allows us to compute accurate upper and lower estimates of the policy premiums.

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## 1 Introduction

Nowadays equity-linked policies have gained a wide popularity in insurance market. The main reason is that they give the opportunity to link the capital invested into the policy to the performance of a portfolio of equities. In this way, the coverage provided in case of death or survival of the insured is coupled with the possibility to obtain higher returns from the capital market than those guaranteed by traditional policies. The problem is that in the latter case the insured bears the risk of a negative performance of the equities considered. To mitigate this risk, insurance companies usually insert into the contract a minimum guarantee that assures the policyholder to receive at maturity (or before in the case of early termination of the contract) at least a prespecified sum.

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In their seminal papers, Brennan and Schwartz [10] and Boyle and Schwartz [9] applied financial mathematics techniques, developed in a Black-Scholes framework to compute fair premiums of equity-linked policies with minimum guarantee. Since then, a huge number of contributions tackled the fair premium evaluation problem in several directions. Among others we mention Aase and Persson [2] and Delbaen [14] that considered periodical premiums computed via Monte Carlo simulations. Bacinello and Ortu [7], at first, extend these models to the case in which the minimum guarantee is endogenous, i.e., it is a function of the premium(s) paid. Then, they derived a closed form formula for the single premium of an endowment equity-linked policy in a framework where interest rates are stochastic [8]. Nielsen and Sandmann [19] developed a model with stochastic interest rates to evaluate periodic premiums computed using numerical procedures.

Often, an equity-linked policy embeds a surrender option that gives the policyholder the chance to escape out of the contract before maturity. The fair premiums evaluation problem may consider the decision to surrender the contract as exogenous. In this case, together with death, early withdrawal is considered as a second exogenous cause of early termination of the contract. Then, given sufficient statistical observations on early withdrawals, it is possible to estimate the periodical probabilities that an individual abandons the contract that are used jointly with death probabilities to compute premiums.

A different approach considers the decision to surrender the policy endogenous within the evaluation model. In other words a policyholder decides to early abandon the contract if it is financially convenient. Hence, the presence of an embedded surrender option makes the policy a Bermudan-style contingent claim and the premiums may be determined using standard techniques for American options evaluation. This second approach has been first proposed by Albizzati and Geman [1] to compute fair single premiums of deferred annuities in a model with stochastic interest rates. Grosen and Jorgensen [17] considered the problem of computing the fair value of a single premium contract similar to an equity-linked policy embedding an early exercisable interest rate guarantee. Bacinello [4] tackled the problem of computing fair premiums of equity-linked policies with an embedded surrender option by considering the Cox-Ross-Rubinstein [13] (CRR hereafter) binomial model to describe the evolution of the reference fund value made up of equities linked to the policy benefit. The CRR model has the advantage of being easy to understand and highly tractable from a mathematical point of view but a problem arises when periodical premiums are considered. Indeed, the binomial tree describing the evolution of the reference fund value is not anymore recombining, hence the evaluation problem becomes computationally unmanageable as the number of time steps increases. To overcome this problem in [11] it has been proposed a model that, instead of considering all the reference fund values, associates to each possible equity value a subset of all of the reference fund values. This trick allows to deeply reduce the computational complexity of the evaluation problem and as a result, accurate values for periodical premiums are obtained.

In this paper, we propose a tree algorithm for evaluating the fair premium of an equity-linked policy with a surrender option and a minimum guarantee that takes into account stochastic interest rates. Indeed, since life insurance policies are usually long term contracts, it is unrealistic to consider interest rates fixed at a certain level during the whole policy lifetime. Hence in order to take into account the correlation between the equity price and the interest rate we construct a bivariate lattice that describes their evolution by adapting the approach proposed by Wey [21] to price plain vanilla options. However, even if the bivariate lattice is recombining, the problem of computing the fair premium of equity-linked policies with maturity guarantee and an embedded surrender option, cannot be treated in a feasible way by standard backward induction. In fact, in this case, the problem is deeply path-dependent leading to a non-recombining tree. For this purpose we propose to use the framework of the singular points technique presented in [16]. In this way we obtain a procedure which allows to get the fair premium with a large number of steps in a reasonable time and the convergence of the discrete approximations to the continuous value in a simple way. In fact, we can get approximations of the *true* lattice policy price (i.e., the price obtained considering all the possible paths) with a prescribed level of error and the convergence of the approximations can be derived from the convergence of the discrete bivariate model to the continuous one. Furthermore

our method provides upper and lower estimates of the premium computed by the lattice procedure.

The paper is organized as follows. In Section 2, we describe the bivariate model for the joint evolution of equity and of the spot interest rate processes in the continuous and discrete settings. In Section 3, we describe the insurance contracts. In Section 4, we propose our tree algorithm for evaluating the fair premium based on the singular points technique. Numerical results relative to the proposed evaluation algorithm are the contents of Section 5. In Section 6, we draw conclusions.

## 2 The bivariate model

This section is devoted to describe the construction of a bivariate lattice corresponding to the joint evolution of spot interest rates and equity values. We shall be concerned with a geometric Brownian motion describing the evolution of the equity value with drift driven by a square root process. Following Wey [21], we transform at first both the processes into two diffusions with unit volatility and then we introduce an auxiliary process orthogonal to the transformed interest rate process. This allows us to build up easily a recombining bivariate lattice to guarantee the tractability of the model from a computational point of view. Finally, we transform back the bivariate lattice into a new one to obtain the joint discretized evolution of the equity and of the spot interest rate.

We consider, under the risk-neutral probability measure, the following dynamics for the equity value

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_S dZ_S(t), \quad S(0) = S_0 > 0, \quad (1)$$

where  $r(t)$  is the short interest rate,  $\sigma_S$  is the constant stock price volatility and  $Z_S(t)$  is a standard Brownian motion.

The risk-neutralized process for the short rate is described, as in the Cox- Ingersoll-Ross model [12] (CIR hereafter), by the following stochastic differential equation

$$dr(t) = k[\theta - r(t)]dt + \sigma_r \sqrt{r(t)}dZ_r(t), \quad r(0) = r_0 > 0, \quad (2)$$

where  $k$  is a constant representing the reversion speed,  $\theta$  is the long term reversion target,  $\sigma_r$  is a constant and  $Z_r(t)$  is a standard Brownian motion whose correlation with  $Z_S(t)$  is  $\rho$ . It is well known that if  $r_0 > 0$  and  $2k\theta \geq \sigma_r^2$  the probability that  $r(t)$  will hit zero is zero.

As indicated in [21], the first step to the construction of a recombining bivariate tree is to transform processes (1) and (2) into two diffusions with unit volatility. This is done by introducing the variables  $X = (\log S)/\sigma_S$  and  $R = 2\sqrt{r}/\sigma_r$ , respectively. The dynamics of  $X$  and  $R$  may be easily derived applying Ito's Lemma. Hence,

$$dX(t) = \mu_X dt + dZ_S(t), \quad X(0) = (\log S_0)/\sigma_S, \quad (3)$$

and

$$dR(t) = \mu_R dt + dZ_r(t), \quad R(0) = 2\sqrt{r_0}/\sigma_r, \quad (4)$$

where

$$\mu_X = \mu_X(R(t)) = \frac{\sigma_r^2 R^2(t)/4 - \sigma_S^2/2}{\sigma_S},$$

and

$$\mu_R = \mu_R(R(t)) = \frac{k(4\theta - R^2(t)\sigma_r^2) - \sigma_r^2}{2R(t)\sigma_r^2}. \quad (5)$$

Both the transformed processes  $X$  and  $R$  have constant volatilities, hence they may be discretized independently using recombining binomial trees. In order to describe easily the joint probabilities of the two diffusions, we introduce a new process orthogonal to  $R(t)$ ,

$$Y(t) = \frac{X(t) - \rho R(t)}{\sqrt{1 - \rho^2}}.$$

Standard calculations give that the processes  $Y(t)$  and  $R(t)$  have zero covariance and that the dynamics of the diffusion process  $Y(t)$  has the form

$$dY(t) = \mu_Y dt + dZ_Y(t), \quad Y(0) = Y_0 = \frac{1}{\sqrt{1-\rho^2}}[(\log S_0)/\sigma_S - 2\rho\sqrt{r_0}/\sigma_r], \quad (6)$$

where  $Z_Y(t)$  is a standard Brownian motion orthogonal to  $Z_r(t)$  and the drift depends by the process  $R(t)$  in the following way

$$\mu_Y = \mu_Y(R(t)) = \frac{\mu_X(R(t)) - \rho\mu_R(R(t))}{\sqrt{1-\rho^2}}. \quad (7)$$

Now, the underlying asset  $S$  and the spot interest rate  $r$  can be obtained from the state variables  $Y$  and  $R$  using the relations:

$$r(t) = \begin{cases} \frac{R^2(t)\sigma_r^2}{4} & \text{if } R(t) > 0 \\ 0 & \text{otherwise} \end{cases} \quad S(t) = \exp[\sigma_S(\sqrt{1-\rho^2}Y(t) + \rho R(t))]. \quad (8)$$

## 2.1 The bivariate discrete model

We are in a position now to build up the bivariate tree discretizing the joint evolution of the processes  $R$  and  $Y$  and consequently the approximating processes for  $r$  and  $S$ .

To construct the discrete approximations of the processes  $R$  and  $Y$ , we divide the time to maturity  $T$  into  $n$  intervals of length  $\Delta t = T/n$  and, since  $R$  and  $Y$  are both processes with unit variance, we set the size of each up step equal to  $\sqrt{\Delta t}$  and the size of each down step equal to  $-\sqrt{\Delta t}$ . As usual, a binomial tree may be considered to describe the evolution of the discrete approximating processes. We label  $(0, 0)$  the starting node where the  $R$ -process has value  $R(0)$ . After  $i$  time steps ( $i = 0, \dots, n$ )  $R$  may be located at one of the nodes  $(i, k)$  ( $k = 0, \dots, i$ ) corresponding to the values

$$R_{i,k} = R_0 + (2k - i)\sqrt{\Delta t}. \quad (9)$$

Analogously, for the discrete process approximating  $Y$ , after  $i$  time steps it may be located at one of the nodes  $(i, j)$  ( $j = 0, \dots, i$ ) corresponding to the values

$$Y_{i,j} = Y_0 + (2j - i)\sqrt{\Delta t}. \quad (10)$$

Transition probabilities have to be specified to assure the matching of the local drift and of the local variance between the discrete and continuous model of  $(Y, R)$ . This will guarantee that the discretized processes converge in distribution to the corresponding diffusions (4) and (6), respectively (see [18] for a detailed description in the one-dimensional setting). To do this we have to take into account that in some regions of the tree it may happen that multiple jumps are needed to satisfy properly the matching conditions. Hence, starting from  $R_{i,k}$  at time  $i\Delta t$ , the process  $R$  may jump at time  $(i+1)\Delta t$  to the value  $R_{i+1,k_d}$  with  $k_d$  defined as

$$k_d \equiv \begin{cases} 0 & \text{if } R_{i,k} + \mu_R \Delta t < R_{i+1,0}, \\ i & \text{if } R_{i,k} + \mu_R \Delta t > R_{i+1,i+1}, \\ \text{the largest integer } k^* \in [0, i] : R_{i,k} + \mu_R \Delta t \geq R_{i+1,k^*} & \text{otherwise,} \end{cases}$$

or it may jump to  $R_{i+1,k_u}$ , with  $k_u$  defined as

$$k_u \equiv k_d + 1.$$

Clearly,  $\mu_R = \frac{k(4\theta - R_{i,k}^2\sigma_r^2) - \sigma_r^2}{2R_{i,k}\sigma_r^2}$  is the drift value of the transformed process  $R$  at  $R_{i,k}$ . Moreover one has

$$k^* = k + \text{int}\left(\frac{\mu_R\sqrt{\Delta t} + 1}{2}\right),$$

where  $\text{int}(x)$  is the integer part of  $x$ .

Now, the probability that the  $R$ -process, located at the node  $R_{i,k}$ , reaches  $R_{i+1,k_u}$ , is well defined by setting

$$p_{i,k} \equiv \begin{cases} 0 & \text{if } R_{i,k} + \mu_R \Delta t < R_{i+1,0}, \\ 1 & \text{if } R_{i,k} + \mu_R \Delta t > R_{i+1,i+1}, \\ \frac{\mu_R \Delta t + R_{i,k} - R_{i+1,k_d}}{R_{i+1,k_u} - R_{i+1,k_d}} & \text{otherwise.} \end{cases}$$

Obviously, the probability to reach  $R_{i+1,k_d}$  is  $1 - p_{i,k}$ . It is worth to notice that in some regions of the tree, particularly near the lower boundary located at zero, the drift of the diffusion,  $\mu_R$ , becomes so large that it cannot be matched by considering the values of the process generated within the approximating binomial grid. As a consequence, in order to consider legitimate transition probabilities, we censure the  $p_{i,k}$  in order to remain within the interval  $[0, 1]$ . Similarly, for the  $Y$ -process we define

$$j_d \equiv \begin{cases} 0 & \text{if } Y_{i,j} + \mu_Y \Delta t < Y_{i+1,0}, \\ i & \text{if } Y_{i,j} + \mu_Y \Delta t > Y_{i+1,i+1}, \\ \text{the largest integer } j^* \in [0, i] : Y_{i,j} + \mu_Y \Delta t \geq Y_{i+1,j^*} & \text{otherwise,} \end{cases}$$

and

$$j_u \equiv j_d + 1.$$

Here  $\mu_Y$  stands for the drift value of the transformed process  $Y$  computed at  $R_{i,k}$  according to (7), and  $j^* = j + \text{int}(\frac{\mu_Y \sqrt{\Delta t + 1}}{2})$ .

The transition probability that the  $Y$ -process, located at  $Y_{i,j}$ , jumps to  $Y_{i+1,j_u}$ , is well defined by setting

$$\hat{p}_{i,j,k} \equiv \begin{cases} 0 & \text{if } Y_{i,j} + \mu_Y \Delta t < Y_{i+1,0}, \\ 1 & \text{if } Y_{i,j} + \mu_Y \Delta t > Y_{i+1,i+1}, \\ \frac{\mu_Y \Delta t + Y_{i,j} - Y_{i+1,j_d}}{Y_{i+1,j_u} - Y_{i+1,j_d}} & \text{otherwise.} \end{cases}$$

Note that the dependence on the index  $k$  in  $\hat{p}$  is due to the specific form of the drift of the process (6) (see equation (7)).

We proceed now describing the discrete approximation scheme for the joint evolution of the processes  $Y$  and  $R$  considering a bivariate tree. For this purpose we construct a recombining structure for the tree (see next picture) by merging the two univariate binomial trees for the state variables  $R$  and  $Y$ . At each time step  $i$  ( $i = 0, \dots, n$ ), we consider  $(i+1)^2$  nodes that we label  $(i, j, k)$  corresponding to the values  $R_{i,k}$  and  $Y_{i,j}$  ( $k, j = 0, \dots, i$ ).

Starting from the node  $(i, j, k)$ , in consideration of possible multiple jumps and taking into account the tree structure, the process may reach one of the four nodes

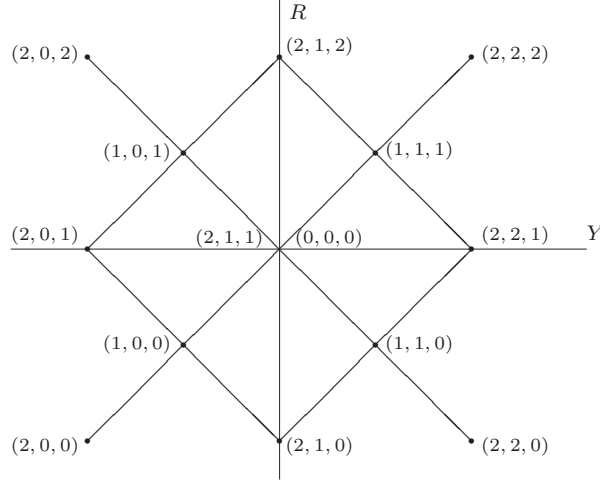
$$\begin{aligned} (i+1, j_u, k_u), & \quad \text{with probability } p_{uu}, \\ (i+1, j_u, k_d), & \quad \text{with probability } p_{ud}, \\ (i+1, j_d, k_u), & \quad \text{with probability } p_{du}, \\ (i+1, j_d, k_d), & \quad \text{with probability } p_{dd}, \end{aligned}$$

where  $j_u, j_d, k_u, k_d$  are the indexes related to the number of multiple jumps on the tree in the  $Y$  and  $R$  directions, and  $p_{uu}, p_{ud}, p_{du}, p_{dd}$  are the transition probabilities. Such probabilities can be computed, due to the orthogonality of the two processes, as follows

$$\begin{aligned} p_{uu} &= p_{i,k} \hat{p}_{i,j,k}, & p_{ud} &= p_{i,k} (1 - \hat{p}_{i,j,k}) \\ p_{du} &= (1 - p_{i,k}) \hat{p}_{i,j,k}, & p_{dd} &= (1 - p_{i,k}) (1 - \hat{p}_{i,j,k}). \end{aligned} \tag{11}$$

Finally, a bivariate tree for the joint evolution of the processes  $r$  and  $S$  is derived simply by applying the transforms (8) to the discrete scheme just defined. Indeed, to each node of the tree

Figure 1: The projection of the tridimensional bivariate tree on the  $(Y, R)$  plane



$(i, j, k)$  it will correspond the value

$$r_{i,k} = \begin{cases} \frac{R_{i,k}^2 \sigma_r^2}{4} & \text{if } R_{i,k} > 0 \\ 0 & \text{otherwise} \end{cases}$$

and the value

$$S_{i,j,k} = \exp[\sigma_S(\sqrt{1 - \rho^2} Y_{i,j} + \rho R_{i,k})].$$

The successor values are easily identified with  $r_{i+1,k_u}$ ,  $r_{i+1,k_d}$  and  $S_{i+1,j_u,k_u}$ ,  $S_{i+1,j_u,k_d}$ ,  $S_{i+1,j_d,k_u}$ ,  $S_{i+1,j_d,k_d}$ . Concerning the transition probabilities we will keep those defined in (11).

Based on the martingale central limit theorem (see [15] p. 354), it follows that the approximating bivariate model defined above converges weakly to the corresponding bivariate continuous diffusion. However, as remarked by Tian [20], the convergence in distribution is guaranteed whenever the condition  $4k\theta > \sigma_r^2$  holds. Empirical estimates for the CIR process rarely violate this condition.

**Remark 1.** The tree structure requires that the denominator in (5) never vanishes, i.e.  $R_0 + j\sqrt{T/n}$  does not vanish for all  $j = -n, \dots, n$ . Taking  $j_0 = \text{int}[\frac{R_0}{\sqrt{T/n}}]$ , we have that the minimum (in absolute value) of the discrete approximation of  $R$  in (5) is

$$\gamma = \min\{|R_0 - j_0\sqrt{T/n}|, |R_0 - (j_0 + 1)\sqrt{T/n}|\}.$$

If  $\gamma$  is too small, say  $\gamma < 10^{-6}$ , we increase the number of steps  $n$  until  $\gamma \geq 10^{-6}$ .

### 3 The equity-linked policies with minimum guarantee and surrender option

We consider an equity-linked policy whose payoff depends on the performance of a portfolio made up of equities of the same kind. We analyze the dynamics of the reference fund generated by investing a fixed contribution,  $D$ , to acquire equities at the beginning of each year until maturity  $T$ . The equity value  $S(t)$  and the spot rate  $r(t)$  dynamics are described by equations (1) and (2), respectively.

At time  $t_0 = 0$ , with the first contribution  $D$ , the insurer buys  $\frac{D}{S(0)}$  equities and at each contribution time  $t_m = m$ ,  $m = 0, \dots, T-1$ , when the equity value is  $S(t_m)$ , the insurer buys  $\frac{D}{S(t_m)}$

equities. Hence, the cumulative number of equities that the insurer has acquired before the time  $t_m$  is  $\sum_{l=0}^{m-1} \frac{D}{S(t_l)}$ . This number depends on the path of the underlying asset value in the time interval  $[0, t_{m-1}]$ . We are interested in describing the evolution of the reference fund at any time  $t$ ,  $0 \leq t \leq T$ . At  $t_0 = 0$  its value is  $F(0) = D$ . At time  $t$ ,  $0 < t \leq T$ , the reference fund value is given by

$$F(t) = S(t) \sum_{m=0}^{m_0} \frac{D}{S(t_m)}, \quad (12)$$

where  $m_0$  is the largest index  $m$  such that  $t_m < t$ . Since  $t_m$  is an integer, we have:  $m_0 = \text{int}(t)$  if  $t$  is not an integer,  $m_0 = t - 1$  otherwise. Remark that the reference fund at time  $t_m$  ( $m > 0$ ) doesn't take into account the contribution due at the same time.

Our goal is to compute fair periodical premiums of equity-linked policies with a minimum guarantee and with a surrender option. The presence of a minimum guarantee protects the policyholder's investment against a negative performance of the reference fund. We analyze at first the case of a policy with periodical premiums  $P$  to be paid at the beginning of each year but not at maturity. A surrender option is embedded into the policy and it allows the policyholder to escape out of the contract at the beginning of each year, just before the premium payment. We consider the decision to surrender the policy as endogenous to the evaluation model, i.e., the policyholder decides to abandon the contract if this is financially convenient. This allows to settle the surrender option as a Bermudan option embedded into the policy.

For the time being we do not consider mortality.

At the policy maturity,  $T$ , if the policyholder has not previously surrendered the contract, the insurer is forced to pay the maximum between the reference fund value  $F(T)$  and the minimum guarantee  $G(T)$ . Among the different types of minimum guarantees, we consider the case

$$G(T) = \sum_{m=0}^{T-1} D e^{(T-m)\delta} = D e^{\delta} \frac{e^{T\delta} - 1}{e^{\delta} - 1}$$

where  $\delta \geq 0$  is the minimum guaranteed interest rate (continuously compounded). This corresponds to the capitalization at the interest rate  $\delta$  of the amounts  $D$  invested by the insurer at the beginning of every year.

The payoff function is therefore

$$\phi(F(T), G(T)) = \max\{F(T), G(T)\}.$$

At every premium payment date  $t_m > 0$ ,  $m = 1, \dots, T - 1$ , the policyholder has two alternatives:

- to continue the contract and in this case pays the premium  $P$ ,
- to surrender the contract receiving the maximum between the accrued reference fund and the contributions paid until  $t$  invested at rate  $\delta$ :

$$\phi(F(t_m), G(t_m)) = \max\{F(t_m), G(t_m)\} \quad (13)$$

where

$$G(t) = \sum_{l=0}^{m_0} D e^{(t-t_l)\delta}. \quad (14)$$

We will consider also the case of an endowment equity-linked policy with a minimum guarantee and an embedded surrender option, i.e. we introduce mortality risk, by assuming that the benefit is paid before maturity, in case of death, and that the periodical premiums are due only in case of survival. As before, the policyholder pays a constant premium at the beginning of each year if he is alive at that date and has not previously surrendered the contract. The insured has the right to early

terminate the contract and this option may be exercised at the beginning of each year, just before the payment of the premium. Moreover, if the insured dies at time  $t \in (0, T]$  the insurer pays at time  $t$  the amount  $\phi(F(t), G(t))$ , closing the contract. Clearly, even more complicated functions specifying the death benefit may be considered and easily adapted to the evaluation framework. We assume the stochastic independence between the lifetime of the insured and the financial state variables. Furthermore we assume that the insurance company is risk-neutral with respect to mortality.

The evaluation of the previous contracts requires the pricing of a Bermudan path-dependent contingent claim in the continuous bivariate model with the interest rate that follows the CIR model. The numerical resolution of this problem is troublesome.

In the next section we propose, based on the framework described in Section 2, a lattice algorithm in order to evaluate the fair premium for the equity-linked policy with a minimum guarantee and a surrender option.

## 4 A tree method for evaluating the fair premium

We shall make use of the bivariate tree described in Section 2.1 to evaluate the fair premium for equity-linked policies described in the previous section. However a direct application of such tree is not feasible since the presence of periodical deemed contributions causes a huge increment in the number of possible values of the reference fund. Indeed, the dynamics of the reference fund value is represented by a non recombining tree with a number of nodes that grows exponentially when the number of time steps in the lattice increases (if the number of steps doubles, passing from  $n$  to  $2n$ , then the number of join paths is multiplied by  $4^n$ ). This is a problem from a computational point of view since it causes a huge increment of the variables to be considered in the evaluation model.

To reduce the computational complexity we settle the problem in the framework of the singular points introduced in [16]. Let us remark first that every piecewise linear function  $f$  of a real variable, is determined by the values  $(x_i, f(x_i))$  where changes the slope of the function. Such points  $(x_i, f(x_i))$  are called the *singular points* of  $f$  (see Appendix). The singular points approach consists in a backward procedure which permits to obtain a continuous representation of the policy value at every node of the bivariate tree, as a piecewise linear continuous function of the reference fund  $F$ . Such value functions are completely characterized by their singular points, hence the pricing procedure depends exclusively by the knowledge of the singular points at every node of the tree.

It is important to note that this procedure provides exactly the *true* lattice value of the policy in the bivariate discrete model described in Section 2.1 but its straightforward application is unfeasible from a computational point of view even for a small number of tree steps. Then, in the paper, we propose a method that permits to obtain an important improvement since it allows to approximate easily the *true* lattice value in a efficient way, giving, in the same time, a control of the error.

### 4.1 The dynamics of reference fund value in the discrete setting

Let  $T$  be the maturity of an equity-linked policy with periodical premium  $P$  paid at every time  $t_m = m$ ,  $m = 0, \dots, T - 1$ . We consider a bivariate tree with  $n$  time steps, where  $n$  is a multiple of  $T$ , hence we take  $n/T$  steps during every year of the life of the contract.

Keeping the notation of Section 2.1, let  $(i, j, k)$ ,  $i = 0, \dots, n$ ,  $j, k = 0, \dots, i$  denote the node of the tree corresponding at time  $i\Delta t$ , to the value  $Y_{i,j}$  of  $Y$  and to the value  $R_{i,k}$  of  $R$  (see (10) and (9)).

We still denote by  $r_{i,k}$  and  $S_{i,j,k}$  the interest rate and the underlying asset price at the node  $(i, j, k)$ , respectively:

$$r_{i,k} = \begin{cases} \frac{R_{i,k}^2 \sigma_r^2}{4} & \text{if } R_{i,k} > 0 \\ 0 & \text{otherwise} \end{cases} \quad S_{i,j,k} = \exp[\sigma_S(\sqrt{1 - \rho^2} Y_{i,j} + \rho R_{i,k})].$$

At time  $t_0 = 0$  the reference fund value  $F$  is equal to the deemed contribution  $D$ .

At any other time  $i\Delta t$ , a reference fund value  $F$  at node  $(i, j, k)$  has four possible successors at time  $(i+1)\Delta t$  with transition probabilities defined by (11) of Section 2. The possible values are

$$F_{uu} = Fm_{uu}, \quad F_{ud} = Fm_{ud}, \quad F_{du} = Fm_{du}, \quad F_{dd} = Fm_{dd}, \quad (15)$$

if either  $i = 0$  or  $i\Delta t$  is not a premium payment date,

and

$$F_{uu} = (F + D)m_{uu}, \quad F_{ud} = (F + D)m_{ud}, \quad F_{du} = (F + D)m_{du}, \quad F_{dd} = (F + D)m_{dd}, \quad (16)$$

if  $i > 0$  and  $i\Delta t$  is a premium payment date,

where

$$m_{uu} = \frac{S_{i+1,j_u,k_u}}{S_{i,j,k}}, \quad m_{ud} = \frac{S_{i+1,j_u,k_d}}{S_{i,j,k}},$$

$$m_{du} = \frac{S_{i+1,j_d,k_u}}{S_{i,j,k}}, \quad m_{dd} = \frac{S_{i+1,j_d,k_d}}{S_{i,j,k}}.$$

Our algorithm is based on the knowledge of lower and upper bounds of the reference fund value with respect to all possible paths of the underlying asset reaching the node  $(i, j, k)$ . Such bounds, denoted by  $F_{i,j,k}^{min}$ ,  $F_{i,j,k}^{max}$ , respectively, can be evaluated inductively on the tree. To do this, let  $S_i^{min}$  and  $S_i^{max}$  be the minimum and the maximum value of the underlying asset at time step  $i$ , i.e.

$$S_i^{min} = \begin{cases} S_{i,0,0} & \text{if } \rho \geq 0 \\ S_{i,0,i} & \text{if } \rho < 0 \end{cases} \quad S_i^{max} = \begin{cases} S_{i,i,i} & \text{if } \rho \geq 0 \\ S_{i,i,0} & \text{if } \rho < 0 \end{cases}.$$

Denote by  $N_i^{min}$  and  $N_i^{max}$  the minimum and the maximum number of equities that can be bought by the insurer until time step  $i$  (included). That is,

$$N_0^{min} = D/S_0, \quad N_i^{min} = \begin{cases} N_{i-1}^{min} & \text{if } i\Delta t \text{ is not a premium payment date} \\ N_{i-1}^{min} + D/S_i^{max} & \text{if } i\Delta t \text{ is a premium payment date} \end{cases} \quad (17)$$

$$N_0^{max} = D/S_0, \quad N_i^{max} = \begin{cases} N_{i-1}^{max} & \text{if } i\Delta t \text{ is not a premium payment date} \\ N_{i-1}^{max} + D/S_i^{min} & \text{if } i\Delta t \text{ is a premium payment date.} \end{cases} \quad (18)$$

Then

$$F_{0,0,0}^{min} = F_{0,0,0}^{max} = D, \quad F_{i,j,k}^{min} = N_{i-1}^{min} S_{i,j,k}, \quad F_{i,j,k}^{max} = N_{i-1}^{max} S_{i,j,k} \quad \text{if } i > 0. \quad (19)$$

Now we are able to construct the lattice procedure for the evaluation of the equity-linked policies.

## 4.2 The lattice algorithm

We proceed now to the description of the policy value function  $v_{i,j,k}(F)$ ,  $i = 0, \dots, n$  and  $j, k = 0, \dots, i$ , in terms of the reference fund value  $F$ , at every node  $(i, j, k)$  of the tree structure. In the first case we do not consider mortality.

At maturity, the policy value  $v$ , as function of the reference fund  $F$ , is continuously defined by

$$v_{n,j,k}(F) = \phi(F, G(T)) = \max\{F, G(T)\},$$

where  $\phi$  is the payoff function. Clearly,  $v_{n,j,k}$  is a piecewise linear convex function characterized by the three singular points  $(F_n^l, E_n^l)$ ,  $l = 1, 2, 3$  ( $F_n^l$  is the reference fund value,  $E_n^l$  is the corresponding equity-linked policy value) given by:

$$\begin{aligned} F_n^1 &= F_{n,j,k}^{min}, & E_n^1 &= G(T); \\ F_n^2 &= G(T), & E_n^2 &= G(T); \\ F_n^3 &= F_{n,j,k}^{max}, & E_n^3 &= F_{n,j,k}^{max}. \end{aligned} \quad (20)$$

If  $G(T) \notin (F_{n,j,k}^{min}, F_{n,j,k}^{max})$  the singular points reduce to two.

Let us consider now the step  $i = n - 1$ . A value  $F$  of the reference fund at node  $(i, j, k)$  generates four reference funds at step  $i + 1$ :  $F_{uu}, F_{ud}, F_{du}, F_{dd}$  described in (15), (16). Therefore we define the policy value at the node  $(i, j, k)$  to be

$$v_{i,j,k}(F) = e^{-r_{i,k}\Delta t} [p_{uu}v_{i+1,j_u,k_u}(F_{uu}) + p_{ud}v_{i+1,j_u,k_d}(F_{ud}) + p_{du}v_{i+1,j_d,k_u}(F_{du}) + p_{dd}v_{i+1,j_d,k_d}(F_{dd})]. \quad (21)$$

Clearly, observe that  $v_{n-1,j,k}(F)$  is piecewise linear and convex as well. The singular values of  $v_{n-1,j,k}$  are  $F_{n-1,j,k}^{min}, F_{n-1,j,k}^{max}$  (the extrema) and  $\frac{G(T)}{m_{uu}}, \frac{G(T)}{m_{ud}}, \frac{G(T)}{m_{du}}, \frac{G(T)}{m_{dd}}$  if they belong to  $(F_{n-1,j,k}^{min}, F_{n-1,j,k}^{max})$ . If  $(n - 1)\Delta t$  is a premium payment date, also these four values should be decreased of the deemed contribution  $D$ . In order to compute the corresponding policy values we have to apply equation (21) using the linearity property.

We then proceed iteratively in the same way for  $i = n - 2, \dots, 0$ . More precisely we compute the singular values of  $v_{i,j,k}(F)$  starting from the singular values of the four nodes:

$$(i + 1, j_u, k_u), \quad (i + 1, j_u, k_d), \quad (i + 1, j_d, k_u), \quad (i + 1, j_d, k_d).$$

Let  $\hat{F}$  be a singular value of the node  $(i + 1, j_u, k_u)$ . This value  $\hat{F}$  is divided by the factor  $m_{uu}$ . In the case  $i\Delta t$  is a premium payment date, we subtract from  $\frac{\hat{F}}{m_{uu}}$  the deemed contribution  $D$ .

The same procedure has to be applied to all the singular values of the nodes  $(i + 1, j_u, k_d), (i + 1, j_d, k_u), (i + 1, j_d, k_d)$ . All the values so obtained become singular values of  $v_{i,j,k}(F)$  if they belong to the domain  $(F_{i,j,k}^{min}, F_{i,j,k}^{max})$ . The computation of the corresponding policy values is obtained again by equation (21) and by linearity.

At the premium payment date the previous procedure needs an additional treatment. Let  $t_m$  be a premium payment date and let  $(F_{i,j,k}^1, E_{i,j,k}^1), \dots, (F_{i,j,k}^L, E_{i,j,k}^L)$  be the singular points associated to this date and evaluated by the previous procedure. Note that these points are referred to an instant after the payment of the premium  $P$ , and hence the policy value includes this premium. Clearly the payment of  $P$  has increased this value of the same amount. Therefore, if we go back to an instant before the premium payment, the new set of singular points becomes

$$(F_{i,j,k}^1, E_{i,j,k}^1 - P), \dots, (F_{i,j,k}^L, E_{i,j,k}^L - P).$$

If a surrender option is embedded into the equity-linked policy, we have to check at time  $t_m > 0$  the convenience of the early exercise. The policy value function  $v_{i,j,k}$  becomes

$$v_{i,j,k}(F) = \max\{\phi(F, G(t_m)), v_{i,j,k}^c(F) - P\}, \quad (22)$$

where  $v_{i,j,k}^c(F)$  is the continuation value computed as in the R.H.S. of (21). At every node  $(i, j, k)$ ,  $v_{i,j,k}(F)$  is still piecewise linear and convex.

The same argument can be applied at every step  $i = n - 1, \dots, 0$ . This allows to compute  $v_{0,0,0}(D) = E_{0,0,0}^1 - P$  which provides the *true* lattice equity-linked policy value in the case of a periodic premium payment  $P$ , with a minimum guarantee and a surrender option, associated to the tree with  $n$  steps.

This algorithm provides the (net) value at time 0 of the contract depending on the premium  $P$ ,  $\psi(P) = v_{0,0,0}(D)$  evaluated by  $n$  time steps. Since the policy has to be fair, we must evaluate the premium  $P^*$  so that  $\psi(P^*) = 0$ , i.e.  $P^* = E_{0,0,0}^1$ . The equation is non linear and has to be solved numerically; its unique solution (see Section 4.4) represents the fair periodical premium required by the insurer to write the contract.

In the case of an endowment equity-linked policy with a minimum guarantee and an embedded surrender option we have to consider the possibility that the insured dies during the interval of time  $(i\Delta t, (i + 1)\Delta t)$ . We use now the assumption of stochastic independence between financial and

demographic risk and the risk-neutrality of the insurer with respect to mortality.

We assume that if the insured dies in this time interval the insurer pays at time  $(i+1)\Delta t$  the amount

$$\phi(F, G((i+1)\Delta t)).$$

As usual, we label  ${}_t p_x$  the probability that an individual of age  $x$  survives the next  $t$  years while  ${}_t q_x = 1 - {}_t p_x$  represents the death probability.

If  $i\Delta t$  is not a premium payment date, the policy value function in the endowment case,  $v_{i,j,k}^E$ , has to be modified as follows:

$$v_{i,j,k}^E(F) = \Delta t \, p_{x+i\Delta t} v_{i,j,k}(F) + \Delta t q_{x+i\Delta t} v_{i,j,k}^d(F) \quad (23)$$

where

$$v_{i,j,k}^d(F) = e^{-r_{i,k}\Delta t} [p_{uu}\phi(F_{uu}, G((i+1)\Delta t)) + p_{ud}\phi(F_{ud}, G((i+1)\Delta t)) + p_{du}\phi(F_{du}, G((i+1)\Delta t)) + p_{dd}\phi(F_{dd}, G((i+1)\Delta t))]$$

and  $v_{i,j,k}(F)$  is computed as in the R.H.S. of (21) but with  $v$  replaced by  $v^E$ .

At a premium payment date  $t_m$ ,  $v_{i,j,k}^E$  becomes

$$v_{i,j,k}^E(F) = \max\{\phi(F, G(t_m)), v_{i,j,k}^c(F) - P\}, \quad (24)$$

where  $v_{i,j,k}^c(F)$  is computed as in the R.H.S. (23).

### 4.3 Approximations of the lattice policy values

The technique previously presented is inefficient from a computational point of view because of the high number of singular points generated by the procedure. Moreover the presence of the premium at payment date increases furthermore such number (the tree becomes not recombining). However simple modifications allow to reduce drastically the number of singular points providing an upper and a lower bound of the *true* lattice value.

In order to get an upper bound of the *true* lattice value we just remove some singular points at every node of the tree. This procedure (detailed in Lemma 1 part a) in the Appendix) ensures that the value obtained in such way is an upper estimate of the *true* lattice value. The criteria to remove the singular points is the same as presented in [16].

More precisely, consider the set of singular points  $C = \{(F_{i,j,k}^1, E_{i,j,k}^1), \dots, (F_{i,j,k}^L, E_{i,j,k}^L)\}$  at node  $(i, j, k)$ , and the corresponding value function  $v_{i,j,k}(F)$  ( $v_{i,j,k}^E(F)$  in the endowment case). Let  $v'_{i,j,k}(F)$  be the value function obtained by removing a point  $(F_{i,j,k}^l, E_{i,j,k}^l)$  from  $C$ . We have (see also Figure 2 in the Appendix)

$$|v_{i,j,k}(F) - v'_{i,j,k}(F)| \leq \epsilon_l, \quad \forall F \in [F_{i,j,k}^{min}, F_{i,j,k}^{max}] \quad (25)$$

where

$$\epsilon_l = v'_{i,j,k}(F_{i,j,k}^l) - v_{i,j,k}(F_{i,j,k}^l). \quad (26)$$

Therefore, given a real number  $h > 0$ , we choose to remove the point  $(F_{i,j,k}^l, E_{i,j,k}^l)$  if  $\epsilon_l < h$ . We repeat sequentially this procedure for all the singular points associated to every node of the tree, avoiding the elimination of two consecutive singular points. In this way the upper estimate of the policy value function so obtained differs from the true one at most for  $h$  at every node. Proceeding with the backward algorithm along the tree of  $n$  time steps, we can conclude that the obtained upper estimate differs from the *true* lattice value of the discrete bivariate model at most for  $nh$ .

The algorithm for the computation of the lower bound is similar and follows from Lemma 1 part b) in the Appendix.

**Remark 2.** (Convergence of the method for the equity-linked policy values) The possibility of obtaining estimates of the lattice price for the bivariate discrete model with a control of the error, allows to prove easily the convergence of our technique to the value in the continuous model. In fact, by applying a convergence result stated in [3] (see also [20]), it follows that the true lattice price with  $n$  steps, converges to the price in the continuous model as  $n \rightarrow \infty$ , whenever the condition  $4k\theta > \sigma_r^2$  holds. Choosing  $h$  depending of  $n$  and such that  $nh(n) \rightarrow 0$  (for example we can choose  $h(n) = O(\frac{1}{n\sqrt{n}})$ ), we have immediately that the corresponding sequences of upper and lower estimates converge to the price in the continuous model.

**Remark 3.** (Black-Scholes model) The methodology introduced for pricing equity-linked policies for the bivariate continuous model can be easily adapted to the case that the underlying equity asset is governed by a univariate log-normal model (Black-Scholes). In this case we can use the Cox-Ross-Rubinstein tree (see [13]). The singular points technique, previously described, becomes simpler and more efficient because the tree is univariate. In fact there are no multiple jumps, every reference fund value produces only two reference funds values at the successive step instead of four and the transition probabilities are constant along all the tree. In the numerical results we will compare such new approach with the ones presented in [11].

## 4.4 Computing the fair premium

In this section we describe how to compute upper and lower estimates of the fair premium of an equity-linked policy. We first deal with the computation of the fair premium using the *true* lattice algorithm described in section 4.2. This leads to an unfeasible procedure because of the computational complexity of the *true* lattice algorithm. Then we adapt such procedure to the case where the *true* lattice value is replaced by the approximated lattice values (upper and lower) described in Section 4.3.

### 4.4.1 Fair premium for the true lattice algorithm

Take again the value function  $v_{i,j,k}$  ( $v_{i,j,k}^E$  in the endowment case) with a fixed number of time steps  $n$  and consider its dependence not only on the reference fund  $F$  but also on the premium  $P$ , that is  $v_{i,j,k}(F, P)$ . At every node of the tree such function is convex as function of the two variables. This follows from the convexity of backward induction relation (21) and since the maximum between convex functions is convex again. As  $\psi(P) = v_{0,0,0}(D, P)$  one has that  $\psi(P)$  is continuous, convex and strictly decreasing (this follows by the backward induction as well) and, for  $P$  large enough,  $\psi(P) = \beta - P$  where  $\beta$  is constant.

On the other hand  $\psi(D) > 0$ , so that we can conclude that the solution  $P^*$  of the equation  $\psi(P) = 0$  exists and is unique. Clearly  $P^*$  is the fair premium evaluated using the *true* lattice policy value with  $n$  time steps.

In order to solve numerically the nonlinear equation  $\psi(P) = 0$ , we propose to use a secant algorithm with a suitable choice of two initial points  $P_1, Q_1$ ,  $P_1 < Q_1$ , such that

$$\psi(P_1) > 0 \text{ and } \psi(Q_1) < 0.$$

To this end we consider the function  $\psi_{Eur}(P)$  representing the policy value at inception without the surrender option, i.e. the function obtained by using only the backward induction given by equation (21). In the case of the endowment  $\psi_{Eur}(P)$  is obtained by using only equation (23). The resulting fair premium  $P_{Eur}^*$  is less than  $P^*$ , so we set  $P_1 = P_{Eur}^*$ . In this "European" case the premium  $P_{Eur}^*$  is easily computable, in fact the function  $\psi_{Eur}(P)$  is linear, i.e.  $\psi_{Eur}(P) = \psi_{Eur}(D) - \alpha(P - D)$ . The slope  $\alpha$  of  $\psi_{Eur}(P)$  can be computed as follows

- if we do not consider mortality,  $\alpha$  is the value of an annuity-certain that pays 1 at every time  $t_m$ ,  $m = 0, \dots, T - 1$ ;

- in the case of an endowment policy,  $\alpha$  is the value of a corresponding life annuity that pays 1 only if the insured is still alive at  $t_m$ .

Let us remark that the time of computation for the evaluation of  $\alpha$  is negligible with respect to the time needed for the computation of  $\psi_{Eur}(D)$ .

Then we take  $Q_1 = P_1 + \psi(P_1)$ . By the convexity of the function  $\psi(P)$  and since  $\psi(P) = \beta - P$  for  $P$  large enough, one has  $\psi(P) - \psi(P_1) \leq P_1 - P$  for all  $P > P_1$ . This implies  $\psi(Q_1) \leq \psi(P_1) + P_1 - Q_1 = 0$ .

The procedure in order to evaluate  $P^*$  is then the following: we consider the secant method starting from the two points  $P_1$  and  $Q_1$ . In this way we obtain a point  $Q_2$  such that  $\psi(Q_2) < 0$  (by convexity), hence  $Q_2 > P^*$ . We repeat this step starting from the points  $P_1$  and  $Q_2$  and so on. In this way we obtain a decreasing sequence of points  $Q_n$ ,  $n \geq 1$ , such that  $Q_n > P^*$ . We stop the procedure when the distance between two consecutive points is less than a fixed level of error  $\epsilon > 0$ . We get in this way an upper estimate  $Q_n$  of  $P^*$ . Now  $P_n = Q_n + \psi(Q_n)$  is a lower estimate of  $P^*$ , obtaining finally an upper and a lower estimate ( $Q_n$  and  $P_n$ ) of the premium evaluated by using the *true* lattice policy values with  $n$  steps.

#### 4.4.2 Fair premium for the approximated lattice algorithm

The procedure just described uses the *true* lattice value  $v_{0,0,0}$ . However, in order to have a feasible procedure for the computation of the premium we need to use approximations of the *true* lattice value. The use of approximations instead of the *true* lattice values requires modifications of the procedure. These will be described in the sequel.

Let us denote by  $\bar{\psi}(P)$ ,  $\psi(P)$ , respectively, the upper and the lower estimates of  $\psi(P)$  obtained by the procedure presented in Section 4.3 and with a fixed level of error  $h > 0$ .

First we evaluate  $\bar{\psi}_{Eur}(D)$ . The zero of the linear function  $\bar{\psi}_{Eur}(D) - \alpha(P - D)$  (the slope  $\alpha$  is the same as before) is less than  $P^*$ , hence we choose as initial point  $P_1$  of the secant algorithm such zero. Then we consider  $Q_1 = \bar{\psi}(P_1) + P_1$ . As before  $Q_1 > P^*$ .

As  $\bar{\psi}(P_1) > \psi(P_1)$  and  $\bar{\psi}(Q_1) > \psi(Q_1)$ , the straight line  $s$  joining the points  $(P_1, \bar{\psi}(P_1))$ ,  $(Q_1, \bar{\psi}(Q_1))$  lies over the graph of the convex function  $\psi(P)$  in all the interval  $[P_1, Q_1]$ . If  $\bar{\psi}(Q_1) \geq 0$  then we stop the procedure, and  $\hat{P}_1 = Q_1 + \bar{\psi}(Q_1)$  and  $Q_1$  are the lower and the upper estimates of  $P^*$ . If  $\bar{\psi}(Q_1) < 0$  then we evaluate the intersection  $Q_2$  of the straight line  $s$  with the  $x$ -axis. We have  $P^* < Q_2 < Q_1$ . We compute  $\bar{\psi}(Q_2)$  and we proceed iteratively as before. The procedure will continue until  $Q_{n-1} - Q_n < \epsilon$  or  $\bar{\psi}(Q_n) \geq 0$ . In both cases one has  $P^* < Q_n$ . Now  $\hat{P}_n = Q_n + \bar{\psi}(Q_n)$  and  $Q_n$  are the lower and the upper estimates of the premium  $P^*$  evaluated with the *true* lattice with  $n$  steps.

## 5 Numerical results

In this section we will test the lattice algorithm presented in Section 4 for computing the fair periodical premiums of equity-linked endowment policies with a surrender option and a minimum guarantee.

It seems that there are no previous papers where this insurance problem was treated using a bivariate model with stochastic interest rate. Hence we propose first to assess the numerical robustness of our algorithm in a case where  $\sigma_r$  is close to zero, so that the equity dynamics mimics the geometric Brownian motion (Black-Scholes model). In this case we can compare our results with the ones obtained in [11] with a complete different technique. As observed in Remark 3, our approach provides also a new univariate lattice method for the Black-Scholes model. The data obtained with this method can be considered a further comparison test for measuring the validity of the technique introduced in this paper.

The parameters of the contract are the following: the fixed deemed contribution is  $D = 100$ , the minimum guaranteed interest rate varies:  $\delta = 0, 0.02, 0.04$ . The maturities  $T$  are 5 and 10 years

and the individual initial age is  $x = 50$ . We model mortality by considering Italian Statistics for Male mortality in 2002. These statistics quoted the annual probabilities of death. Since we need to compute death probabilities on time periods smaller than one year, we assume uniformity of the deaths, in the sense that in any fraction of width  $\Delta t$  of one year it is expected the same fraction  $\Delta t$  of the deaths related to that age. Hence, the death probability on a fraction  $\Delta t$  of one year,  ${}_1q_x\Delta t$ , is equal to  ${}_1q_x\Delta t$ .

The parameters of our bivariate model defined in equations (1), (2) are: the initial equity value  $S_0$ , the volatility of the underlying asset  $\sigma_S$ , the initial interest rate  $r_0$ , the long term mean of the interest rate  $\theta$ , the speed of mean reversion  $k$ , the volatility of the interest rate  $\sigma_r$ , while the correlation between the two Brownian motions  $Z_r$  and  $Z_S$  is  $\rho$ .

We will consider 3 different choices of the parameters of the model. Case 1 and Case 2 are chosen to assess the reliability of the lattice algorithm comparing it with the Black-Scholes model. In Case 3 we consider the same parameters modifying only  $\sigma_r$ . This provides an example of the effect of the change on the premium obtained by using the bivariate model.

In Cases 2 and 3 we apply the singular points algorithm presented in Section 4 for evaluating an upper and a lower estimate of the fair periodical premium of the equity-linked policy. We chose  $h = 0.001$  (the maximal error for the upper and lower estimates at each node) and  $\epsilon = 0.001$  (the control error in the secant method). In Case 1, for benchmark purposes, we choose  $h = 0.0001$ , which permits to obtain very tiny estimates of the premium.

- Case 1. Black-Scholes model. We use the univariate tree discussed in Remark 3.

$$S_0 = 100, \sigma_S = 0.1358, r = 0.04.$$

- Case 2. Bivariate model which mimics Black-Scholes.

$$S_0 = 100, \sigma_S = 0.1358, r_0 = 0.04, \theta = 0.04, k = 1, \sigma_r = 10^{-6}, \rho = 0.$$

- Case 3. Bivariate model.

$$S_0 = 100, \sigma_S = 0.1358, r_0 = 0.04, \theta = 0.04, k = 1, \sigma_r = 0.2, \rho = 0.$$

In Table 1, we report the fair annual premiums for equity-linked endowment policies with an embedded surrender option and a minimum guarantee computed for different values of time steps  $n = 50, 100, 200, 500, 1000$ . In Case 1 (univariate case) we can use easily 500 or more steps, in Cases 2 and 3 the computation for  $n = 500, 1000$  is not available because of memory and time requirements. In Cases 1 and 2 we compare our algorithm with the one presented in [11] computed with a univariate tree of 500 steps (CMR 500).

To further illustrate the reliability of the proposed bivariate approximation technique, in Table 2 and Table 3 we present some examples of interest rate and asset parameters corresponding to two different values of  $\sigma_r$  and  $\rho$ . In both tables we take

- $S_0 = 100, \sigma_S = 0.25, r_0 = 0.08, \theta = 0.05, k = 0.5, \sigma_r = 0.08, 0.16, \rho = -0.25, 0.25$ .

In Table 2 the minimum guaranteed interest rate is  $\delta = 0$ , in Table 3 we take  $\delta = 0.04$ .

The previous data have been performed in double precision on a PC with processor Centrino 2 at 2.4 Ghz with 4 Gb of RAM. The time of computation for the bivariate tree, in the case  $T = 5$ , is approximately 15 seconds for  $n = 50$ , 120 seconds for  $n = 100$ , 400 seconds for  $n = 150$  and 1300 seconds for  $n = 200$ . In the case  $T = 10$  the times of computation double. In the univariate case our technique is clearly much more efficient and the computation of the fair periodical premium requires few seconds (3 seconds for  $n = 200$  and 26 seconds for  $n = 500$ ).

## 6 Conclusions

This paper considers the problem of computing, in a bivariate continuous model, the fair periodical premium of equity-linked policies embedding a surrender option and a minimum guarantee. The

Table 1: Upper and lower estimates of the fair annual premium in the case  $\delta = 0.02$ 

	n	Case 1	Case 2	Case 3
$T = 5$	50	106.753 - 106.753	106.743 - 106.743	107.105 - 107.108
	100	106.745 - 106.745	106.738 - 106.741	107.100 - 107.108
	150	106.742 - 106.742	106.734 - 106.740	107.100 - 107.112
	200	106.747 - 106.747	106.741 - 106.747	107.096 - 107.113
	500	106.741 - 106.742		
	1000	106.741 - 106.743		
	CMR 500	106.716	106.716	
$T = 10$	50	108.132 - 108.132	108.094 - 108.098	108.549 - 108.555
	100	108.131 - 108.131	108.107 - 108.115	108.631 - 108.641
	150	108.131 - 108.132	108.109 - 108.122	108.636 - 108.652
	200	108.131 - 108.132	108.115 - 108.123	108.632 - 108.664
	500	108.131 - 108.133		
	1000	108.130 - 108.133		
	CMR 500	108.078	108.078	

Table 2: Upper and lower estimates of the fair annual premium in the case  $\delta = 0$ 

	n	$\sigma_r = 0.08$		$\sigma_r = 0.16$	
		$\rho = -0.25$	$\rho = 0.25$	$\rho = -0.25$	$\rho = 0.25$
$T = 5$	50	109.866 - 109.871	110.005 - 110.010	110.171 - 110.176	110.544 - 110.549
	100	109.865 - 109.875	110.080 - 110.090	110.154 - 110.164	110.588 - 110.598
	150	109.855 - 109.870	110.088 - 110.103	110.146 - 110.161	110.599 - 110.615
	200	109.849 - 109.868	110.082 - 110.100	110.142 - 110.160	110.598 - 110.618
$T = 10$	50	111.686 - 111.693	111.840 - 111.847	112.187 - 112.193	112.579 - 112.586
	100	111.704 - 111.717	111.948 - 111.961	112.205 - 112.217	112.711 - 112.724
	150	111.704 - 111.721	111.976 - 111.994	112.203 - 112.220	112.749 - 112.767
	200	111.702 - 111.724	112.002 - 112.021	112.199 - 112.218	112.768 - 112.792

policy pay-off depends on the value of a reference fund made up of equities accrued by investing fixed contributions at every premium payment date. The surrender option is modeled endogenously in the evaluation framework as a Bermudan option that is exercised by the insured only if it is financially convenient. Since insurance policies are usually long term contracts, we propose to describe the equity dynamics by means of a bivariate model which takes into account stochastic interest rates. We construct a bivariate tree which approximates the joint evolution of the equity price and of the interest rate associating to each node of the tree a set of reference fund values. The presence of periodical premiums causes a huge number of possible reference fund values and, as a consequence, the fair premium evaluation becomes computationally burdensome even when a small number of steps is used. In order to overcome this obstacle, we use the singular points framework which permits to treat efficiently this deeply path-dependent problem. With this methodology we obtain accurate upper and lower estimates of the fair premium evaluated with the lattice structure. The numerical results confirm the reliability of this approach showing the robustness in dependence of the model parameters.

## 7 Appendix: Singular points

We recall the definitions of the singular points approach introduced in [16].

Table 3: Upper and lower estimates of the fair annual premium in the case  $\delta = 0.04$ 

	n	$\sigma_r = 0.08$		$\sigma_r = 0.16$	
		$\rho = -0.25$	$\rho = 0.25$	$\rho = -0.25$	$\rho = 0.25$
$T = 5$	50	114.075 - 114.081	114.237 - 114.243	114.594 - 114.600	115.008 - 115.015
	100	114.066 - 114.080	114.282 - 114.296	114.574 - 114.586	115.041 - 115.054
	150	114.058 - 114.078	114.293 - 114.313	114.565 - 114.584	115.048 - 115.067
	200	114.054 - 114.078	114.299 - 114.323	114.560 - 114.585	115.048 - 115.073
$T = 10$	50	118.850 - 118.860	119.022 - 119.032	119.825 - 119.835	120.304 - 120.315
	100	118.871 - 118.890	119.152 - 119.172	119.854 - 119.872	120.460 - 120.479
	150	118.877 - 118.904	119.193 - 119.220	119.851 - 119.877	120.507 - 120.534
	200	118.877 - 118.910	119.226 - 119.244	119.863 - 119.883	120.538 - 120.562

**Definition 1.** Let us consider a set of points:  $(x_1, y_1), \dots, (x_n, y_n)$ , such that

$$a = x_1 < x_2 < \dots < x_n = b,$$

and the piecewise linear function  $f(x)$ ,  $x \in [a, b]$ , obtained by interpolating linearly the given points. The points  $(x_1, y_1), \dots, (x_n, y_n)$  (which characterize completely  $f$ ), will be called the singular points of  $f$ , while  $x_1, \dots, x_n$  will be called the singular values of  $f$ .

Let us remark that  $f$  is convex if and only if the slopes

$$\alpha_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i},$$

are increasing, i.e.  $\alpha_i \leq \alpha_{i+1}$  for all  $i = 1, \dots, n-2$ .

The approach of singular points allows to construct upper and lower bounds of the value of an equity-linked policy with a minimum guarantee and a surrender option in a simple way, as pointed out in the next lemma (see also the geometrical interpretation in Figure 2 and 3).

**Lemma 1.** Let  $f$  be a piecewise linear and convex function defined on  $[a, b]$ , and let  $C = \{(x_1, y_1), \dots, (x_n, y_n)\}$  be the set of its singular points. Then:

a) Removing a point  $(x_i, y_i)$ ,  $2 \leq i \leq n-1$ , from the set  $C$ , the resulting piecewise linear function  $\tilde{f}$ , whose set of singular points is  $C \setminus \{(x_i, y_i)\}$ , is again convex in  $[a, b]$  and we have:

$$f(x) \leq \tilde{f}(x), \quad \forall x \in [a, b].$$

b) Let  $2 \leq i \leq n-2$ , and denote by  $(\bar{x}, \bar{y})$  the intersection between the straight line joining  $(x_{i-1}, y_{i-1})$ ,  $(x_i, y_i)$  and the one joining  $(x_{i+1}, y_{i+1})$ ,  $(x_{i+2}, y_{i+2})$ . If we consider the new set of  $n-1$  singular points

$$\{(x_1, y_1), \dots, (x_{i-1}, y_{i-1}), (\bar{x}, \bar{y}), (x_{i+2}, y_{i+2}), \dots, (x_n, y_n)\},$$

the associated piecewise linear function  $\tilde{f}$  is again convex on  $[a, b]$  and we have:

$$f(x) \geq \tilde{f}(x), \quad \forall x \in [a, b].$$

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Figure 2: Upper estimate:  $x_4$  has been removed.

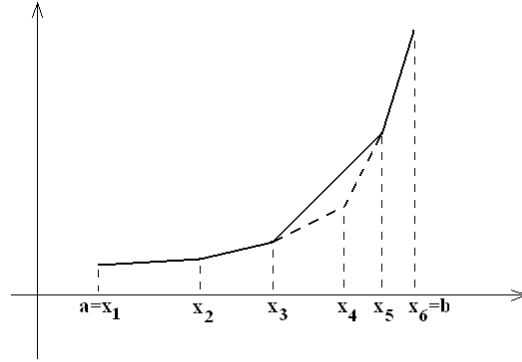
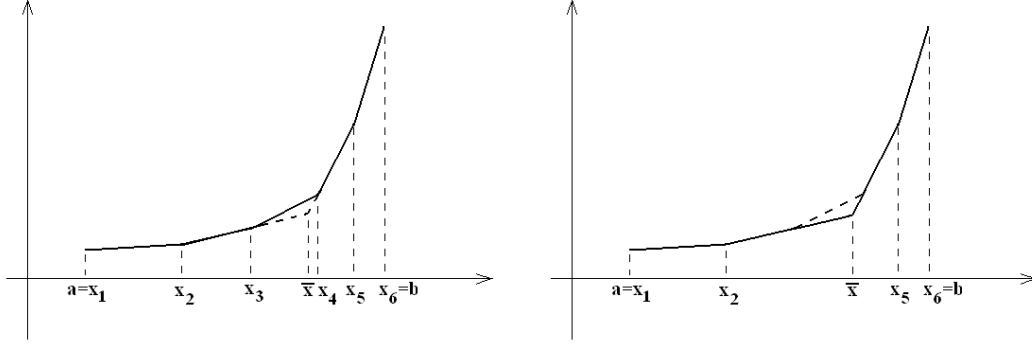


Figure 3: Lower estimate:  $x_3$  and  $x_4$  have been removed,  $\bar{x}$  has been inserted.



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